Adjacency algebra of unitary Cayley graph

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Abstract
Let \( A \) be the adjacency matrix of a graph \( X \). The set of all polynomials in \( A \) with coefficients from the field of complex numbers \( \mathbb{C} \) forms an algebra called the adjacency algebra of \( X \), denoted by \( \mathcal{A}(X) \). In this study, we show that the adjacency algebra of every unitary Cayley graph is a coherent algebra and its consequences.

Keywords: Adjacency algebra, circulant graphs, coherent algebra, distance polynomial graphs.

1 Introduction and preliminaries

For a fixed positive integer \( n \), let \( U_n = \{ k : 1 \leq k \leq n, \gcd(k, n) = 1 \} \). If \( |S| \) denotes the cardinality of the set \( S \), then \( |U_n| = \varphi(n) \). The function \( \varphi(n) \) is known as the Euler-totient function. Let \( \zeta_n \in \mathbb{C} \) denote a primitive \( n \)-th root of unity, i.e., \( \zeta^n = 1 \) and \( \zeta^k \neq 1 \) for \( 0 < k < n \). Let \( N_n = \{ \zeta_n^k : 0 \leq k \leq n - 1 \} \) be the set of all the \( n \)-th roots of unity, forms a cyclic group with respect to multiplication. Also for each positive divisor \( d \) of \( n \), \( M_d = \{ \zeta_n^{bk} | k \in U_d \} \) denote the set of all primitive \( d \)-th roots of unity. Then \( M_d = M_d^{-1} \), \( |M_d| = \varphi(d) \) and \( N_n = \cup_{d|n} M_d \). We denote a circulant graph by \( X_d^n \) and define it as \( X_d^n = \text{Cay}(N_n, M_d) \) (where \( \text{Cay}(G, S) \) is a Cayley graph on the group \( G \), with connection set \( S = S^{-1} \)) \(^1\) with \( n \) vertices and the degree of every vertex is \( |M_d| = \varphi(d) \). Throughout this paper we fix \( n \), hence we denote the graph \( X_d^n \) simply by \( X_d \). Note that \( X_d \) is a graph with \( n \) vertices. When \( d = n \), the

\(^1\) for more information about Cayley graphs and circulant graphs refer Biggs [1] and Chris D. Godsil & Gordon Royle [3]
The unitary Cayley graph $X_n$ has been studied as an object of independent interest (see, for example [Koltz & Sander [4]]).

Let $A(X)$ (or simply $A$) be the 0-1 adjacency matrix of a graph $X$. The set of all polynomials in $A$ with coefficients from the complex number field $\mathbb{C}$ forms an algebra called the adjacency algebra of $X$, denoted by $\mathcal{A}(X)$. For any two vertices $u$ and $v$ of a connected graph $X$, let $d(u, v)$ denote the length of the shortest path from $u$ to $v$. Then the diameter of a connected graph $X = (V, E)$ is $\max\{d(u, v) : u, v \in V\}$. It is shown in Biggs [1] that if $X$ is a connected graph with diameter $\ell$, then

$$\ell + 1 \leq \dim(\mathcal{A}(X)) \leq n. \quad (1)$$

where $\dim(\mathcal{A}(X))$ is the dimension of $\mathcal{A}(X)$ as a vector space over $\mathbb{C}$.

**Lemma 1.1 (Biggs [1])** A graph $X$ is connected regular if and only if $J \in \mathcal{A}(X)$. Where $J$ is a square matrix of appropriate size with all entries '1's.

**Definition 1.2** Hadamard product of two $n \times n$ matrices $A$ and $B$ is denoted by $A \odot B$ and is defined as $(A \odot B)_{xy} = A_{xy}B_{xy}$.

Two $n \times n$ matrices $A$ and $B$ are said to be disjoint if their Hadamard product is the zero matrix.

**Definition 1.3** A sub algebra of $M_n(\mathbb{C})$ is called coherent if it contains the matrices $I$ and $J$ and if it is closed under conjugate-transposition and Hadamard multiplication.

**Theorem 1.4** Every coherent algebra contains unique basis of mutually disjoint $0,1$-matrices (matrices with entries either 0 or 1).

**Definition 1.5** Let $X = (V, E)$ be a graph with adjacency matrix $A$ then any coherent algebra which contains $A$ is called coherent algebra of $X$.

**Definition 1.6** If $X = (V, E)$ be a graph and $A$ is its adjacency matrix then coherent closure of $X$, denote by $\langle\langle A \rangle\rangle$ or $CC(X)$, is the smallest coherent algebra containing $A$.

Let $X = (V, E)$ be a connected graph with diameter $\ell$. The $k$-th distance matrix $D_k(0 \leq k \leq \ell)$ of $X$, is defined as follows $(D_k)_{rs} = \begin{cases} 1, & \text{if } d(v_r, v_s) = k \\ 0, & \text{otherwise}. \end{cases}$

It is clear from the definition that the matrices $D_k$ for $1 \leq k \leq \ell$ are adjacency matrices of graphs and are called distance graphs with respect to the given
graph. It follows that

\[ D_0 = I \text{ (Identity matrix)}, \quad D_1 = A, \quad D_0 + D_1 + \ldots + D_\ell = J. \]

A connected graph \( X \) is said to be a distance polynomial graph if \( D_k \in \mathcal{A}(X) \) for \( 1 \leq k \leq \ell \). From the Lemma 1.1 the following note is evident.

**Note 1** • Every distance polynomial graph is regular connected graph. But converse is not true.

• Every connected regular graph of diameter 2 is a distance polynomial graph.

**Theorem 1.7** If \( X \) be a connected graph, then \( \mathcal{CC}(X) \) contains all the distance matrices of \( X \).

### 2 Main results

**Theorem 2.1** \( \mathcal{A}(X^n) \) is a coherent algebra for every \( n \).

Consequently from Theorem 1.7, we have following result.

**Corollary 2.2** Every unitary Cayley graph is a distance polynomial graph.

**Lemma 2.3** Let \( B_n = \{ A_d|d \text{ divides } n \} \), then \( L(B_n) \) is a coherent subalgebra of \( M_n(\mathbb{C}) \) of dimension \( |B_n| = \tau(n) \), where \( L(S) \) is the linear span of the set \( S \), \( A_d \) is the adjacency matrix of the graph \( X_d \) and \( \tau(n) \) is the number of devisors of \( n \).

We also see some of the consequences of above results.

### References


