Random Walks on Planar Graphs

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Harmonic functions

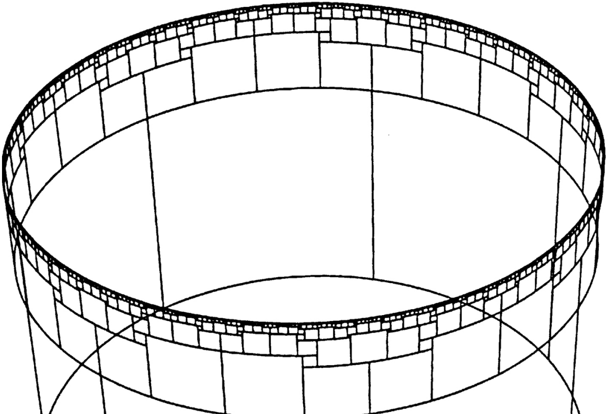
Recall, a function h from the vertices of a graph to $\mathbb{R}$ is harmonic, if for every vertex,

$$h(v) = 1/d_v \sum_{u \sim v} h(u).$$

Harmonic functions

Assume G is an infinite bounded degree transient planar graph, then G admits non constant bounded harmonic functions.

The proof with Oded Schramm, almost twenty years ago, uses a discrete uniformization via an infinite version of the square tiling of Brooks, Smith, Stone and Tutte (1940). Note the contrast between transient trees and the 3 dimensional grid.



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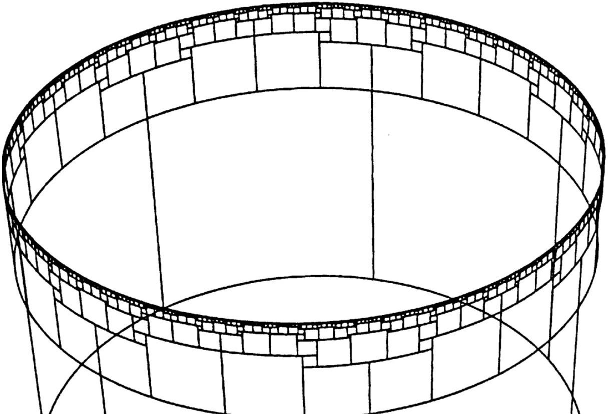

Harmonic functions

With Oded we *conjectured* that for transient, bounded degree planar graphs with one end, the boundary of the tiling above, is a realization of the Poisson boundary.

That is, any bounded harmonic function is realized as a solution to the Dirichlet problem with respect to a bounded measurable function on the boundary, and uniquely absorbing means having one transient end.

Theorem (Agelos Georgakopoulos (2013))

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Isoperimetric dichotomy

Let

$$i(G) = \inf_{0<|S|<\infty} \frac{|\partial S|}{|S|}.$$

An infinite graph G with $i(G) > 0$ is called *non-amenable*. Otherwise it is called amenable.

See Brian Bowditch (95) for a short proof of the fact (suggested by Gromov) that infinite planar graphs are either non-amenable or there is $C < \infty$ and arbitrarily large sets, S, with boundary ∂S of size smaller than $C|S|^{1/2}$.

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Speed dichotomy

By Markov type theory developed recently in Ball–Naor, Peres, Sheffield and Schramm –Ding, Lee and Peres:

Theorem

The simple random walk on an infinite planar graph is either ballistic or there is $C < \infty$ so that for any time t, there is a starting vertex v, so that

$$E(dist(X_t, v)) < Ct^{1/2} \log t.$$

Where X_t is a random walk starting at v.

Some perspective: Speed in vertex transitive graphs

A graph is vertex transitive if for any pair of vertices there is an isometry of the graph mapping one to the other.

Anna Erschler proved that on infinite vertex transitive graphs the simple random walk is at least diffusive and constructed examples (lamplighter graphs) where $\mathrm{dist}(o, X_n) \asymp n^\alpha$ where $X(n)$ denotes the simple random walk starting at o for $\alpha = 1 - 2^{-n}$ for all n.

Gidi Amir and Balint Virag constructed examples for any $\alpha \geq 3/4$. It is still open to construct examples with $1/2 < \alpha < 3/4$.

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Uniform growth

A graph admits uniform volume growth f, if there is $1 \leq C < \infty$, all balls of radius r has volume between $(1/C)f(r)$ and $Cf(r)$.

Try to *imagine* a planar triangulation with uniform polynomial volume growth of balls which is faster than quadratic.

