# Multiplicative Functions and Small Divisors*

by

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* A report of joint work of K. Alladi, P. Erdös, J. D. Vaaler

and later work of

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# Background and Motivation



Let $S \subseteq \mathbb{Z}^+$ and

$$g(n) = \prod_{p|n,\ p=\text{prime}} g(p) \quad (\text{strongly mult.}),\ g \geq 1$$

**Problem**: Obtain upper bound for

$$S(g;x) = \sum_{n \leq x,\ n \in S} g(n)$$

**Idea**: Write $\quad g(n) = \sum_{d|n} h(d)$

Then $h$ is multiplicative, $0 \leq h$,

$$h(p) = g(p) - 1,\quad h(p^\alpha) = 0,\ \text{for } \alpha \geq 2.$$

Thus

$$S(g;x) = \sum_{n \leq x,\ n \in S} \ \sum_{d|n} h(d)$$

$$= \sum_{d \leq x} h(d) \sum_{n \in S,\ n \equiv 0 \pmod d} 1 = \sum_{d} h(d) |S_d(x)|$$

where $S_d(x) = \{ m \in S \mid m \equiv 0 \pmod d),\ m \leq x \}$

Suppose we have the estimate

$$|S_d(x)| \ll |S_1(x)| \frac{\omega(d)}{d}, \quad \omega\text{-}\underline{\text{mult.}} \qquad (1)$$

then we would get (with $X = |S_1(x)|$

$$S(g;x) \ll X \cdot \sum_d \frac{h(d)\,\omega(d)}{d} \ll X \cdot \prod_{p \le x} \left(1 + \frac{\omega(p)\,h(p)}{p}\right)$$

It turns out that (1) usually holds only for

$$d \le x^\beta, \quad \text{with some } \beta < 1$$

For example

a) if $S = \{ p+a \mid p \le x \}$, shifted primes,

then $\beta = \frac{1}{2}$ (Brun-Titchmarsh)

b) if $S = \{ P(n) \mid P(n) \le x \}$, $P(t) \in \mathbb{Z}^+[t]$,

then $\beta = \dfrac{1}{\deg P}$.

So what we require are inequalities like

$$\sum_{d \mid n} h(d) \ll \sum_{d \mid n,\, d \le n^\beta} h(d), \qquad (2)$$

with $0 < \beta < 1$.

To lead us to such estimates, we formulate

Meta theorem : "Large divisors of an integer are more composite than the smaller ones."

We will characterize this meta theorem in various ways.

On the basis of the meta theorem, I made

Weak conjecture (Alladi, 1983 Asilomar Conf.)

For each integer $k \geq 2$, there exists $c_k$ such that if $0 \leq h(p) \leq c_k$, then

$$\sum_{d|n}{}' h(d) \ll_k \sum_{d|n,\ d \leq n^{1/k}}{}' h(d)$$

An example:

Let $r$ be large & $p_1 \sim p_2 \sim \cdots \sim p_r$ primes.

Let $n = p_1 p_2 \cdots p_r$. Then $d|n$ satisfies

$$d \leq n^{1/k} \iff \nu(d) \lesssim \frac{r}{k}$$

Suppose $h(p) = c$, $\forall\, p$. Then

$$\sum_{d|n}{}' h(d) = (1+c)^r$$

and

$$\sum_{d|n,\, d \le n^{1/k}}{}' h(d) \simeq \sum_{\ell \le \frac{r}{k}}{}' \binom{r}{\ell} c^{\ell} \qquad (3)$$

The peak of $\binom{r}{\ell} c^{\ell}$ occurs when $\ell \sim \frac{rc}{1+c}$.

So if $\frac{rc}{1+c} > \frac{r}{k} \iff c > \frac{1}{k-1}$,

then the sum in (3) is $o((1+c)^r)$. This example shows that $c_k \le \frac{1}{k-1}$.

Strong Conjecture: (Alladi, 1983)

$$c_k = \frac{1}{k-1}, \text{ for } k = 2, 3, \ldots\ldots$$

Remark: For purpose of applications to Probabilistic Number Theory that I had, I needed only inequalities like (2) for $0 \le h(p) \le \delta$, with some $\delta > 0$, and so the truth of the weak conjecture was sufficient

# A mapping for sets and divisors



If $n \neq \square$, then trivially we have that half the divisors of $n$ are $< \sqrt{n}$ and half are $> \sqrt{n}$, as seen by the correspondence

$$d \mid n \iff \frac{n}{d} \mid n.$$

## A more interesting (deeper) mapping

**Conjecture** (Alladi, 1982)

There exists a mapping $m$ from the set $S^{<}_{\sqrt{n}}$ of divisors of $n$ which are $< \sqrt{n}$ to the set $S^{>}_{\sqrt{n}}$ of divisors of $n$ which are $> \sqrt{n}$ such that

