Closures of exponential families, and generalized MLE

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For an *exponential family*

$$\mathcal{E} = \mathcal{E}_\mu = \left\{ Q_\vartheta : \frac{dQ_\vartheta}{d\mu}(x) = \exp\left[\langle \vartheta, x\rangle - \Lambda(\vartheta)\right], \quad \vartheta \in \mathrm{dom}(\Lambda) \right\}$$

of probability measures on $\mathbb{R}^d$, and $a \in \mathbb{R}^d$ such that the maximum $\Lambda^*(a)$ of the *log-likelihood function* $\ell_a(\vartheta) = \langle a, \vartheta\rangle - \Lambda(\vartheta)$ is attained, the measure $Q_a^* \in \mathcal{E}$ parametrized by a maximizing ϑ satisfies

$$D\left(Q_a^* \| Q_\vartheta\right) \leq \Lambda^*(a) - \ell_a(\vartheta) \quad \text{for all} \quad Q_\vartheta \in \mathcal{E}.$$

If $\Lambda^*(a) = \sup \ell_a(\vartheta)$ is finite but not attained, there still exists a unique Q_a^* satisfying the last inequality. This Q_a^*, called *generalized MLE*, is not in $\mathcal{E}$ but in its variation distance closure $c\ell(\mathcal{E})$.

Not requiring any regularity conditions, explicit descriptions of $c\ell(\mathcal{E})$ and of the set $\mathrm{dom}(\Lambda^*) = \{a : \Lambda^*(a) < +\infty\}$ will be provided, in terms of the *convex core* of μ, and its faces.

To solve these problems, it was necessary to solve also their analogues for *canonically convex subfamilies* of $\mathcal{E}$, where the parameter ϑ is restricted to a convex subset of $\mathrm{dom}(\Lambda)$.

The reported result were obtained jointly with F. Matúš.