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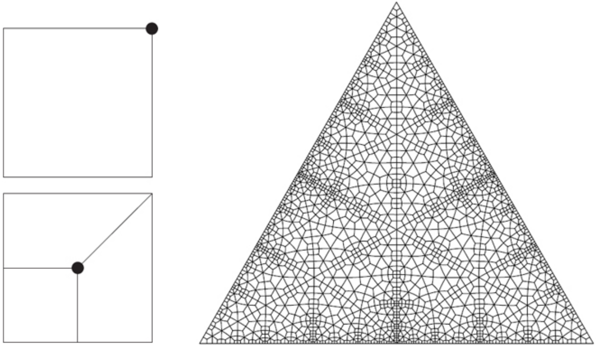
Uniform growth, an example



Cuts in planar graphs

Theorem (with Panos Papasoglu)

Let G be a planar graph such that $|B(v, 2n)| < C|B(u, n)|$ for some constant $C > 0$ for any two balls. Then for every vertex v of G and integer n, there is a domain Ω such that $B(v, n) \subset \Omega$, $\Omega \subset B(v, 6n)$ and the size of the boundary of Ω is at most order n.

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Cuts in planar graphs

This suggests that, a volume doubling planar graphs even with polynomial growth faster than quadratic, has a fractal structure of cactus like folds at all scales, in order to account for the volume together with the small cuts as seen from every point.

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Subdiffusive

Assume G planar and has uniform polynomial volume growth r^β with $\beta > 2$.

Conjecture

The simple random walk spends a long time in such traps and hence is subdiffusive (that is, $\mathrm{dist}(o, X_n) \asymp n^\alpha$ *where* $X(n)$ *denotes the simple random walk starting at* o *and* $\alpha < 1/2$*) ?*

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We know one example supporting the conjecture, Angel and Schramm's *Uniform infinite planar quadrangulation*.

For the UIPQ, which has asymptotic volume growth r^4, subdiffusivity holds. The proof (with Nicolas Curien) uses spatial Markovity, to get an upper bound of $1/3$, conjecturally the exponent is $1/4$.

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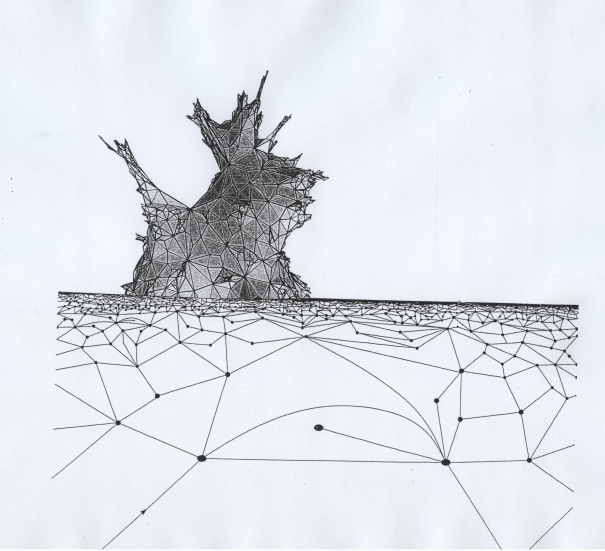
The uniform infinite planar quadrangulation



Limits

A limit of finite graphs G_n is a random rooted infinite graph (G, ρ) with the property that neighborhoods of G_n around a random vertex converge in distribution to neighborhoods of G around ρ.

Formally, let (G, o) and $(G_1, o_1), (G_2, o_2), \ldots$ be random connected rooted locally finite graphs. We say that (G, o) is the limit of (G_j, o_j) as $j \to \infty$ if for every $r > 0$ and for every finite rooted graph (H, o'), the probability that (H, o') is isomorphic to a ball of radius r in G_j centered at o_j converges to the probability that (H, o') is isomorphic to a ball of radius r in G centered at o.

Limits

Exercise: What is the limit of n-levels full binary trees?

Hint: A random uniform vertex in the n-levels full binary, is near the leaves.

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The limit of the n-levels full binary tree is the *canopy* tree, consisting of half line $\mathbb{N}$ with a binary tree of height n rooted at n and the root is on the line with geometric$(1/2)$ distance to 0.

This example illustrates the following: With Oded Schramm we proved (2001) that limits of bounded degree finite planar graphs are a.s. recurrent for the simple random walk.

The proof uses circle packing and the fact that near a random point the circle packing "looks" as if it has at most one accumulation point.