(i) $m$ is a bijection

(ii) If $d' = m(d)$ for $d \in S^{<}_{\sqrt{n}}$ & $d' \in S^{>}_{\sqrt{n}}$, then

$$m(d) = d' \equiv 0 \pmod{d}.$$

**Remark** (i) The conjecture, if true, would immediately imply

<u>Lemma 1</u>: Let $n$ be sq. free and $h$ mult. such that $0 \le h \le 1$. Then

$$\sum_{d|n}' h(d) \le 2 \sum_{d|n,\, d<\sqrt{n}}' h(d) .$$

<u>Proof</u>: Write

$$\sum_{d|n}' h(d) = \sum_{d|n,\, d<\sqrt{n}}' h(d) + \sum_{d|n,\, d>\sqrt{n}}' h(d)$$

$$= \sum_{d \in S^{<}_{\sqrt{n}}}' h(d) + \sum_{d' \in S^{>}_{\sqrt{n}}} h(d')$$

The second sum is less than the first because

$$h(d') = h(d d'') \le h(d). \quad //$$

(ii) One does not need the conj. to prove Lemma 1 which follows from a <u>monotonicity principle</u> namely

<u>Lemma 2</u>: If $n$ is sq. free & $h, h'$ mult. such that

$$0 \le h' \le h,$$

then

$$\sum_{d|n} h'(d) \Big/ \sum_{d \in S^{<}_{\sqrt{n}}} h'(d) \le \sum_{d|n}' h(d) \Big/ \sum_{d \in S^{<}_{\sqrt{n}}} h(d)$$

and the fact that the conj. holds when $h = 1$.

Lemma 1 was precisely what I needed (7) to establish the Erdos-Kac-Kubilius thm. on the set of shifted primes (KA, Pac. J. Math., 1983)

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I informed Erdös about my conjecture and Said that I could not prove it. He replied that if we strengthen the conjecture, it could be proved by induction on the number of prime factors:

Theorem (Erdos, 1982) - private communication.

Let $n$ be sq. free, and $t \in [1, \sqrt{n}]$. Then there exists a mapping $m_t = m_{t,n}$ from the set $S_t^<$ of the divisors of $n$ which are $< t$ to the set $S_{n/t}^>$ of the divisors of $n$ that are $> n/t$ such that

(i) $m_t$ is a bijection

(ii) If $d' = m_t(d)$, $d \in S_t^<$, $d' \in S_{n/t}^>$, then

$$m_t(d) = d' \equiv 0 \pmod{d}.$$

At this stage Vaaler noticed that this could (8)
be formulated more generally in terms of sets and measures:

Theorem$'$: Let $S$ be a finite set and $\lambda$ a finite measure on the set of all subsets of $S$. For each $t \geq 0$, define

$$A(t,S) = \{ E \subseteq S \mid \lambda(E) \leq t \}$$

Then there is a permutation

$$\pi_{t,S} : A(t,S) \longrightarrow A(t,S)$$

such that

$$\pi_{t,S}(E) \cap E = \phi, \ \forall\, E \in A(t,S).$$

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Theorem$'$ is proved in Alladi - Erdös - Vaaler, Prog. Math. Vol. 70, (Proc. 1984 Oklahoma Conf.), 1987.

Cor: Let $S, \lambda$ as above. Define

$$B(t,S) = \{ E \subseteq S \mid \lambda(S) - t \leq \lambda(E) \}.$$

Then there is a bijection

$$\sigma_{t,S} : A(t,S) \longrightarrow B(t,S)$$

s.t. $E \subseteq \sigma_{t,S}(E), \ \forall\, E \in A(t,S).$

In that same paper, we prove

**Theorem 1:**

Let $h$ be sub-multiplicative and satisfy

$$0 \le h(p) \le c < \frac{1}{k-1}.$$

Then

$$\sum_{d|n}{}' h(d) \le \left\{1 - \frac{kc}{1+c}\right\}^{-1} \sum_{d|n,\ d\le n^{1/k}}{}' h(d).$$

(This settles the weak conjecture with any $c_k < \frac{1}{k-1}$)

Proof: Begin with the decomposition

$$\sum_{d|n}{}' h(d) = \sum_{d|n/p}{}' h(d) + \sum_{d|n/p}{}' h(pd)$$

for any $p|n$, $p$ = prime. Consequently

$$h(p) \sum_{d|n}{}' h(d) = h(p) \sum_{d|n/p}{}' h(d) + h(p) \sum_{d|n/p}{}' h(pd)$$

$$\ge \sum_{d|n/p}{}' h(pd) + h(p) \sum_{d|n/p}{}' h(pd) = (1+h(p)) \sum_{d|n/p}{}' h(pd).$$

