ON THE CONTINUED FRACTION EXPANSION OF ALGEBRAIC NUMBERS

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$0 < \xi < 1$ irrational.

$$\xi = \xi_{\underline{a}} = [0; a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}.$$

$(a_k \geq 1)$

$\underline{a} = (a_k)_{k \geq 1}$ sequence of partial quotients.

$\frac{p_n}{q_n} := [0; a_1, \dots, a_n]$: n^{th} convergent.

$\underline{a}$ viewed as the infinite word $a_1 a_2 a_3 \dots$

$\underline{a} = (a_k)_{k \geq 1}$ is ultimately periodic

if and only if

$\xi_{\underline{a}}$ is a quadratic real number.

PROBLEM: What can be said on $(a_k)_{k\geq 1}$ when ξ is algebraic of degree $\geqslant 3$?

It is expected that, when ξ is algebraic of degree ≥ 3, then $(a_k)_{k\geq 1}$ should be unbounded and every finite block of digits on $\{1, 2, 3, \dots\}$ should occur in the word $a_1 a_2 a_3 \dots$

Note that the continued fraction expansion of almost all numbers has both properties.

Our goal : To find conditions on $(a_k)_{k\geq 1}$ ensuring that the real number $[0; a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$ is transcendental.

LIOUVILLE (1844) : When $(a_k)_{k\geq 1}$ grows fast enough, then $[0; a_1, a_2, \dots]$ is transcendental.

[Ex. : $a_k = 10^{k!}$]

MAILLET (1906) : Explicit examples of transcendental numbers with bounded partial quotients.

Let $\xi := [0; a_1, a_2, \dots]$ be not quadratic. If $a_k \leq M$ for $k \geqslant 1$ and if there is an increasing sequence $(k_\ell)_{\ell \geqslant 1}$ s.t.

$$a_{k_\ell} = a_{k_\ell + 1} = \dots = a_{\ell k_\ell} = 1 \qquad (\ell \geqslant 1),$$

then ξ is transcendental.

Maillet : If θ is a real algebraic number of degree $d \geqslant 3$, then there exists $c(\theta) > 0$ s.t.

$$|\theta - \alpha| \geqslant c(\theta)\, H(\alpha)^{-d}$$

for every real quadratic number α.

$H(\alpha)$ is the max. of the absolute values of the coeff. of the minimal polynomial of α over $\mathbb{Z}$ (naïve height of α).

Idea : ξ is too close to algebraic numbers to be algebraic.

Indeed, for $\ell \geq 1$, the quadratic number $\xi_\ell := [0; a_1, a_2, \ldots, a_{k_\ell - 1}, 1, 1, 1, \ldots, 1, \ldots]$ is very close to ξ:

$$|\xi - \xi_\ell| \leq q_{\ell k_\ell}^{-2} \leq 2^{-\ell k_\ell}.$$

Height of ξ_ℓ : $H(\xi_\ell) \leq (M+2)^{2k_\ell}$.

$$|\xi - \xi_\ell| \leq H(\xi_\ell)^{-\ell (\log 2)/(2 \log(M+2))}.$$

Since ℓ is arbitrary, we are done.

Improvements of Maillet's result by A. Baker (1962) by means of Roth's theorem for number fields.

Further transcendence results, using the Schmidt Subspace Theorem, by Davison (1989), M. Queffélec (1998), Allouche - Davison - Queffélec - Zamboni (2001), Adamczewski - B. (2005, 2007), B. (2013).

Complexity of a sequence / an infinite word.

Let $\underline{w} = w_1 w_2 w_3 \dots$ be an inf. word over $\mathbb{Z}_{\geq 1}$.

For $n \geq 1$, set

$$p(n, \underline{w}) := \operatorname{Card}\{w_{k+1} \dots w_{k+n} : k \geq 0\}.$$

Obvious : $1 \leq p(n, \underline{w}) \leq +\infty$.

