Lower Bounds on The Differential of a Graph

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1 Introduction

Let $\Gamma = (V, E)$ be a graph of order $n$ and let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set $D$. The \textit{differential} of $D$ is defined as $\partial(D) = |B(D)| - |D|$ and the differential of a graph is equal to $\max\{\partial(D) : D \subseteq V\}$. The graph parameter $\partial$ was introduced in [3]. There, also several basic properties were derived. Notice that for a graph $\Gamma$ of order $n$, $0 \leq \partial(\Gamma) \leq n - 2$.

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As explained in [3], this parameter is related to the well-known parameter $\gamma(\Gamma)$ denoting the minimum size of a dominating vertex set in $\Gamma$. Firstly [7],

$$\Psi(\Gamma) := \max\{|B(D)| : D \subseteq V\} = n - \gamma(\Gamma),$$

and the parameter $\Psi$ is known as the enclaveless number of a graph, and the corresponding set $B(D)$ is also known as the nonblocker set, see [2,7]. Secondly, for any graph without isolated vertices,

$$\Psi(\Gamma) - \gamma(\Gamma) = n - 2\gamma(\Gamma) \leq \partial(\Gamma) \leq \Psi(\Gamma) - 1,$$

(1) see [3]. We have shown in another paper that computing $\partial(\Gamma)$ is of a complexity similar to computing $\Psi(\Gamma)$, being NP-complete on rather restricted graph classes but solvable using parameterized algorithms (with a standard parameterization).

In this paper, we are studying lower bounds on the differential of a graph, obtaining results that nicely complement what is known about the enclaveless number. For that parameter, the following is known:

• [5] For any connected graph $\Gamma$ of order $n \geq 2$, $\Psi(\Gamma) \geq n/2$.

• [1,4] For any connected graph $\Gamma$ of order $n \geq 8$ and minimum degree $\delta(\Gamma) \geq 2$, $\Psi(\Gamma) \geq \frac{3n}{8}$. Moreover, there are seven exceptional connected graphs at all that violate this bound.

• [6] For any graph $\Gamma$ of order $n(\geq 3)$ satisfying $\delta(\Gamma) \geq 3$, $\Psi(\Gamma) \geq \frac{5n}{8}$.

The second item immediately implies, when combined with Eq. (1), that (under the same conditions as listed in that item), $\partial(\Gamma) \geq \frac{n}{5}$. Here, we derive the following main results, improving this immediate bound:

• For any connected graph $\Gamma$ of order $n \geq 3$, $\partial(\Gamma) \geq n/5$.

• For any connected graph $\Gamma$ of order $n \geq 9$ and minimum degree $\delta(\Gamma) \geq 2$, $\partial(\Gamma) \geq \frac{3n}{11}$. Moreover, there are four exceptional connected graphs at all that violate this bound.

2 Preliminaries

An alternative way of defining the differential of a graph is the following, which is based on the notion of a big star, i.e., some star $S_d$ with $d \geq 2$. Given the graph $\Gamma = (V, E)$, a big star packing is given by a vertex-disjoint collection $S = \{X_i | 1 \leq i \leq k\}$ of (not necessarily induced) big stars $X_i \subseteq V$, i.e., $\Gamma[X_i]$ contains some $S_d$ with $d = |X_i| - 1 \geq 2$. If $S$ is a big star packing of $\Gamma$, we also denote this by $SP(\Gamma, S)$. 


Proposition 1 \( \partial(\Gamma) = \max\{\sum_{S \in \mathcal{S}}(|S| - 2) : SP(\Gamma, S)\} \).

Proposition 2 There exists an infinite family of connected graphs \( \Gamma_k, k \geq 1 \), of order \( n \) with a partial of \( n/5 \).

We will say that a vertex \( v \in V \) is a critical vertex if \( v \in D \cup B(D) \) for every set \( D \subseteq V \) such that \( \partial(D) = \partial(\Gamma) \).

Lemma 1 Let \( \Gamma_1 = (V_i, E_i) \) be a graph which has a critical vertex \( u_i \), for \( i = 1, 2 \). If \( \Gamma = (V, E) \) is a graph such that \( V = V_1 \cup V_2 \) and \( E = E_1 \cup E_2 \cup u_1 u_2 \), then \( u_1 \) and \( u_2 \) are critical vertices of \( \Gamma \) and \( \partial(\Gamma) = \partial(\Gamma_1) + \partial(\Gamma_2) \).

Proposition 3 There exists an infinite family of connected graphs \( \Gamma_k, k \geq 1 \), of minimum degree two and of order \( 11k \) with a partial of \( 3k \).