Thus

$$\sum_{d|n/p}{}' h(pd) \le \frac{h(p)}{1+h(p)} \sum_{d|n}{}' h(d). \qquad (4)$$

Next write

$$\sum_{d|n,\, d\le n^{1/k}} h(d) \ge \sum_{d|n}{}' h(d) \frac{\log(n^{1/k}/d)}{\log n^{1/k}}$$

$$= \sum_{d|n}{}' h(d) - \frac{k}{\log n} \sum_{d|n}{}' h(d) \log d \qquad (5)$$

Note that (4) implies

$$\sum_{d|n}{}' h(d) \log d = \sum_{d|n}{}' h(d) \sum_{p|d} \log p = \sum_{p|n}{}' \log p \sum_{d|n/p}{}' h(pd)$$

$$\le \Big(\sum_{d|n}{}' h(d)\Big)\Big(\sum_{p|n}{}' \frac{h(p)\log p}{1+h(p)}\Big) \le \frac{c}{1+c} \log n \sum_{d|n}{}' h(d) \quad (6)$$

Theorem 1 follows from (5) & (6).

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<u>Theorem 2</u>: (Alladi-Erdös-Vaaler, J.N.T. 1989)

If $k \ge 2$ is an integer and $0 \le h(p) \le \frac{1}{k-1}$, where $h$ is ~~mu~~ strongly multiplicative, then for all sq. free $n$

$$\sum_{d|n}{}' h(d) \le (2k + o(1)) \sum_{d|n,\, d\le n^{1/k}}{}' h(d),$$

where $o(1) \to 0$ as $\nu(n) \to \infty$.

(<u>This settles the strong conjecture.</u>)

To prove this we use a powerful theorem on hypergraphs due to Baranyai (1973):

Baranyai's thm: Let $k, m$ be positive integers. Let $S$ be a set with $km$ elements. Then the $\binom{km}{m}$ subsets of $S$ having $m$ elements each, can be grouped $k$ at a time such that in every such group, the $k$ subsets of size $m$ generate a partition of $S$.

Proof of Thm 2: In view of the monotonicity, it suffices to prove Thm 2 in the case $h(p) = c = \frac{1}{k-1}$.

Let $\nu(n) = km + \ell$, $0 \le \ell \le k-1$.

For $j < m$ consider a divisor $N$ of $n$ with $\nu(N) = k(m-j)$. By Baranyai's thm, the divisors $d$ of $N$ having $\nu(d) = m-j$, can be grouped $k$ at a time such that they are mutually coprime and the product is $N$. In every such group, one of these $k$ divisors is $\le n^{1/k}$.

So there are

$$\geq \frac{1}{k}\binom{k(m-j)}{m-j} \qquad (7)$$

divisors of $N$ which are $\leq n^{1/k}$.

The number of ways of choosing such $N$ is

$$\binom{km+l}{k(m-j)}$$

Every such divisor $d$ can be the divisor of <u>at most</u>

$$\binom{km+l-m+j}{(k-1)(m-j}$$

such numbers $N$. Thus we have <u>at least</u>

$$\frac{\frac{1}{k}\binom{k(m-j)}{m-j}\binom{km+l}{k(m-j)}}{\binom{km+l-m+j}{(k-1)(m-j)}} = \frac{1}{k}\binom{km+l}{m-j} \quad \text{(a miracle}$$

divisors of $n$ with $\nu(d)=m-j$ & $d \leq n^{1/k}$. Thus

$$\sum_{d|n,\, d\leq n^{1/k}} h(d) \geq \frac{1}{k}\sum_{j=0}^{m}\binom{km+l}{m-j}\left(\frac{1}{k-1}\right)^{m-j}$$

$$\sim \frac{1}{k.2}(1+c)^{\nu(n)} = \frac{1}{2k}\sum_{d|n} h(d)$$

and this proves Theorem 2.

**Question**: Can the implicit "constant" $2k+o(1)$ in Thm 2 be replaced effectively by an expression (possibly depending on $n$) so that the inequality is valid for <u>all</u> $n$?

Theorem 1 has an expression valid for all $n$, but $\left\{1-\frac{kc}{1+c}\right\}^{-1} = \infty$ when $c = \frac{1}{k-1}$.

An answer to the above question is given by

<u>Theorem 3</u>: (Alladi - Erdős - Vaaler, JNT 1989)

Let $k \geq 2$. Let $h$ be multiplicative, and satisfy $0 \leq h(p) \leq \frac{1}{k-1}$. Then for all sq. free $n$

$$\sum_{d|n} h(d) \leq \frac{k\,\nu(n)}{k-1} \sum_{d|n,\ d\leq n^{1/k}} h(d).$$

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To prove Thm 3, we use the more general monotonicity given by

<u>Lemma M</u>: Let $n$ be sq-free & $0<\alpha<1$. For fixed $\alpha$ and $h$ the quantity

$$R_{\alpha,n}(h) = \Big(\sum_{d|n,\ d\leq n^{\alpha}} h(d)\Big)\Big/\sum_{d|n} h(d)$$ decreases as $n$ increases.