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UIPT

Angel and Schramm constructed the uniform infinite planar triangulation (UIPT), a rooted infinite random triangulation which is the limit (in the sense above) of finite random triangulations: the uniform measure on all nonisomorphic triangulations of the sphere of size n.
The UIPQ is a similar construction with quadrangulation, due to Krikun and Chassaing.

UIPT

The UIPT/Q looks very different from random perturbations of the plane as in the Poisson-Voronoi triangulation and has a rather surprising geometry at first encounter, e.g. volume growth of balls in the UIPT is asymptotically r^4.
The UIPT is recurrent for the simple random walk (Gurel-Gurevich and Nachmias (2012)).

UIPQ

Le Gall and Miermont (2011) proved that the random triangulations scaled Gromov-Hausdorff converge to a random compact metric space of dimension 4. This limiting surface called the Brownian map can be seen as the two-dimensional sphere equipped with a random metric which induces the usual topology but makes it a fractal space of Hausdorff dimension 4.
It is of interest to obtain quantitative estimates on the rate of convergence as in the Hungarian coupling of random walks and Brownian motion.

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Progress, due among others to Krikun, Duplantier and Sheffield and Curien.
Further establish the KPZ relation (Knizhnik-Polyakov-Zamolodchikov) between random and Euclidean geometries.

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Limits of triangulations with no genus restriction

Guth, Parlier and Young (2010) studied pants decomposition of random closed surfaces obtained by randomly gluing N Euclidean triangles (with unit side length) together.

Their work indicates that the injectivity radius around a typical point is growing to infinity. That is, the growing neighborhoods around a random vertex are simply connected.

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Limits of triangulations with linear genus

Take a uniform measure on triangulations with N triangles conditioned on the genus to be CN for some fixed $C < 1/4$ and a uniformly chosen root.

We *conjecture* that as N grows, this random surface converges to a rooted random triangulation of the hyperbolic plane with average degree $6/(1-4C)$.

In particular we believe that the local injectivity radius around a typical vertex will go to infinity on such a surface as N grows.

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Scaling limits

There is growing interest in establishing a rigorous theory of two dimensional continuum quantum gravity. Heuristically, quantum gravity is a metric chosen on the sphere uniformly among all possible metrics. Although there are successful discrete mathematical quantum gravity models, we do not yet have a satisfactory continuum definition of a planar random length metric space (rather than random measure).

Random subdivisions

One possible toy model is to start with a unit square divide it four squares and now recursively at each stage pick a square uniformly at random from the current squares (ignoring their sizes) and divide it to four squares and so on.

Random subdivisions

Look at the minimal number of squares needed in order to connect the bottom left and top right corner with a connected set of squares.

It is *conjectured* that there is a deterministic scaling function, such that after dividing the random minimal number of squares needed after n subdivisions by it, the result is a non degenerate random variable. Establishing the conjecture will provide a random planar length metric space.

Question

Does geodesic stabilize, as we further divide?

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Very recently Mikhail Khristoforov , Victor Kleptsyn and Michele Triestino, established existence of scaling limit in simpler random geometric set up. They were able to construct a random metric on the fractal objects that are the limit on finite graphs: this is an intermediate difficulty problem, since contrary to an interval, the number of possible geodesic paths joining any two points is infinite, but still it saves some peculiarity of the one-dimensional world.

Generalization of planarity

Excluded minor, Separation, *Packability*.

Generalization of planarity

Question

Which graphs can be realized as the nerve graph of a sphere packing in Euclidean d-dimensional space?

Where vertices correspond to spheres with disjoint interiors and edges to tangent spheres.

Packing

Theorem (with Oded Schramm)

The grid $\mathbb{Z}^4$, $T_3 \times \mathbb{Z}$ and lattices in hyperbolic 4-space cannot be sphere packed in Euclidean $\mathbb{R}^3$.

The proof is an adaptation of the result above, that bounded degree transient planar graph admits non constant harmonic functions, to p-potential theory.

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The following is an extension to higher dimension of the theorem of Bowditch.

Theorem (with Nicolas Curien)

Let G be an infinite locally finite connected graph which admits a regular packing in $\mathbb{R}^d$. Then we have the following alternative: either G has a positive Cheeger constant, or there are arbitrarily large subsets S of G such that $|\partial S| < |S|^{\frac{d-1}{d}+o(1)}$.

The proof uses sparse graphs limits: Local limits of bounded degree finite planar graphs are a.s. recurrent for the simple random walk, adapt the proof to show that local limit of finite graphs that are regularly packed in $\mathbb{R}^d$, are d-parabolic, which is the key to results above.

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