Invariant cones in infinite dimensional Lie algebras and unitary representations

In this talk we describe a systematic approach to unitary representations of infinite-dimensional Lie groups in terms of boundedness conditions on spectra in the derived representations. For any unitary representation $\pi\colon G \to U(H)$ which is smooth in the sense that it has a dense set of smooth vectors (which is automatic for finite-dimensional groups), one can associate its momentum set I_π which is a convex weak-$*$-closed subset of the dual space $\mathbf{L}(G)'$, invariant under the coadjoint action. It encodes the information on the spectral bounds of the derived representation. We call π bounded if I_π is equicontinuous and semi-bounded if I_π has a weaker property which we call semi-equicontinuity and which implies in particular that the convex cone $B(I_\pi)$ of all elements in $\mathbf{L}(G)$ for which the spectrum of $i \cdot d\pi(x)$ is bounded from below has interior points, which leads to an invariant open convex cone in $\mathbf{L}(G)$. For finite-dimensional groups, the semi-bounded representations are precisely the unitary highest weight representations and only groups with compact Lie algebras have bounded representations. For infinite-dimensional groups, the picture is much more colorful. There are many interesting bounded representations, in particular all those coming from representations of C^*-algebras, and most of the unitary representations appearing in physics are semibounded. We present old and new results connecting convexity properties of the coadjoint representation and unitary representations.