# Generalized Sierpiński graphs [1]

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**Abstract**

Sierpiński graphs, $S(n,k)$, were defined originally in 1997 by Klavžar and Milutinović. The graph $S(1,k)$ is simply the complete graph $K_k$ and $S(n,3)$ are the graphs of Tower of Hanoi problem. We generalize the notion of Sierpiński graphs, replacing the complete graph appearing in the case $S(1,k)$ with any graph. The newly introduced notion of generalized Sierpiński graphs can be seen as a criteria to define a graph to be self-similar. We describe the automorphism group of those graphs and compute their distinguishing number. We also study existence of perfect codes in those graphs and give a complete characterization of the existence of perfect codes in the case when the basic graph is a power of a cycle.

*Keywords:* Sierpiński graphs, automorphism, perfect codes, distinguishing number

## 1 Introduction

Klavžar and Milutinović introduced in 1997 in [5] graphs $S(n,k)$ that generalize the graphs of Tower of Hanoi problem. Later, those graphs have been called *Sierpiński graphs* in [6], since their introduction was first motivated by topological studies of Lipscomb's space [9,10]. Those graphs have been well studied since their introduction, see for example [2,4,7,6]. They can be defined recursively with the following process: $S(1,k)$ is isomorphic to the complete graphs on $k$ vertices, $S(n+1,k)$ is constructed from $S(n,k)$ by copying $n$ times graph $S(n,k)$ and adding exactly one edge between each pair of copies. When $k=3$, those graphs are exactly Tower of Hanoi graphs and for larger $k$, they

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can be seen as graph of a variation of Hanoi problem [5]. In this paper, we generalize this construction for any graph $G$, by defining *generalized Sierpiński graphs*, $S(n, G)$. $S(1, G)$ is isomorphic to the graph $G$ and we can construct $S(n+1, G)$ by copying $|G|$ times $S(n, G)$ and adding one edge between copy $x$ and copy $y$ of $S(n, G)$ whenever $(x, y)$ is an edge of $G$. When $G$ is the complete graph on $k$ vertices, we obtain graphs $S(n, k)$.
In Section 2 we give a formal definition of $S(n, G)$ and some preliminairy results on those graphs. In Section 3, we study automorphism group of $S(n, G)$. We compute the number of automorphisms and the distinguishing number of $S(n, G)$ generalizing some results of [3,7]. In [6], it is shown that always there exists a perfect code in $S(n, k)$. In Section 4, we study perfect codes in $S(n, G)$. The existence of such codes depends of the graph $G$ and we give a complete characterization when $G$ is a power of cycle.

## 2 Preliminaries

Let $k$ be an integer and $G$ be a finite undirected graph on a vertex set $\{1, \ldots, k\}$. In the following, vertices of graphs will be identified with words on integers. We denote by $\{1, \ldots, k\}^n$ the set of words of size $n$ on alphabet $\{1, \ldots, k\}$. The letters of a word $u$ of $\{1 \ldots, k\}^n$ are denoted by $u = u_1 u_2 \cdots u_n$. The concatenation of two words $u$ and $v$ is denoted by $uv$.

The *generalized Sierpiński graph of $G$ of dimension $n$* denoted by $S(n, G)$ is the graph with vertex set $\{1, \ldots, k\}^n$ and edge set defined by: $\{u, v\}$ is an edge if and only if there exists $i \in \{1, \ldots, n\}$ such that:

(i) $u_j = v_j$ if $j < i$,

(ii) $u_i \neq v_i$ and $(u_i, v_i) \in E(G)$,

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

In other words, if $\{u, v\}$ is an edge of $S(n, G)$, there is an edge $\{x, y\}$ of $G$ and a word $w$ such that $u = wxy \cdots y$ and $v = wyx \cdots x$. We says that edge $\{u, v\}$ is *using* edge $\{x, y\}$ of $G$.
Graphs $S(n, G)$ can be constructed recursively from $G$ with the following process: $S(1, G)$ is isomorphic to $G$. To construct $S(n, G)$ for $n > 1$, copy $k$ times $S(n-1, G)$ and add to labels of vertices in copy $x$ of $S(n-1, G)$ the letter $x$ at the beginning. Then for any edge $\{x, y\}$ of $G$, add en edge between vertex $xy \cdots y$ and vertex $yx \cdots x$. See Figures 1 and 2 for some examples. For any word $u$ of length $d$, with $1 \leq d < n$, the subgraph of $S(n, G)$ induced by vertices with label beginning by $u$, is isomorphic to $S(n-d, G)$. We will call this subgraph the *copy $u$ of $S(n-d, G)$*. For a vertex $x$ of $G$, we call *extreme*