Furthermore, if $\underline{w}$ is ultimately periodic, then there exists C s.t. $p(n, \underline{w}) \leq C$ for $n \geq 1$.

If $\underline{w}$ is not ultimately periodic, then $p(n, \underline{w}) \geq n+1$ for $n \geqslant 1$.

Note that there are uncountably many $\underline{w}$ s.t. $p(n, \underline{w}) = n+1$ for $n \geqslant 1$ (Sturmian words).

THEOREM (Allouche et al., 2001): If $\xi = [0; a_1, a_2, \ldots]$ is algebraic of degree ≥ 3, then $p(n, \underline{a}) - n \xrightarrow[n \to +\infty]{} +\infty$, where $\underline{a} = (a_k)_{k \geqslant 1}$.

THEOREM (B. 2013):

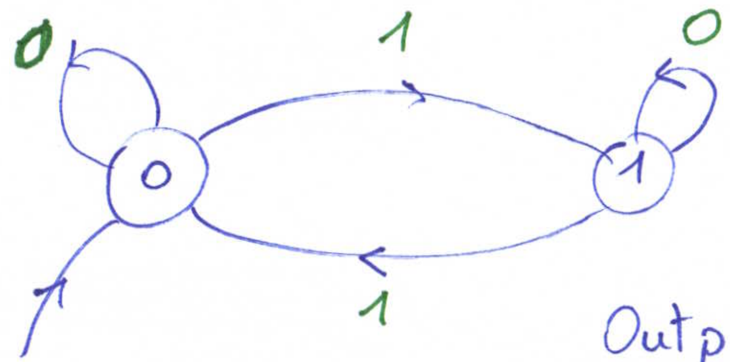
If $\xi := [0; a_1, a_2, a_3, \dots]$ is algebraic of degree ≥ 3, then

$$\frac{p(n, \underline{a})}{n} \xrightarrow[n \to +\infty]{} +\infty .$$

In other words: the sequence of partial quotients of a real algebraic number of degree $\geqslant 3$ cannot be « too simple ».

COROLLARY: The continued fraction expansion of an algebraic number of degree $\geqslant 3$ cannot be generated by a finite automaton.

[Indeed, seq. $\underline{a}$ generated by finite automata satisfy $p(n, \underline{a}) = O(n)$, Cobham 1972]



Input: n in base 2

Output: $u_n \in \{0, 1\}$

ex: $n = 12 = (1100)_2$

Output: 0

TRANSCENDENCE CRITERION:

Let $(a_k)_{k\geq 1}$ be a bounded sequence of positive integers. Assume that, for some integer $m \geq 1$ and arbitrarily large integers N, there exists a word W_N of length N having two occurrences in $a_1 a_2 \ldots a_{mN}$. Then, the real number $[0; a_1, a_2, \ldots]$ is either quadratic or transcendental.

This criterion implies the theorem. Indeed, if there are an integer $C \geqslant 2$ and $n_1 < n_2 < \dots < n_j < \dots$ such that $p(n_j, \underline{a}) \leq C n_j$ for $j \geqslant 1$, then, for $j \geq 1$, there is a block of length n_j having two occurrences in $a_1 a_2 \dots a_{(C+1) n_j}$ (apply Dirichlet's Schubfachprinzip).

Idea of the proof of the criterion:

By assumption, there is an infinite set $\mathcal{N} \subset \mathbb{Z}_{\geq 1}$ and, for N in $\mathcal{N}$, there are finite words U_N, V_N, W_N such that:

- $|W_N| = N$ [$|\cdot|$ is the length];
- $\underline{a}$ starts with $U_N W_N V_N W_N$;
- $|U_N W_N V_N W_N| \leq m N$.

