Erdős 100

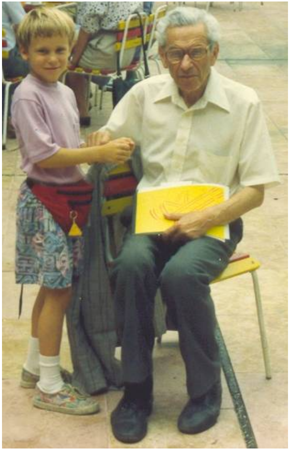
Divergent square averages and related topics

Zoltán Buczolich

Eötvös University, Budapest

`www.cs.elte.hu/∼buczo`

(partly joint work
with I. Assani and D. Mauldin)



The Birkhoff Ergodic Theorem

Assume that $(X, \mathcal{B}, \mu)$ *is a probability space,* $T : X \to X$ *is invertible and measure preserving* ($\mu(T^{-1}A) = \mu(A)$, $\forall A \in \mathcal{B}$) *and* $f \in L^1(X, \mathcal{B}, \mu)$*.*

Then $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^k x) = \overline{f}(x)$ *exists* μ *a.e.*

$\overline{f}(Tx) = \overline{f}(x)$ *a.e.*

This also implies $\frac{f(T^N x)}{N} \to 0$ μ a.e.

Weak $(1,1)$ inequality $\lambda > 0$:

$$\mu\{x : \sup_N \frac{1}{N} \sum_{k=1}^{N} f(T^k x) > \lambda\} \leq \frac{\int |f| d\mu}{\lambda}.$$

$A \in \mathcal{B}$ is **T-invariant** if
$0 = \mu(T^{-1}A \Delta A) = \mu((T^{-1}A \setminus A) \cup (A \setminus T^{-1}A))$.
T is **ergodic** when A is T-invariant $\Leftrightarrow \mu(A) = 0$ or 1.
If T is ergodic in the above thm. then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(T^k x) = \int_X f d\mu.$$

Recall:

Banach's principle: Let $1 \leq p < \infty$ and let T_n be a sequence of bounded linear operators on L^p.

If $\sup_n |T_n f| < \infty$ a.e. $\forall f \in L^p$

then the set of f for which $T_n f$ converges a.e. is closed in L^p.

e.g.: $T_n f = (1/n) \sum_{k=1}^{n} f(T^k x).$

D.: An infinite set $A \subset \mathbb{N}$ is of zero Banach density

if $$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \frac{\#(A \cap [n, n+k])}{k+1} = 0.$$

Results of Bourgain imply that if $f \in L^p(\mu)$, for some $p > 1$, the ergodic means

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{n^2}(x)) \qquad (1)$$

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D.: A sequence $\{n_k\}_{k=1}^{\infty}$ is L^1-*universally bad* if for all aperiodic ergodic dynamical systems there is some $f \in L^1$ such that $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^{n_k} x)$ fails to exist for all x in a set of positive measure.

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By the Conze principle and the Banach principle of Sawyer a sequence $\{n_k\}_{k=1}^{\infty}$ is *not* L^1-universally bad if and only if there exists a constant $C < \infty$ such that for all systems (X, Σ, μ, T) and all $f \in L^1(\mu)$ we have the following weak $(1,1)$ inequality for all $t > 0$

$$\mu\left(\left\{x : \sup_{N\geq 1}\left|\frac{1}{N}\sum_{k=1}^{N} f(T^{n_k}x)\right| > t\right\}\right) \leq \frac{C}{t}\int |f| d\mu.$$

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T.: **(Z.B. & D. Mauldin)** *The sequence $\{k^2\}_{k=1}^{\infty}$ is L^1-universally bad.*

This theorem is proved by showing that there is no constant C such that the above weak $(1,1)$ inequality holds.

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(*) $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^{n_k} x)$ conv. μ a.e.

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$\Rightarrow$ Conjecture (†) is false. This also provides an explanation why it was so difficult to obtain the result that $n_k = k^2$ is L^1-universally bad.

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Since $\frac{f(T^n x)}{n} \to 0$ a.e. $\mathbf{N}_n(f)(x) = \# \left\{ k : \frac{f(T^k x)}{k} > \frac{1}{n} \right\}$ is finite a.e..

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If $f \in L^p_+$ for $p > 1$, or $f \in L \log L$ and the transformation T is ergodic, then $\frac{\mathbf{N}_n(f)(x)}{n}$ converges a.e to $\int f d\mu$.

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Hence, the limit is the same as the limit of the ergodic averages $\frac{1}{N}\sum_{n=1}^{N} f(T^n x)$.
It is natural to ask whether $\frac{\mathbf{N}_n(f)(x)}{n}$ also converges a.e., when $f \in L^1(\mu)$.
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By using a generalized version of the Stein-Sawyer result (Assani [1997]) one can state the following equivalent problem.

Counting Problem II. *Does there exist a finite positive constant C such that for all measure preserving systems and all $\lambda > 0$*

$$\mu \left\{ x : \sup_n \frac{\mathbf{N}_n(f)(x)}{n} > \lambda \right\} \leq \frac{C}{\lambda} \|f\|_1 ?$$

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Averages along the squares on the torus

On $\mathbb{T}^2$ consider the erg. tr. $T(x,y) = (x+\alpha, y+2x+\alpha)$ with $\alpha \notin \mathbb{Q}$.

Suppose $f \in L^1(\mathbb{T})$ and $\widetilde{f}(x,y) = f(y)$.

Then

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$$\frac{1}{N}\sum_{n=1}^{N}(\widetilde{f} \circ T^n)(x, y) = \frac{1}{N}\sum_{n=1}^{N} f(y + 2nx + n^2\alpha) \to \int_{\mathbb{T}^2} \widetilde{f} = \int_{\mathbb{T}} f.$$

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What are the values x for which the averages

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$$D_{\alpha,f} = \left\{ x \in \mathbb{T} : \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(y+nx+n^2\alpha) \text{ d. n. e. for a.e. } y \right\}.$$

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The Hausdorff dimension of a set A will be denoted by $\dim_H A$.

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