Concentration of the Domination number of $\mathcal{G}(n, p)$

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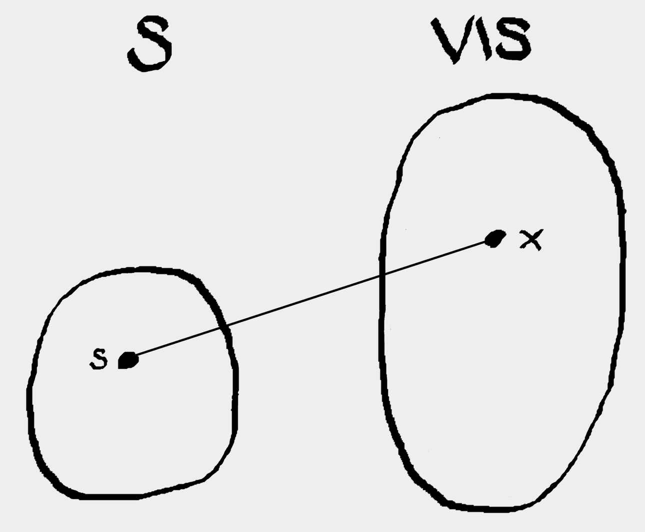
Introduction

- For a graph G, we call $S \in V(G)$ a *dominating set* if for all $x \in V \setminus S$ there is an $s \in S$ such that $xs \in E(G)$. We denote by $D(G)$ the *domination number of* G, the smallest possible size of a dominating set.



- Throughout, we work in the random graph model $\mathcal{G}(n, p)$, that is, in the space of all graphs where edges are inserted with probability p, all choices being made independently.
- We show that for $G \sim \mathcal{G}(n, p)$, $D(G)$ is sharply concentrated for a certain range of p.
- *Notation.* We denote by $\ln n$ the natural logarithm, and for $p \in [0, 1)$, set $q = \frac{1}{1-p}$.

Related work

- In the article *On the Domination Number of a Random Graph* (2001) B. Wieland and A.P. Godbole prove that the domination number is concentrated on two points asymptotically almost surely (a.a.s.): let $p_0(n)$ be the smallest p for which
$$\frac{p^2}{40} \geq \frac{\ln\left(\ln^2 n/p\right)}{\ln n}.$$
Let $p = p(n)$ be either constant, or tend to 0 with $p(n) \geq p_0(n)$. For $G_n \sim \mathcal{G}(n, p)$, a.a.s.
$$D(G_n) = \lfloor \log_q n - \log_q(\log_q n \cdot \ln n) \rfloor + 1 \text{ or}$$
$$D(G_n) = \lfloor \log_q n - \log_q(\log_q n \cdot \ln n) \rfloor + 2.$$

Result 1: 2-point-concentration

We extend the result of Wieland and Godbole to a wider range of p: We show a 2-point-concentration of the domination number even if p tends to 0 almost as fast as $n^{-1/2}$.

- **Theorem 1.** Let $K < 1/2$ be a constant, $p = n^{-K}$, and let $G_n \sim \mathcal{G}(n, p)$. Then there exists $r = r(n) \in \mathbb{R}$ such that a.a.s. $D(G_n) = \lfloor r \rfloor + 1$ or $D(G_n) = \lfloor r \rfloor + 2$. One can check that r is of the form
$$r = \log_q n - \log_q\left(\log_q n \cdot \ln n \cdot (1-K)^2(1 + o(1))\right).$$
- *Proof (sketch).* For $r \in \mathbb{N}$, consider the expected number of dominating sets of size r and form its continuous extension to $\mathbb{R}$. That is, consider the function $\mathcal{E}(x) := \binom{n}{x} \cdot (1 - (1-p)^x)^{n-x}$. Set r to be the unique positive solution of $\mathcal{E}(x) = 1$ ($\mathcal{E}$ is inreasing).
It follows by standard first moment arguments that a.a.s. $D(G_n) \geq \lfloor r \rfloor + 1$. Second moment methods and careful analysis of the asymptotics yield that a.a.s. $D(G_n) \leq \lfloor r \rfloor + 2$.

