Ramsey-Turán numbers of graphs and hypergraphs.

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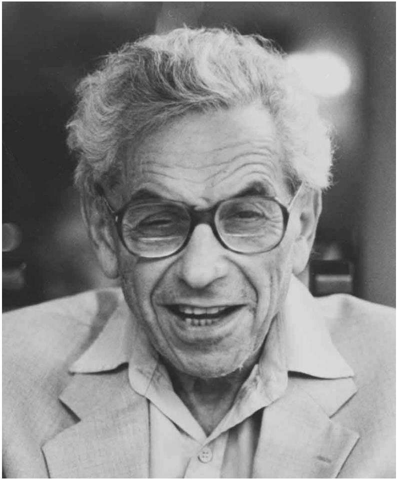
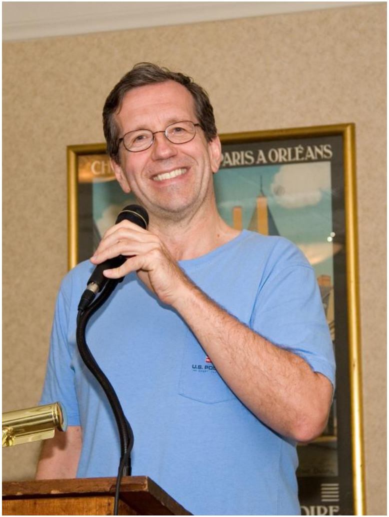
Pál Erdős 100!



Ákos Seress (1958–2013)



Balázs Montágh (1967–2013)

Turán Theorem

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Denote $T_{n,r}$ the complete r-partite, n-vertex graph with almost equal part sizes. Let

$$\text{ext}(n, H) := \max\{e(G) : \; v(G) = n, H \not\subset G\}.$$

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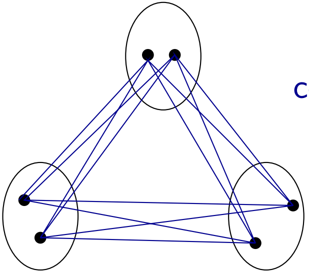
Turán's Theorem [1941]

$$\text{ext}(n, K_{r+1}) = e(T_{n,r}).$$



complete

K_4-free

Ramsey Problem

Graph Ramsey Problem [1929]

Let $\mathbf{R}(s, t)$ be the smallest n such that every graph on n vertices either contains a clique K_s or an independent set I_t.
What is $\mathbf{R}(s, t)$?

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What is $\mathbf{R}(s,t)$?

$$(2+o(1))^{s/2} \ \leq \ \mathbf{R}(s,s) \ \leq \ (4-o(1))^s.$$

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$$(2+o(1))^{s/2} \le \mathbf{R}(s,s) \le (4-o(1))^s.$$

$$\mathbf{R}(3,s) \sim \frac{s^2}{\log s}.$$

It is believed that the best possible structures are randomlike.

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- K_5-free G_n exists with $\alpha(G_n) < n/100$.
- How many edges such graph can have?

Ramsey and Turán Flavour!

Ramsey-Turán Numbers of graphs

Erdős, V.T. Sós [1970]

The Ramsey-Turán function for a graph H, function $f(n)$,

$$\mathbf{RT}(n, H, f(n)) := \max_{G_n} \{e(G_n) : H \not\subset G_n,\ \alpha(G_n) \le f(n)\}.$$

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Definition

$$\mathbf{RT}(n, H, o(n)) := \left(\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\mathbf{RT}(n, H, \epsilon n)}{n^2} \right) n^2 + o(n^2).$$

Definition

The Ramsey-Turán Density (Number) of H is the constant defined by the double limit.

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Complete Graphs

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$$\mathbf{RT}(n, K_{2s+1}, o(n)) = \frac{1}{2}\left(1 - \frac{1}{s}\right) n^2 + o(n^2).$$

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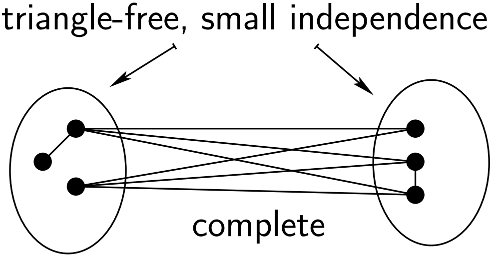
Example

$$\mathbf{RT}(n, K_5, o(n)) = \frac{1}{4} n^2 + o(n^2).$$



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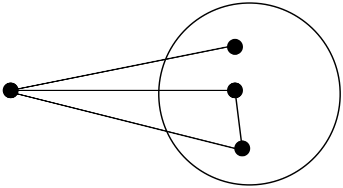
$$\mathbf{RT}(n, K_3, f(n)) \le \frac{1}{2} f(n) \cdot n.$$

$$\mathbf{RT}(n, K_3, o(n)) = o(n^2).$$



The inverse problem and applications I.