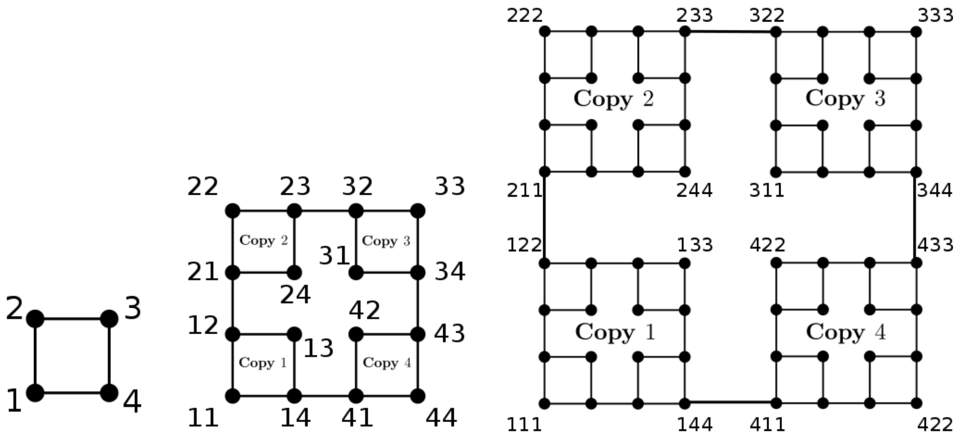


Figure 1. $S(1, C_4)$, $S(2, C_4)$ and $S(3, C_4)$

*vertex* $x$ of $S(n, G)$ the vertex with label $x \cdots x$. Next proposition generalizes a result of [5]:

**Proposition 2.1** *Let* $x$ *and* $y$ *be two vertices of a graph* $G$*, then the distance between extreme vertices* $x$ *and* $y$ *in* $S(n, G)$ *is* $(2^n - 1)d_G(x, y)$.

Connectivity of $G$ gives information on connectivity of $S(n, G)$. For example, if $G$ is not connected then clearly, $S(n, G)$ is also not connected. If $G$ is 1-edge-connected we have the following result, useful for proving results of next session:

**Proposition 2.2** *Let* $\{x, y\}$ *be a cutting edge of a graph* $G$*. Then each edge of* $S(n, G)$ *using edge* $\{x, y\}$ *is a cutting edge of* $S(n, G)$.

# 3 Automorphism group of $S(n, G)$

We denote by $Aut(G)$ the automorphism group of a graph $G$. In this section, we study $Aut(S(n, G))$ for any graph $G$. We denote by $Aut(S(n, G)/x)$ (resp. $Aut(S(n, G)/[x])$) the set of automorphisms of $S(n, G)$ that fix extreme vertices $y$ with $y$ adjacent to $x$ (resp. $x = y$ or $y$ adjacent to $x$). The next proposition gives a decomposition of any automorphism of $Aut(S(n, G))$:

**Proposition 3.1** *Let* $G$ *be a connected graph,* $\tau \in Aut(S(n, G))$*. Let* $x$ *be a vertex of* $G$*. There is a vertex* $\sigma(x)$ *of* $G$ *such that all vertices of copy* $x$ *of* $S(n-1, G)$ *are sent to vertices of copy* $\sigma(x)$ *of* $S(n-1, G)$*. The function* $\sigma$ *is an automorphism of* $G$ *and the function* $\sigma_x : \{1, \ldots, k\}^{n-1} \to \{1, \ldots, k\}^{n-1}$ *defined by* $\tau(xv) = \sigma_d(x)\sigma_x(v)$ *for* $v \in \{1, \ldots, k\}^{n-1}$ *is an automorphism*

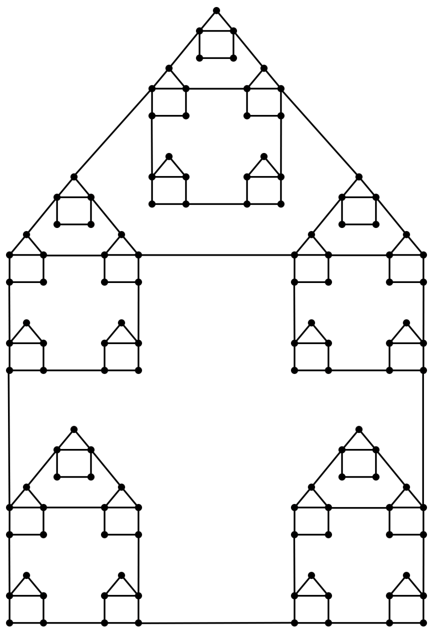

Figure 2. $S(3, G)$ where $G$ is a house

*of $S(n-1, G)$. This implies a bijection between $Aut(S(n, G))$ and the set $Aut(G) \times \prod_{x \in V} Aut(S(n-1, G)/x)$.*