**Proof of Lemma M:** The lemma is trivial when $\nu(n) = 1$. So let $\nu(n) \geq 2$.

Define

$$\chi_\alpha(x) = \begin{cases} 1, & \text{if } x \leq \alpha \\ 0, & \text{if } x > \alpha. \end{cases}$$

Then

$$R_{\alpha,n}(h) = \sum_{d|n}{}' \chi_\alpha\left(\frac{\log d}{\log n}\right) \frac{h(d)}{\prod_{q|n,\ q = \text{prime}} (1 + h(q))}$$

$$= \sum_{d|\frac{n}{p}}{}' \left\{ \chi_\alpha\left(\frac{\log d}{\log n}\right) \frac{h(d)}{1+h(p)} + \chi_\alpha\left(\frac{\log p + \log d}{\log n}\right) \frac{h(pd)}{1+h(p)} \right\} \times \frac{1}{\prod_{q|n,\ q\neq p} (1+h(q))}$$

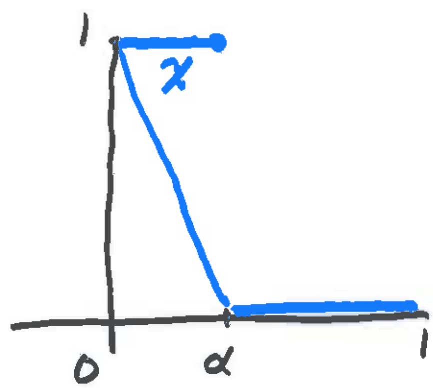
$$= \sum_{d|n/p}{}' \left\{ \chi_\alpha\left(\frac{\log d}{\log n}\right)\left(1 - \frac{h(p)}{1+h(p)}\right) + \chi_\alpha\left(\frac{\log p + \log d}{\log n}\right) \frac{h(p)}{1+h(p)} \right\} \times \frac{h(d)}{\prod_{q|n,\ q\neq p} (1+h(q))}, \qquad (8)$$

for some $p | n$. Note that

$$\chi_\alpha\left(\frac{\log d}{\log n}\right) \geq \chi_\alpha\left(\frac{\log p + \log d}{\log n}\right)$$

and so $R_{\alpha,n}(h)$ decreases by increasing $h(p)$, and by not changing $h(q)$ for $q \neq p$. Then by increasing $h(q)$ in succession for other primes $q$, we get Lemma M.

<u>Proof of Theorem 3:</u> Let $F(\alpha, c, n)$ denote the value of $R_{\alpha,n}(h)$ when $h(p) = c$, $\forall p$. To get a lower bound for $F$ we consider bounding $\chi_\alpha(x)$ from below, where $x = \frac{\log d}{\log n}$.




$\chi_\alpha(x)$ in blue

$1 - \frac{x}{\alpha}$ in green

One possibility is to take $y = 1 - \frac{x}{\alpha}$. This was the choice in the proof of Thm 1.

So we now seek a <u>quadratic polynomial</u>

To this end, let

$$-\frac{1}{\alpha} \le t \le \frac{1}{\alpha}$$

Then

$$\chi_\alpha(x) \geq f(x) = tx^2 - \left(\alpha t + \frac{1}{\alpha}\right)x + 1$$

With this choice

$$F(\alpha, c, n) \geq \frac{1}{H(n)} \sum_{d|n}{}' \left\{ \frac{t \log^2 d \; h(d)}{\log^2 n} - \left(\alpha t + \frac{1}{\alpha}\right) \frac{\log d}{\log n} h(d) + h(d) \right\},$$

where

$$H(n) = \sum_{d|n}{}' h(d).$$

Note that

$$\frac{1}{\log n} \sum_{d|n}{}' h(d) \log d = \sum_{d|n} \frac{h(d)}{\log n} \sum_{p|n}{}' \log p$$

$$= \sum_{p|n} \frac{\log p}{\log n} \sum_{d|n/p}{}' h(pd) = \frac{H(n)}{\log n} \sum_{p|n} \frac{h(p) \log p}{1 + h(p)} = \frac{c H(n)}{1+c}. \qquad (9)$$