Remarks

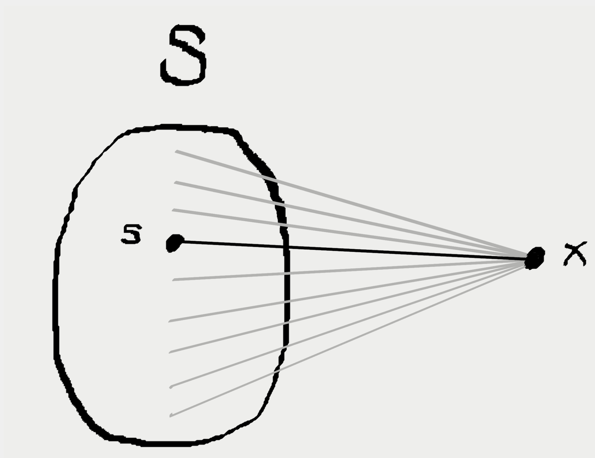
- The calculations carry through even when K tends to $1/2$ from below sufficiently slowly. That is, we can actually push p down to $p(n) = \frac{\ln^c n}{\sqrt{n}}$, or $K(n) = \frac{1}{2} - \frac{c \cdot \ln \ln n}{\ln n}$ respectively, where c is some small constant.
- When p tends to 1, then the asymptotics of $\log_q n$ change drastically. However, adjusting the estimates to this case, we get the same result: Let $p(n) = 1 - o(1)$ and $G_n \sim \mathcal{G}(n, p)$. Then a.a.s.
$$D(G_n) = \lfloor \log_q n - \log_q\left(\log_q n \cdot \ln n\right) \rfloor + 1 \text{ or}$$
$$D(G_n) = \lfloor \log_q n - \log_q\left(\log_q n \cdot \ln n\right) \rfloor + 2.$$

Result 2: Tight concentration for $p < n^{-1/2}$

- **Theorem 2.** Let $1/2 < K < 1$ be a constant, let $p = n^{-K}$, and $G_n \sim \mathcal{G}(n, p)$. Then there exists $r = r(n) \in \mathbb{R}$ such that a.a.s. $D(G_n) = r + \mathcal{O}^*\left(r \exp(n^{K-1})\right)$. As in Theorem 1, r is of the form
$$r = \log_q n - \log_q\left(\log_q n \cdot \ln n \cdot (1-K)^2(1 + o(1))\right).$$
- The proof essentially uses Talagrands inequality and was inspired by the concentration results for the independence number, as it has been done in *Random graphs* by Janson, Łuczak and Ruciński.

Result 3: Non-concentration on intervals of constant length

- **Theorem 3.** Let $2/3 < K < 1$ be a constant, let $p = n^{-K}$, and $G_n \sim \mathcal{G}(n, p)$. Then for all (constants) $C \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for any interval I of length C and for any $n \in \mathbb{N}$ large enough:
$$\Pr(D(G_n) \in I) < 1 - \varepsilon.$$
- *Proof sketch.* Assume the opposite and suppose that a.a.s. $d := D(G_n)$ lies in an interval I of constant length C. From Theorem 2 we know that I must lie in $r + \mathcal{O}^*\left(r \exp(n^{K-1})\right)$. For a dominating set S of size d, we call $e = xs \in E(G_n)$ (for $x \in V \setminus S$, $s \in S$) *crucial w.r.t.* S if for all $s' \in S - s$, $xs' \notin E(G_n)$. That is, in $G_n - e$, S is not dominating anymore.



Consider the graph $F_n \sim \mathcal{G}(n, p')$, where $p' = p - \sqrt{p}/n$. Note that we obtain the same distribution if in G_n, we delete every edge with probability $\frac{1}{\sqrt{pn}}$. It can be shown that under those assumptions, a.a.s. $D(F_n)$ lies in I, as well. Hence, our strategy is to delete edges in G_n with probability $\frac{1}{\sqrt{pn}}$, and to show that with (at least) constant positive probability a crucial edge has been destroyed for every dominating set of size d. That is, with positive probability, the domination number has gone up. We repeat the process C times, and finally get
$$\Pr(D(F_n) \notin I) > \varepsilon$$
for some absolute constant $\varepsilon > 0$.

Remarks and open problems

- For $p = \frac{n^{-4/3}}{\ln n}$, there is a simpler argument on concentration. Now, G_n is a.a.s. a collection of stars (since a.a.s. no triangles and no paths of length 3). In that case, $D(G_n) = n - e(G)$. But $e(G_n)$ enjoys a binomial distribution, and so its variance is $\frac{n^{1/3}}{\sqrt{2 \ln n}}$.
- There is still an enormous gap between the values of p where we can show a 2-point-concentration, and where we can show 'non-concentration' (on an interval of constant length). It is desirable to close this gap. We conjecture that for $p = o(n^{-1/2})$, the domination number is not concentrated on two values anymore.

References

- N.Alon and J.Spencer, *The Probabilistic Method,* John Wiley, New York, 1992.
- B. Bollobás, *Random graphs,* Cambridge University Press, 2001.
- S. Janson, T. Łuczak and A. Ruciński, *Random graphs,* John Wiley, New York, 2000.
- B. Wieland and A.P. Godbole, *On the Domination Number of a Random Graph,* Electr. J. Comb., vol. 8, no. 1, 2001.