The 'DECOMPOSITION THEOREM'

ANTAL BALOG,

Alfréd Rényi Institute of Mathematics, Budapest,

joint work (in progress) with TREVOR WOOLEY,

University of Bristol.

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Let A be a finite set of real numbers, the sum set and the product set are defined by

$$A+A=\{a+b;\ a,b\in A\},\quad A\cdot A=\{ab;\ a,b\in A\}.$$

In size they are between $|A|$ and $|A|^2$.

$A+A$ can be small if A has some 'additive structure', similarly $A\cdot A$ can be small if A has some 'multiplicative structure'.

The Erdős–Szemerédi sum–product conjecture claims that these structures cannot come together, namely for any $\epsilon>0$ and for any sufficiently big finite set A

$$\max\{|A+A|,|A\cdot A|\}\geq|A|^{2-\epsilon}.$$

The best result is due to Solymosi (2009),

$$\max\{|A+A|,|A\cdot A|\}\geq\frac{|A|^{4/3}}{(2\log|A|)^{1/3}}.$$

Another notion to measure the 'additivity' (or 'multiplicativity') of a set is the additive energy (or multiplicative energy).

$$E_+(A) = \#\{a_1 + a_2 = a_3 + a_4;\ a_1, a_2, a_3, a_4 \in A\},$$
$$E_\times(A) = \#\{a_1 a_2 = a_3 a_4;\ a_1, a_2, a_3, a_4 \in A\}.$$

In size they are between $|A|^2$ and $|A|^3$. Sometimes better to view the energy as the quadratic moment of the representation function,

$$s_A(r) = \#\{r = a_1 + a_2;\ a_1, a_2 \in A\},$$
$$p_A(r) = \#\{r = a_1 a_2;\ a_1, a_2 \in A\}.$$

We have

$$E_+(A) = \sum_r s_A(r)^2, \quad E_\times(A) = \sum_r p_A(r)^2,$$

and a straightforward Cauchy–Schwarz inequality gives

$$|A|^4 \le |A + A|\, E_+(A), \quad |A|^4 \le |A \cdot A|\, E_\times(A).$$

If A is 'additive', i.e. $A + A$ is small, then $E_+(A)$ is big,

If A is 'multiplicative', i.e. $A \cdot A$ is small, then $E_\times(A)$ is big.

One may believe that the sum–product conjecture appears as one of the two energies should be small. However, this is very far from true.

This is because $E_+(A)$ or $E_\times(A)$ being big does not require that $A+A$ or $A \cdot A$ being small. For example, if half of A has some additive structure, while the other half has some multiplicative structure then both energies are big.

We will show that this is the only example (in a weak sense). Every finite set of real numbers A can be split into two disjoint parts $A = B \cup C$ with both $E_+(B)$ and $E_\times(C)$ are small.

The sum–product conjecture would follow if these energies are as small as $|A|^{2+\epsilon}$. However, this is not true either.

THEOREM 1: There are arbitrarily large finite sets of integers A such that for any split A into two parts $A = B \cup C$ one has

$$\max\{E_+(B), E_\times(C)\} \gg |A|^{7/3}.$$

CONSTRUCTION: Let N be a large integer, and let

$$A = \{m2^n\,;\, m \leq 2N^{2/3},\, m \text{ is odd},\, n \leq N^{1/3}\}.$$

Note that A is an arithmetic progression for any fixed n, as well as a geometric progression for any fixed m. Obviously $|A| \sim N$ and one can show that both the additive and multiplicative energy of A are big. This is also true for any big subset of A, if $B \subset A$, $|B| \geq |A|/2$ then $E_+(B) \gg N^{7/3}$, and $E_\times(B) \gg N^{7/3}$.

THEOREM 2: Any finite set of real numbers A can be split into two sets $A = B \cup C$ such that

$$\max\{E_+(B), E_\times(C)\} \ll |A|^{3-2/33} \log |A|.$$

Key ingredients:

We have mentioned that $E_+(A)$ big does not imply $A+A$ small.