Similarly it can be shown that

$$\frac{1}{\log^2 n} \sum_{d|n}{}' h(d) \log^2 d$$

$$= H(n) \left\{ \left(\frac{c}{1+c}\right)^2 + \frac{c}{(1+c)^2 \log^2 n} \sum_{p|n} \log^2 p \right\}, \qquad (10)$$

by writing

$\log^2 d = \left(\sum_{p|d} \log p\right)^2$ and expanding the square

Thus (9) & (10) yield

$$F(\alpha,c,n) \geq f\left(\frac{c}{1+c}\right)^2 + \frac{tc}{(1+c)^2\log^2 n}\sum_{p|n}' \log^2 p. \qquad (11)$$

By the Cauchy-Schwartz inequality

$$1 = \sum_{p|n}' \frac{\log p}{\log n} \leq \nu(n)^{1/2}\left(\sum_{p|n}' \frac{\log^2 p}{\log^2 n}\right)^{1/2}$$

and so

$$\sum_{p|n}' \frac{\log^2 p}{\log^2 n} \geq \frac{1}{\nu(n)}.$$

Therefore

$$F(\alpha,c,n) \geq f\left(\tfrac{c}{1+c}\right)^2 + \frac{tc}{(1+c)^2\nu(n)}$$

Obviously we wish to make $t$ maximal.

Since $\alpha = \frac{1}{k}$ in Thm. 2, we take $t = k$, $c = \frac{1}{k-1}$.

With these choices it turns out that

$$f(c/(1+c)) = 0 \quad (!)$$

Thus

$$F\left(\frac{1}{k}, \frac{1}{k-1}, n\right) \geq \frac{k-1}{k\,\nu(n)}$$

and this proves Theorem 2.

Remarks:

(i) One would expect to get better bounds by increasing the degree of the minorizing polynomial. This would involve expressions

$$\frac{1}{\log^j n} \sum_{p \mid n} \log^j p \quad , \quad j = 3, 4, \ldots$$

These would be complicated for large j and might yield worse bounds unless the cancellation among the higher moments are calculated properly.

(ii) In Thm 3., we do not require k to be an integer

# Improvements and simplification by later authors

(I) What we needed to prove Theorem 2 was inequality (7) which we deduced as a consequence of Baranyai's theorem. Suryamohan (Glasgow Math. J., 2004) showed that (7) could be deduced directly by a simple counting argument without appeal to Baranyai's theorem. We show his proof now:

<u>Proposition</u>: Let $k, \ell \geq 1$ be integers and let $N = p_1 p_2 \cdots p_{k\ell}$ where $p_1 < p_2 < \cdots < p_{k\ell}$ are primes. Then the number of $d \mid N$ with $d \leq N^{1/k}$ and having exactly $\ell$ prime divisors is

$$\geq \frac{1}{k}\binom{k\ell}{\ell}.$$

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<u>Note</u>: The proposition is our inequality (7) with $\ell$ in place of $(m-j)$.

**Proof** (Suryamohan)

Let $S_{k\ell} = \{1, 2, 3, \cdots, k\ell\}$ and $\pi = (\sigma_1, \sigma_2, \cdots, \sigma_{k\ell})$ be any permutation of $S_{k\ell}$. We set

$\xi_\pi = \{A_1, A_2, \cdots, A_k\}$, where $A_j = (\sigma_{(j-1)\ell+1}, \sigma_{(j-1)\ell+2}, \cdots, \sigma_{j\ell})$.

For each set $B$ with $|B| = \ell$, let

$$\delta_\pi(B) = \begin{cases} 1, & \text{if } B \in \xi_\pi \\ 0, & \text{otherwise} \end{cases}$$

For each $A \subseteq S_{k\ell}$, let $d_A$ be the associated divisor of $N$ given by the product $\prod_{i \in A} p_i$.

Let

$$\mathcal{S}_1 = \left\{ A \subseteq S_{k\ell} \;\middle|\; |A| = \ell,\ d_A \leq N^{1/k} \right\}$$

$$\mathcal{S}_2 = \left\{ B \subseteq S_{k\ell} \;\middle|\; |B| = \ell,\ d_B > N^{1/k} \right\}$$

Clearly if $C \subseteq S_{k\ell}$, $|C| = \ell$, the $C$ belongs exactly to one of $\mathcal{S}_1$ or $\mathcal{S}_2$. Thus

$$|\mathcal{S}_1| + |\mathcal{S}_2| = \binom{k\ell}{\ell} \qquad (12)$$

Note that

$$\prod_{i=1}^{k} d_{A_i} = N$$

Thus $\exists$ some $i$ such that $d_{A_i} \leq N^{1/k}$. Thus

$$|\mathcal{S}_2 \cap \xi_{\mathcal{S}_\pi}| \leq (k-1)\,|\mathcal{S}_1 \cap \xi_{\mathcal{S}_\pi}|\,, \ \forall \pi$$