Ajtai, Komlós, Szemerédi [1980]

There is a $c > 0$ so that if $t = 2e(G_n)/n$ and G_n is triangle-free, then

$$c\frac{n}{t}\log t < \alpha(G_n).$$

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Using

$$t \leq \alpha(G_n),$$

Corollary

The Ramsey number $R(3, m) = O\left(\frac{m^2}{\log m}\right)$.

The inverse problem and applications II.

Komlós, Pintz, Szemerédi [1981, 1982]

Heilbronn's Conjecture is false!
There exists n points in the unit disc in the plane such that the area of each of the $\binom{n}{3}$ triangles is at least $\Omega\left(\frac{\log n}{n^2}\right)$.

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Ajtai, Komlós, Szemerédi [1981]

A set of integers is a **Sidon set** if all pairwise sums are distinct.
There exists a Sidon set $\subset [n]$ of size $\Omega((n \log n)^{1/3})$.

Fox [2010]

If $\alpha(G_n) = 2$ then G_n contains a clique-minor of size

$$\frac{n}{3} + \frac{1}{9} n^{4/5} \log n.$$

The case K_4: Upper bound

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$$\mathbf{RT}(n, K_3, o(n)) = o(n^2).$$

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Szemerédi [1972]

$$\mathbf{RT}(n, K_4, o(n)) \leq \frac{1}{8}n^2 + o(n^2).$$

- In the cluster graph the density of each regular pair is at most

$$1/2 + o(1).$$

- The cluster graph is triangle-free.

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$V(G) := V_1 \cup V_2$.

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Let $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$.

- For $i \in \{1, 2\}$ let $x_i y_i \in E(G)$ if $d(x_i, y_i) > 2 - \theta$.

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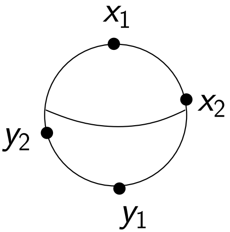
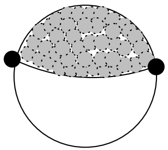
- For $i \in \{1, 2\}$ let $x_i y_i \in E(G)$ if $d(x_i, y_i) > 2 - \theta$.
- Let $x_1 x_2 \in E(G)$ if $d(x_1, x_2) < \sqrt{2} - \theta$.





K_t-independence

Definition [Hajnal]

K_t-independence number of G:

$$\alpha_t(G) := \max\{|S| : S \subseteq V(G), G[S] \text{ is } K_t\text{-free}\}.$$

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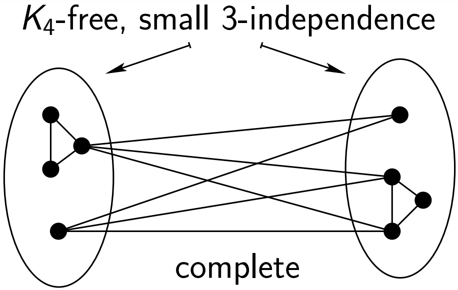
- $\mathbf{RT}_3(n, K_7, o(n)) = \frac{1}{4}n^2 + o(n^2)$.



K_t-independence

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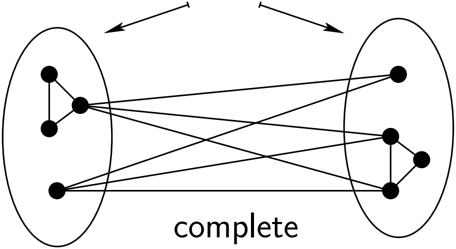
- $\mathbf{RT}_3(n, K_4, o(n)) = o(n^2)$.
- $o(n^2) \le \mathbf{RT}_3(n, K_5, o(n)) \le \frac{1}{12}n^2 + o(n^2)$.
- $o(n^2) \le \mathbf{RT}_3(n, K_6, o(n)) \le \frac{1}{6}n^2 + o(n^2)$.
- $\mathbf{RT}_3(n, K_7, o(n)) = \frac{1}{4}n^2 + o(n^2)$.

K_4-free, small 3-independence



complete

K_t-independence

Erdős, Hajnal, Simonovits, Sós, Szemerédi [1994]:

- *Is* $\mathbf{RT}_3(n, K_5, o(n)) = \Omega(n^2)$*?*

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- *Is* $\mathbf{RT}_3(n, K_5, o(n)) = \Omega(n^2)$*?*
- *What is the smallest* ℓ *such that* $\mathbf{RT}_t(n, K_{t+\ell}, o(n)) = \Omega(n^2)$*?*
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Balogh, Lenz [2011]

For every $t \geq 2$

$$\mathbf{RT}_t(n, K_{t+2}, o(n)) = \Omega(n^2).$$

More problems:

Erdős, Hajnal, Simonovits, Sós, Szemerédi [1994]: Should be true!

$$\mathbf{RT}_t(n, K_{2t}, o(n)) \geq \frac{1}{8}n^2 + o(n^2),$$

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Balogh, Lenz [2013]

For every $t \geq 2$

$$\mathbf{RT}_t(n, K_{t+\lceil t/2 \rceil+1}, o(n)) \geq \frac{1}{8}n^2 + o(n^2).$$

Known to be sharp for $t \leq 8$ *even, should be for every even* t.