However $E_+(A)$ big DOES imply that $A' + A'$ small for a big subset $A' \subset A$.

The quantitative statement is known as the B–Sz–G lemma. There are several versions, the lemma below is due to Thomas Schoen.

LEMMA 1: Let $1 > \theta > 0$ be a fixed real number and A be a subset of the real numbers with $|A| \leq N$. If $E_+(A) > N^{2+\theta}$, then there is an $A' \subset A$ such that

$$\text{i.)}\ |A' + A'| \ll N^{8-7\theta}, \quad \text{ii.)}\ |A'| \gg N^{(1+3\theta)/4}.$$

Note that the result is not trivial for θ close to 1 only. The proof is a complex averaging argument.

The next lemma is the key ingredient of Solymosi in his sum–product estimate. It is actually estimating the multiplicative energy by the size of the sum set.

LEMMA 2: Let A be a large finite subset of the real numbers. We have

$$E_{\times}(A) \ll |A+A|^2 \log |A|.$$

The proof is elementary geometry and is well described in the talk of Solymosi tomorrow.

Finally we need the fact that the multiplicative energy (or the additive energy as well) shows some subadditive behavior.

LEMMA 3: Let A_j, $j = 1, \ldots, K$ be finite subsets of the real numbers. We have

$$E_\times(\bigcup_{j=1}^{K} A_j) \le K^3 \sum_{j=1}^{K} E_\times(A_j).$$

This is a simple consequence of the Cauchy–Schwarz inequality.

The idea of the proof:

The optimal chice is $\theta = 31/33$, write $|A| = N$.
If $E_+(A) \leq N^{2+\theta}$ then $B \cup C = A \cup \emptyset$ and we are done.
If $E_+(A) > N^{2+\theta}$ then there is an $A_1 \subset A$ large with small doubling by Lemma 1.
If $E_+(A \setminus A_1) \leq N^{2+\theta}$ then $B \cup C = (A \setminus A_1) \cup A_1$ and we are done since $E_\times(A_1) \ll |A_1 + A_1|^2 \log N \ll N^{16-14\theta} \log N$ by Lemma 2 and Lemma 1.
If $E_+(A \setminus A_1) > N^{2+\theta}$ then we repeat this argument with $A \setminus A_1$ in place of A getting an $A_2 \subset A \setminus A_1$, next we repeat with $A \setminus (A_1 \cup A_2)$ in place of A, and so on.

We arrive at large disjoint subsets $A_1, \ldots, A_K \subset A$ such that for all $j = 1, \ldots, K$ $A_j + A_j$ has small doubling and

$$E_+(A \setminus \bigcup_{j=1}^{K} A_j) \leq N^{2+\theta}.$$

We get the decomposition

$$B \cup C = \big(A \setminus \bigcup_{j=1}^{K} A_j\big) \cup \big(\bigcup_{j=1}^{K} A_j\big).$$

$E_\times(C)$ is small by Lemma 3 and Lemma 2.

Some remarks on further works:

– Similar results can be derived in finite fields of prime order.
– Similar results can be derived for higher degree energies, for higher moments of the representation functions.
– Similar results can be derived for multiple energies, for the number of additive (multiplicative) six–tuples, eight–tuples, etc.
– For the decomposition $A = B \cup C$ in Theorem 2, the mutual energies of B and C are also small,

$$\max\{E_+(B,C), E_\times(B,C)\} \ll |A|^{3-1/33} \log^{1/2} |A|.$$

Here

$$E_+(B,C) = \#\{b_1 + c_1 = b_2 + c_2;\ b_1, b_2 \in B, c_1, c_2 \in C\},$$
$$E_\times(B,C) = \#\{b_1 c_1 = b_2 c_2;\ b_1, b_2 \in B, c_1, c_2 \in C\}.$$

Finally:

– I have NO ANY idea what is the right exponent in the theorems. There is probably room for improvement in both the construction of a lower bound and in the upper bound.

THANK YOU FOR YOUR ATTENTION.