Consequently

$$\sum_{B \in \mathcal{S}_2}{}' \delta_\pi(B) \leq (k-1) \sum_{A \in \mathcal{S}_1}{}' \delta_\pi(A)\,. \qquad (13)$$

If we sum the expressions in (13) over all $\pi$, we get

$$\sum_{B \in \mathcal{S}_2}{}' \sum_{\pi} \delta_\pi(B) \leq (k-1) \sum_{A \in \mathcal{S}_1}{}' \sum_{\pi} \delta_\pi(A) \qquad (14)$$

It is now <u>crucial</u> to note that

$$\sum_{\pi}{}' \delta_\pi(C) = k \times \ell!\,(k\ell - \ell)!\,, \ \forall C \text{ with } |C| = \ell \qquad (15)$$

is <u>independent of</u> $C$. Thus (14) & (15) imply

$$|\mathcal{S}_2| \leq (k-1)\,|\mathcal{S}_1|\,,$$

which together with (12) yields

$$k\,|\mathcal{S}_1| \geq \binom{k\ell}{\ell}$$

which is the assertion of the proposition.

(II) Just as our Theorem 1 holds for all sub-multiplicative functions satisfying suitable bounds, S. Srinivasan (Glasgow Math. J., 1994) showed that our Theorem 2 holds for sub-multiplicative functions as well.

(III) K. Soundararajan (J. N. T., 1992) achieved several improvements of our results in JNT, 1989 paper.

First Improvement: In Theorem 2, with $h$ multiplicative, $0 \le h(p) \le \frac{1}{k-1}$, for $k \ge 2$, he showed

$$\sum_{d \mid n} h(d) \le (k + o(1)) \sum_{d \mid n,\ d \le n^{1/k}}' h(d),$$

where $o(1) \to 0$ as $\nu(n) \to \infty$.

Remark: Thus Soundararajan cut the implicit constant in Thm. 2 by half. This is crucial because when $k = 2$ it corresponds better with the bound

$$\sum_{d \mid n} h(d) \le 2 \sum_{d \mid n,\ d < \sqrt{n}} h(d), \text{ for } 0 \le h \le 1$$

<u>Second Improvement</u>: In Theorem 3 for $k \geq 2$, and $h$ multiplicative satisfying $0 \leq h(p) \leq \frac{1}{k-1}$, he showed that

$$\sum_{d|n}' h(d) \ll \sqrt{\nu(n)} \sum_{d|n,\ d<\sqrt{n}}' h(d), \quad \forall \text{ sq. free } n,$$

where the implicit constant is absolute.

<u>Third Improvement</u>: He extended Theorem 2 to rational values $k$ as follows: Let $k \geq 2$ be <u>rational</u>, $h$ mult., and $0 \leq h(p) \leq \frac{1}{k-1}$. Then

$$\sum_{d|n}' h(d) \leq (\gamma_k + o(1)) \sum_{d|n,\ d \leq n^{1/k}}' h(d),$$

for sq. free $n$, where

$$\gamma_k = 1 + a_0 + a_1 + \cdots + a_r,$$

with $k-1 = [a_0, a_1, \cdots, a_r]$ being the continued fraction of $k-1$. Here also $o(1) \to 0$ as $\nu(n) \to \infty$.

<u>Remark</u>: If $k \geq 2$ is an integer, then $r = 0$, and $a_0 = k-1$. Thus $1 + a_0 = k$ which is what one has in the First Improvement of Thm. 2.

# Soundararajan's approach/ideas

For $t \geq 0$, let $F_t$ denote the set of mult. functions $F : \mathbb{Z}^+ \to [0,\infty)$ such that $F(p) \geq t$ for all primes $p$.

Let $G_t$ denote the set of multiplicative functions $G : \mathbb{Z}^+ \to [0,\infty)$ such that $0 \leq G(p) \leq t$, $\forall p$.

For square-free $n$, put

$$a(t,n) = \inf\left\{ \left( \sum_{d|n,\ d \geq n^{t/(t+1)}} F(d) \Big/ \sum_{d|n} F(d) \right) \Big| F \in F_t \right\}$$

and

$$b(t,n) = \sup\left\{ \left( \sum_{d|n,\ d > n^{t/(t+1)}} G(d) \Big/ \sum_{d|n} G(d) \right) \Big| G \in G_t \right\}$$

Also, let

$$A(t) = \inf\{ a(t,m) \mid m \text{ sq. free} \}$$

$$B(t) = \sup\{ b(t,m) \mid m \text{ sq. free} \}$$

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For $F \in F_t$, we have by definition

$$\sum_{d|n,\ d \geq n^{t/(t+1)}} F(d) \geq A(t) \sum_{d|n} F(d)$$

Our Theorem 2 is equivalent to

$$A(k) \geq \frac{1}{2k+2+o(1)}$$

for <u>integers</u> $k \geq 1$.