Thus, our number $\xi := [0; a_1, a_2, \dots]$ is close to the quadratic number

$$[0; U_N W_N V_N W_N V_N W_N V_N \dots W_N V_N \dots].$$

THEOREM (SCHMIDT, 1967):

Let $\varepsilon > 0$. If θ is a real algebraic number of degree $\geqslant 3$, then there exists $c(\theta, \varepsilon)$ s.t.

$$|\theta - \alpha| \geqslant c(\theta, \varepsilon)\, H(\alpha)^{-3-\varepsilon}$$

for every real quadratic nb α.

[In Maillet's result, we have the exponent $-\deg(\theta)$ instead of $-3-\varepsilon$].

Not sufficient to get our result!

Schmidt Subspace Theorem

Let $m \geq 2$ be an integer. Let $L_1, \dots, L_m$ be linearly independent linear forms in $\underline{x} = (x_1, \dots, x_m)$ with algebraic, real coefficients. Let $\varepsilon > 0$. Then, the set of solutions $\underline{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ to

$$|L_1(\underline{x}) \times \dots \times L_m(\underline{x})| \leq \left(\max\{|x_1|, \dots, |x_m|\}\right)^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbb{Q}^m$.

In particular, if θ is real algebraic of degree $\geqslant 3$, then the set of integer triples (a_0, a_1, a_2) s.t.

$$\left|(a_2\theta^2 + a_1\theta + a_0) \times a_1 \times a_2\right| \leqslant \max\{|a_0|, |a_1|, |a_2|\}^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbb{Q}^3$.

$$\longmapsto \quad |P(\theta)| \leqslant H(P)^{-2-\varepsilon}$$ [$H(P)$: height of $P(X)$]

$\longmapsto$ α root of $P(X)$: $|\theta - \alpha| \leqslant H(\alpha)^{-3-\varepsilon}$.

First ingredient :

Apply the Schmidt Subspace Theorem with 4 linear forms (and not only 3).

The minimal polynomial of the quadratic nb $[0; \overline{a_1, a_2, \dots, a_s}]$ is

$$q_{s-1} X^2 - (p_{s-1} - q_s) X - p_s$$

(x_1, x_2, x_3) $=$ $(q_{s-1}, p_{s-1} - q_s, p_s)$

$\leadsto$ linear form $x_1 \xi^2 - x_2 \xi - x_3$

or better $x'_1 \xi^2 - x'_2 \xi + x'_3 \xi - x'_4$

$$(x'_1, x'_2, x'_3, x'_4) = (q_{s-1}, p_{s-1}, q_s, p_s)$$

Second ingredient :

Let $\alpha := [0; a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}]$ be a quadratic real number. Assume that $r \geq 3$ and $s \geq 1$. Let α' denote its Galois conjugate.

If $a_r \neq a_{r+s}$, then

$$|\alpha - \alpha'| \ll a_r^2 \max\{a_{r-1}, a_{r-2}\} q_r^{-2},$$

where q_r is the denominator of $[0; a_1, \ldots, a_r]$.

Notation : if $W = w_1 \dots w_n$, then $\overleftarrow{W} = w_n \dots w_1$.

A further transcendence criterion (B. 2013):

Let $(a_k)_{k \geq 1}$ be a bounded sequence of positive integers. Assume that, for some integer $m \geq 1$ and arbitrarily large integers N, there exists a word W_N of length N s.t. W_N and $\overleftarrow{W_N}$ occur without overlapping in $a_1 a_2 \dots a_{mN}$. Then, the real number $[0; a_1, a_2, \dots]$ is either quadratic or transcendental.

TRANSCENDENCE MEASURES (B. 2012).

Let $(a_k)_{k\geq 1}$ be a bounded, not ultimately periodic, sequence of positive integers such that $\limsup_{n\to+\infty} \frac{p(n,\underline{a})}{n} < +\infty$.

Set $\xi := [0; a_1, a_2, \ldots]$.

There exists $f : \mathbb{Z}_{\geq 1} \longrightarrow \mathbb{R}_{>0}$ s.t.

$$|\xi - \alpha| \geq H(\alpha)^{-f(\text{degree of } \alpha)}$$

for every real algebraic number α.