More Precise Result

Balogh, Lenz [2013]

Let $2 \leq s \leq r$. Let ℓ be the maximum positive integer such that $\lceil r \cdot 2^{-\ell} \rceil < s$. Then

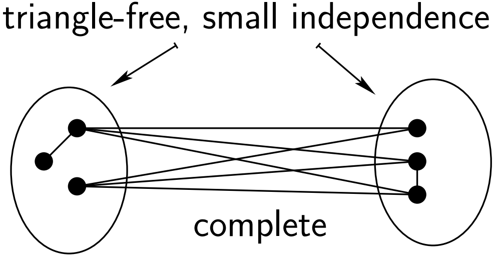
$$\mathbf{RT}_r(n, K_{r+s}, o(n)) \geq \frac{1}{2^{\ell+2}} n^2 + o(n^2) \approx \frac{s-1}{4r} n^2 + o(n^2).$$

Known to be sharp, when $4r/(s-1)$ is a power of 2 and $s \leq 5$.

Other direction

Example

$$\mathbf{RT}(n, K_5, o(n)) = \frac{1}{4}n^2 + o(n^2).$$



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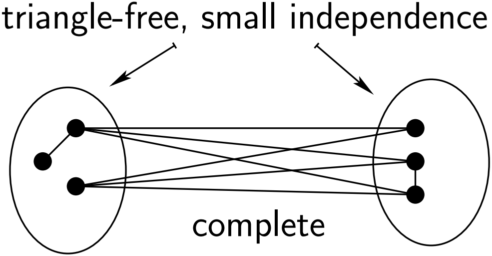

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J. H. Kim [1995]

$$\mathbf{R}(3, m) = \Theta\left(\frac{m^2}{\log m}\right).$$

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$$\left(\frac{1}{4} - o(1)\right) \frac{m^2}{\log m} \leq \mathbf{R}(3, m).$$

$$\mathbf{R}_3^*(n) := x \quad \text{such that} \quad \mathbf{R}(3, x) = n.$$

Corollary

$$\left(1\Big/\sqrt{2} - o(1)\right) \sqrt{n \log n} \leq \mathbf{R}_3^*(n) \leq \left(\sqrt{2} + o(1)\right) \sqrt{n \log n}.$$

Phase Transitions of K_5

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$\mathbf{RT}(n, K_5, c\sqrt{n \log n}) = n^2/4 + o(n^2)$ for every $c > 1$.

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Balogh-Hu-Simonovits [2013+]

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K_5 has a phase transition at $\sqrt{n \log n}$.

Phase Transitions of cliques

Balogh-Hu-Simonovits [2013+]

Phase transitions of K_r are at around inverse Ramsey numbers!

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Balogh-Hu-Simonovits [2013+]

Phase transitions of K_r are at around inverse Ramsey numbers!

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Dependent Random Choice Lemma

Let a, d, m, n, r be positive integers. Let $G = (V, E)$ be a graph with n vertices and average degree $d = 2e(G)/n$. If there is a positive integer t such that

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a,$$

then G contains a subset U of at least a vertices such that every r vertices in U have at least m common neighbors.

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- Assume there is a K_5-free graph G on n vertices with εn^2 edges and $\alpha(G) = o\left(\sqrt{n \log n}\right)$.

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- So there is an edge in their common neighborhood.
- Find a K_5! Contradiction.

Ramsey Graphs are sparse!

Non-symmetric Ramsey:

For a fixed s as $t \to \infty$

$$\left(\frac{t}{\log t}\right)^{(s+1)/2} \leq \mathbf{R}(s,t) \leq \frac{t^{s-1}}{\log^{s-2} t}.$$

Balogh-Hu-Simonovits [2013+]

If $K_s \not\subset G_n$, $\alpha(G_n) \leq t$ *and* $n \approx \mathbf{R}(s,t)$, *then*

$$e(G_n) = o(n^2).$$

Open Problems

- Erdős, Hajnal, Simonovits, Sós, and Szemerédi [1994]:

$$\mathbf{RT}(n, H, o(n)) \leq \mathbf{RT}(n, K_{\gamma(H)}, o(n)),$$

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Open Question:

$$\mathbf{RT}\left(n, K_{2,2,2}, o(n)\right) = o(n^2)?$$

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Open Question:

$$\mathbf{RT}\left(n, K_5, (1-\varepsilon)R_3^*(n/2)\right) = o(n^2)?$$

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Open Question:

$$\mathbf{RT}(n, K_6, \varepsilon\sqrt{n \log n}) > \frac{n^2}{100} \; ?$$