(<u>Note</u>: Soundararajan has replaced $k$ in our Theorem 2 by $k+1$).

He establishes three results:

<u>Theorem $S_1$</u>: For all $t \geq 0$

$$A(t+1) \geq \frac{A(t)}{A(t)+1}$$

In particular,

$$A(k) \geq \frac{1}{k+1} \quad \forall\, k \geq 0,\ k \in \mathbb{Z}.$$

---

<u>Theorem $S_2$</u>: For all $t \geq 0$,

$$B(t+1) \leq \frac{1}{2-B(t)}.$$

In particular,

$$B(k) = \frac{k}{k+1}, \ \forall\, k \in \mathbb{Z}^+.$$

<u>Theorem $S_3$</u>: For all $t > 0$,

$$A\left(\frac{1}{t}\right) + B(t) = 1.$$

Using Theorem 3, he extends our Theorem 2 (more specifically the assertion $A(k) \geq \frac{1}{k+1}$ in his Theorem $S_1$) to rational $k$ as follows:

<u>Theorem</u> $S_4$: Let $k > 0$ be rational and

$$k = [a_0, a_1, \cdots, a_r]$$

its continued fraction expansion. Then

$$A(k) \geq \frac{1}{1 + a_0 + a_1 + \cdots + a_r}$$

and

$$B(k) \leq \frac{a_0 + a_1 + \cdots + a_r}{1 + a_0 + a_1 + \cdots + a_r}.$$

---

<u>Note</u>: Even if we write

$$k = [a_0, a_1, \cdots, a_{r-1}, a_r - 1, 1]$$

we have

$$1 + a_0 + \cdots + a_{r-1} + a_r - 1 + 1 = 1 + a_0 + \cdots + a_r$$

and so the above inequalities for $A(k)$ and $B(k)$ do not change.

(IV) Ritabrata Munshi (Ramanujan J., 2011) considered the problem of obtaining bounds of the type

$$\tau(n) \ll_{\delta,\beta} \sum_{d|n,\ d \leq n^{\delta}}' \tau(d)^{\beta},$$

where $\tau(n)$ is the divisor function. He was motivated by applications of such inequalities in analytic number theory.

Landreau (Bull. LMS, 1989) had shown

$$\tau(n) \leq k^{k/(k-1)} \sum_{d|n,\ d \leq n^{1/k}}' \tau(d)^{k}$$

(related to earlier work of Wolke (JLMS 1972))

For certain small k, Friedlander & Iwaniec improved on Landreau by showing

$$\tau(n) \leq 9 \sum_{d|n,\ d \leq n^{1/3}} \tau(d)$$

&

$$\tau(n) \leq 256 \sum_{d|n,\ d \leq n^{1/4}} \tau(d)^{\log_2 9}$$

Munshi (2011) improves these results as follows:

For $\delta \in (0, \frac{1}{2})$, define

$$\beta(\delta) = \frac{\log \delta}{\log 2} + \frac{1}{\delta}\left\{1 + (1-\delta)\frac{\log(1-\delta)}{\log 2}\right\}$$

Then $\beta(\delta)$ is a strictly decreasing function of $\delta$.

Moreover

$$\tau(n) \ll_{\delta,\beta} \sum_{d\mid n,\ d \le n^{\beta}}' \tau(d)^{\beta}, \quad \forall \beta > \beta(\delta). \quad (16)$$

<u>Remark</u>: Munshi observes that (16) is optimal by taking $n = p_1 p_2 \cdots p_r$, with $p_1 \sim p_2 \sim \cdots \sim p_r$ large primes, as we did. Then

$$\sum_{d\mid n,\ d \le n^{1/k}}' \tau(d)^{\beta} \asymp \binom{r}{r/k} 2^{\beta r/k}$$

This leads to the requirement

$$\frac{\beta r}{k} - \frac{r}{k}\log_2\left(\frac{1}{k}\right) - \left(r - \frac{r}{k}\right)\log_2\left(1-\frac{1}{k}\right) > r$$

$\Longleftrightarrow$

$$\beta > \beta(\delta).$$

# Applications to Probabilistic Number Theory

## - Distribution (moments) of additive functions

For $S \subseteq \mathbb{Z}^+$, define

$$S_d(x) = \sum_{\substack{n \le x,\, n \in S \\ n \equiv 0 \pmod{d}}}' 1$$

Write

$$S_d(x) = \frac{X\omega(d)}{d} + R_d(x), \quad X = S_1(x),$$

where $\omega(d)$ is multiplicative.