**Proof** [Sketch of the proof] We prove by induction that in fact each copy of $S(d, G)$, for $1 \leq d < n$ is preserved by the automorphism. We first prove that an automorphism of $S(n, G)$ must send any copy of $S(d, G)$ on at most two different copies of $S(d, G)$. Then we prove that if it is the case, then $G$ must have a cutting edge and by Proposition 2.2, there is a cutting edge of $S(n, G)$ that is sent to another cutting edge. We finish the proof by a counting argument. $\square$

**Corollary 3.2** *The number of automorphisms of $S(n, G)$ is:*

$$|Aut(S(n,G))| = |Aut(G)| \prod_{x \in V} \left( |Aut(G/[x])|^{d(x)P(k,n)} |Aut(G/x)|^{(k-d(x)-1)P(k,n)+(n-1)} \right)$$

*with $P(k, n) = \sum_{i=0}^{n-3} (n-2-i)k^i$ and $d(x)$ is the degree of $x$.*

For example, if $G = C_4$, then $|Aut(G)| = 2^3$, $|Aut(G/x)| = 2$, $|Aut(G/[x])| = 1$ and we obtain: $|Aut(C_4, n)| = 2^{4P(4,n)+4n-1}$.

A *distinguishing coloring* of a graph $G$ is a coloring of vertices (not necessarily a proper coloring) such that the only automorphism of $G$ that fixes the coloring is the identity. See Figure 3 for an example of a distinguishing coloring of $S(2, C_4)$. The *distinguishing number* $D(G)$ of a graph $G$ is the minimal number of colors required in any distinguishing coloring (see [1]). We denote by $D(G/x)$ the minimal number of colors required when the only automorphism of $G$ that fixes the open neighborhood of $x$ and the coloring is the identity. We have:

**Theorem 3.3** *For $d \geq 2$ and $G$ such that $D(G) > 1$:*

$$D(S(n,G)) = \max(\max_{x \in \{1,\ldots,k\}} D(G/x), 2)$$

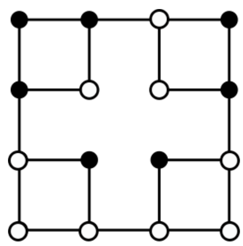

Figure 3. The distinguishing number of $S(2, C_4)$ is 2

# 4 Perfect codes

We say that a vertex $x$ is dominated by a vertex $y$ if $d(x, y) \leq 1$. A *perfect code* of $G$ is a subset of vertices of $G$ that is both a packing (no vertex is dominated twice) and a dominating set (every vertex is dominated). Existence of perfect codes is an NP-complete problem for general graphs [8]. It has been shown in [6] that there is always a perfect code in $S(n, K_k)$. This is not anymore true for general graphs $S(n, G)$. We first give a necessary and sufficient condition for existence of perfect codes when $G$ has no perfect code:

**Proposition 4.1** *Let $G$ be a graph with no perfect code. The following statements are equivalent:*

(i) *There is a integer $n > 1$ such that $S(n, G)$ has a perfect code.*

(ii) *For all integers $n > 1$, $S(n, G)$ has a perfect code.*

(iii) *$S(2, G)$ has a perfect code.*

(iv) *There is an oriented 2-factor $H$ in $G$ such that for any vertex $x$ of $G$, $x$ has exactly one ingoing neighbor $u(x)$ and one outgoing neighbor $c(x)$*

*and there is packing $S_x$ of $G$ containing $c(x)$ such that $u(x)$ is the only vertex not dominated by $S_x$.*

We finally give a complete characterization of the existence of perfect codes $S(n, G)$ when $G$ is a power of cycle, providing more infinite families of graphs with perfect codes.

**Theorem 4.2** *Let $k, r, n$ be integers with $k \geq 3$, $1 \leq r < \frac{k-1}{2}$, $n > 1$. There is a perfect code in $S(n, C_k^r)$ if and only if one of the following statements holds:*

(i) *$r + 1$ is even and $k \equiv 1 \mod (2r + 1)$, or*

(ii) *$r = 1$ and $k \equiv 0 \mod (2r + 1)$, or*

(iii) *$k \equiv 0 \mod (2r + 1)$ and $n = 2$.*

To prove this theorem, we use Proposition 4.1 when $k \neq 0 \mod (2r + 1)$ (there is no perfect codes) and show that the statement $(iv)$ of the proposition is true only when $r + 1$ is even and $k \equiv 1 \mod (2r + 1)$. Whenever $k \equiv 0 \mod (2r+1)$, we study *weak perfect codes*, i.e. packings where extreme vertices are not necessarily dominated and we show that there is no weak perfect code when the dimension $n$ is larger or equal to 3 and $r > 1$.

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