A Poisson-type summation formula with automorphic weights

András Biró

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Let f be a "nice" even function on $\mathbb{R}$, let $w(n) := 1$ for every n, then the sum

$$\sum_{n=-\infty}^{\infty} w(n)f(n)$$

is invariant to the change

$$f \to g,$$

where g is the Fourier transform of f.

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- The Fourier transform is replaced by the Wilson function transform of type II (Groenevelt, 2003), and:
  it is its own inverse and it is an isometry on a Hilbert space.
- We have triple product integrals investigated very intensively in the theory of automorphic forms as weights (and the integers n are replaced by another discrete point set).

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$(f_1, f_2) := \int_{D_4} f_1(z)\overline{f_2(z)}d\mu_z$.

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$$B_0(\gamma z) = \nu(\gamma)\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{1/2} B_0(z) \text{ for } \gamma \in \Gamma_0(4).$$

Maass forms and cusp forms of weight $2n$:

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If f decays exponentially at cusps, then f is called a cusp form.

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  If $g(z)$ is such a form, then $g(z)\,(\mathrm{Im}\ z)^{-\frac{1}{4}-n}$ is holomorphic.

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- If $n \geq 1$, let $g_{n,1}, g_{n,2}, \ldots, g_{n,s_n}$ be an orthonormal basis of Maass cusp forms of weight $2n + \frac{1}{2}$ and $\Delta_{2n+\frac{1}{2}}$-eigenvalue $-\frac{1}{4} + \left(n - \frac{1}{4}\right)^2$.

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- If u is a cusp form of weight 0 for $SL(2,\mathbf{Z})$, and $\Delta_0 u = s(s-1)u$, then for $n \geq 0$ define

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- $$\zeta_a(f, r) := \int_{D_4} f(z)\overline{E_a\left(z, \frac{1}{2} + ir, \frac{1}{2}\right)}d\mu_z.$$

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- $\Gamma(X \pm Y)$ is the abbreviation of

  $$\Gamma(X+Y)\Gamma(X-Y),$$

  and similarly, $\Gamma(X \pm Y \pm Z)$ is the abbreviation of

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The sum below is invariant to the changes $u_1 \to \overline{u_2}$, $u_2 \to \overline{u_1}$, $f \to g$, where g is the type II Wilson function transform of f:

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$$\sum_{j=1}^{\infty} f\left(T_j\right) \Gamma\left(\frac{3}{4} \pm iT_j\right) \left(B_0\kappa_0\left(u_1\right), u_{j,\frac{1}{2}}\right) \overline{\left(B_0\kappa_0\left(u_2\right), u_{j,\frac{1}{2}}\right)}$$

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$$\frac{1}{4\pi} \sum_{a=0,\infty} \int_{-\infty}^{\infty} f(r) \Gamma\left(\frac{3}{4} \pm ir\right) \zeta_a\left(B_0 \kappa_0\left(u_1\right), r\right) \overline{\zeta_a\left(B_0 \kappa_0\left(u_2\right), r\right)} dr$$

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$$\sum_{n=1}^{\infty} f\left(i\left(\frac{1}{4} - n\right)\right) \Gamma\left(2n + \frac{1}{2}\right) \sum_{j=1}^{s_n} \left(B_0\kappa_n\left(u_1\right), g_{n,j}\right) \overline{\left(B_0\kappa_n\left(u_2\right), g_{n,j}\right)}$$

We write $(w)_n = \Gamma(w+n)/\Gamma(w)$, and we define the generalized hypergeometric function in the usual way:

$$ {}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix};z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{n!\,(b_1)_n\ldots(b_q)_n} z^n. $$

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Then define the Wilson function $\phi_\lambda(x)$ as the sum of

$$ \frac{\Gamma(2it_1)\,{}_4F_3\left(\begin{array}{c} \frac{1}{4} - it_1 + ix, \frac{1}{4} - it_1 - ix, \frac{1}{4} - it_1 + i\lambda, \frac{1}{4} - it_1 - i\lambda \\ \frac{1}{2} - it_1 + it_2, \frac{1}{2} - it_1 - it_2, 1 - 2it_1 \end{array}; 1\right)}{\Gamma\left(\frac{1}{2} - it_1 \pm it_2\right)\Gamma\left(\frac{1}{4} + it_1 \pm ix\right)\Gamma\left(\frac{1}{4} + it_1 \pm i\lambda\right)} $$

and the same expression with $-t_1$ in place of t_1.

Let

$$H(x) = \frac{\Gamma\left(\frac{1}{4} \pm it_1 \pm ix\right)\Gamma\left(\frac{1}{4} \pm it_2 \pm ix\right)\Gamma\left(\frac{1}{4} \pm ix\right)\Gamma\left(\frac{3}{4} \pm ix\right)}{\Gamma\left(\frac{1}{2} \pm it_1\right)\Gamma\left(\frac{1}{2} \pm it_2\right)\Gamma\left(\pm 2ix\right)}.$$

Define the measure dh for functions F on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ as

$$\int F(x)dh(x) := \frac{1}{2\pi}\int_0^\infty F(x)H(x)dx + i\sum_{x \in D^+} F(x)\mathrm{Res}_{z=x}H(z).$$

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$$H(x) = \frac{\Gamma\left(\frac{1}{4} \pm it_1 \pm ix\right)\Gamma\left(\frac{1}{4} \pm it_2 \pm ix\right)\Gamma\left(\frac{1}{4} \pm ix\right)\Gamma\left(\frac{3}{4} \pm ix\right)}{\Gamma\left(\frac{1}{2} \pm it_1\right)\Gamma\left(\frac{1}{2} \pm it_2\right)\Gamma\left(\pm 2ix\right)}.$$

Define the measure dh for functions F on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ as

$$\int F(x)dh(x) := \frac{1}{2\pi}\int_0^\infty F(x)H(x)dx + i\sum_{x \in D^+} F(x)\mathrm{Res}_{z=x}H(z).$$

Then the Wilson function transform of type II is defined as

$$(\mathcal{G}F)(\lambda) = \int F(x)\phi_\lambda(x)\,dh(x).$$

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$$\left(B_0\kappa_0\left(u_1\right), u_{j,\frac{1}{2}}\right) = \int_{D_4} B_0\left(z\right) u_1\left(4z\right) \overline{u_{j,\frac{1}{2}}\left(z\right)} d\mu_z.$$

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The right-hand side here is very closely related to

$$\int_{SL(2,\mathbf{Z})\backslash H} \left|u_1(z)\right|^2 \left(\mathrm{Shim}\, u_{j,1/2}\right)(z)\, d\mu_z,$$

where $\mathrm{Shim}\, u_{j,1/2}$ (the Shimura lift of $u_{j,1/2}$) is a cusp form of weight 0.

References

References

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Thank you for your attention!