**Assumptions:**

(i) $|R_d(x)| \ll \frac{X\omega(d)}{d}$, $1 \le d \le x^\beta$, for some $\beta < 1$

(ii) Bombieri Type condition

$$\sum_{d \le x^\beta/\log^V x}' |R_d(x)| \ll_U \frac{X}{\log^U X}$$

---

Next, let $f$ be a strongly additive function:

$$f(n) = \sum_{p|n}' f(p)$$

We will focus on real valued $f$ and even $f \ge$

Consider

$$A(x) = \sum_{p \le x} \frac{f(p)\,\omega(p)}{p} \quad \text{(mean of } f(n), n \in S)$$

and

$$B(x) = \sum_{p \le x} \frac{|f^2(p)|\,\omega(p)}{p} \quad \text{(variance of } f(n), n \in S)$$

Problem: Obtain bounds for the moments

$$\sum_{n \le x,\, n \in S} |f(n) - A(x)|^{\ell}$$

Elliott (Canadian J. Math. 1980) has considered and solved this problem elegantly when $S = \mathbb{Z}^+$.

Reductions: Use $|a+b|^{\ell} \ll_{\ell} |a|^{\ell} + |b^{\ell}|$

So we assume $f \ge 0$ because, if $f$ is real, we may write

$$f = f^+ - f^-,$$

where

$$f^+(p) = \max(0, f(p))$$

&

$$f^-(p) = \min(0, f(p))$$

Consider the distribution function

$$F_x(\lambda) = \frac{1}{x} \sum_{\substack{n \le x,\ n \in S \\ f(n) - A(x) < \lambda\sqrt{B(x)}}} 1 .$$

Then

$$\frac{1}{B(x)^{\ell/2}} \sum_{n \in S,\ n \le x} (f(n) - A(x))^{\ell} = \int_{-\infty}^{\infty} \lambda^{\ell} \, d F_x(\lambda) \qquad (*)$$

Next consider the bilateral Laplace transform

$$T_u(x) = \int_{-\infty}^{\infty} e^{u\lambda} \, d F_x(\lambda)$$

If $|T_u(x)| \ll 1$ for $|u| \le R$ for some $R > 0$, then we can say the expression in $(*)$ is bounded for each $\ell$.

Note that

$$T_u(x) = \frac{1}{x} \sum_{n \le x,\ n \in S} e^{u(f(n) - A(x))/\sqrt{B(x)}}$$

$$= \frac{e^{-uA(x)/\sqrt{B(x)}}}{x} \sum_{n \le x,\ n \in S}{}' g(n) ,$$

where

$g(n) = e^{u f(n)/\sqrt{B(x)}}$ is strongly multiplicative

Case 1: $u \le 0$. Here $0 \le g \le 1$.

I used Sieve methods (Springer Lecture Notes, # 1122, (1984)) to show

$$S(g,x) = \sum_{n \le x,\, n \in S} g(n) \ll x \prod_{p \le x} \left(1 - \frac{(1-g(p))\omega(p)}{p}\right)$$

uniformly for $0 \le g \le 1$.

Case 2: $u \ge 0$. Here $g \ge 1$.

Assumption: $\max_{p \le x} f(p) \ll \sqrt{B(x)}$

With R chosen sufficiently small, we can make

$$1 \le g(p) \le 1 + \frac{1}{2(k-1)}$$

$$\Rightarrow \quad 0 \le h(p) = g(p) - 1 \le \frac{1}{2(k-1)}.$$

Thus we will have

$$g(n) = \sum_{d \mid n} h(d) \ll_k \sum_{d \mid n,\ d \le n^{1/k}} h(d)$$

This will lead to the estimate

$$S(g,x) \ll x \prod_{p \le x} \left(1 + \frac{h(p)\omega(p)}{p}\right).$$

All these estimates lead to

**Theorem A**: If $f$ is strongly additive and

$$0 \le |f(p)| \ll \sqrt{B(x)},$$

then

$$\sum_{n \le x,\, n \in S} |f(n) - A(x)|^{\ell} \ll_{\ell} X\, B(x)^{\ell/2}, \text{ for } \ell > 0,$$

---

This extends Elliott's result for $\mathbb{Z}$ to more general sets $S$.

**Remark:** We have indicated only bounds for moments here. The bilateral Laplace Transform approach along with sieve methods yields asymptotic estimates for the moments of additive functions in special sets of integers $S$, and this in turn leads to Erdös-Kac-Kubilius type theorems. (see KA, Springer Lecture Notes, # 1122 (1984), or # 1395 (1989)).

# Open Problems



(New) <u>Weak Conjecture</u>: The implicit constant in Theorem 2 is absolute.

(New) <u>Strong Conjecture</u>: For $k = 2, 3, \dots$, the implicit constant in Theorem 2 is

$$\left(1 + \frac{1}{k-1}\right)^{k-1}.$$