Proof. Consider the following graph \( \Gamma^i = (V^i, E^i) \) of order 11.

\[ \text{The differential of this graph is 3 and } v_i \text{ is the only critical vertex. Now, we consider a connected graph } \Gamma_k = (V_k, E_k) \text{ such that } V_k = \bigcup_{i=1}^{k} V^i \text{ and } E_k = \bigcup_{i=1}^{k} E^i \cup \{v_1 v_2, v_2 v_3, ..., v_{k-1} v_k\}. \]

By Lemma 1, we know that \( \partial(\Gamma) = 3k \). \( \square \)

3 Main Results

We are going to exhibit the following main results:

Theorem 1 For any connected graph \( \Gamma \) of order \( n \geq 3 \), \( \partial(\Gamma) \geq n/5 \).

Theorem 2 For any connected graph \( \Gamma \) of order \( n \) that has minimum degree two, \( \partial(\Gamma) \geq 3n/11 \) except for the following graphs

Moreover, the bounds in both theorems are best possible, as certified by the infinite families of graphs presented in the previous section. As the proof of Theorem 1 is pretty straightforward, we will sketch the proof of Theorem 2 only in the following.

If our theorem was false, then there should exist an example \( \Gamma = (V, E) \) with \( |V| = n \), \( \delta(\Gamma) \geq 2 \) and \( \partial(\Gamma) < 3n/11 \). If such a counterexample exists, we
could also ask for a proof, i.e., we are also given a vertex set \( D \) with \( \partial(D) = \partial(\Gamma) \). Let \( D_1(\Gamma) = \{ D \subseteq V \mid \partial(D) = \partial(\Gamma) \} \). Define \( C(D) = V \setminus (D \cup B(D)) \).

**Lemma 2** If \( D \in D_1(\Gamma) \), then the induced graph \( \Gamma[C(D)] \) decomposes into \( K_1 \)- and \( K_2 \)-components.

For convenience, we can also assume that, apart from \( D \in D_1(\Gamma) \), we are also given a corresponding big star packing \( S(D) \), where \( D \) collects all centers of stars in \( S(D) \). Hence, \( \partial D = \sum_{S \in S(D)} (|S| - 2) \) and \( D \cup B(D) = \bigcup_{S \in S(D)} S \).

Notice that there might be several big star packings that testify the differential claimed for \( D \), but we will fix one of these big star packings, denoted as \( S(D) \), in the following discussion. For every \( j = 2, \ldots, \Delta \), we will denote by \( S_j(D) \) the set of all stars \( S \) in \( S(D) \) such that \( |S| = j + 1 \).

Since there could be several differential sets that attain the differential \( \partial(\Gamma) \), we will furthermore ask for a differential set \( D \) that maximizes \( |D| \) among all those with \( \partial(D) = \partial(\Gamma) \). Let \( D_2(\Gamma) = \{ D \in D_1(\Gamma) \mid \forall D' \in D_1(\Gamma) \ (|D'| \leq |D|) \} \).

**Lemma 3** If \( D \in D_2(\Gamma) \), then any vertex \( x \) in \( B(D) \) has at most one neighbor in \( C(D) \).

As a possible application of Lemma 2 and Lemma 3, we state:

**Lemma 4** For any connected graph \( \Gamma \) of order \( n \leq 8 \) that has minimum degree two, if \( \partial(\Gamma) < \frac{3n}{11} \), then \( \Gamma \) is one of the four graphs listed in Theorem 2.

Due to Lemma 2, we could also establish a third priority; let \( k_2(D) \) denote the number of \( K_2 \)-components in \( \Gamma[C(D)] \). Let \( D_3(\Gamma) = \{ D \in D_2(\Gamma) \mid \forall D' \in D_2(\Gamma) \ (k_2(D') \leq k_2(D)) \} \). Fix some arbitrary \( D \in D_3(\Gamma) \) in the following discussion. In order to explain the importance of the given sequence of priorities, we establish:

**Lemma 5** If \( S \in S(D) \) with \( |S| \geq 4 \), then no \( x \in S \setminus D \) is neighbor of a \( K_2 \)-component in \( \Gamma[C(D)] \).

A sequence of pairwise distinct adjacent vertices \( v_1, \ldots, v_t \) is called an \( S_2 \) sequence if it obeys the following recursive definition:

- either \( t = 1 \) and \( v_1 \in C(D) \), or
- \( t = 3 \), \( \{v_1, v_2, v_3\} \in S(D) \), and \( v_2 \in D \), or
- \( t > 1 \), \( v_t \in C(D) \), and \( v_1, \ldots, v_{t-1} \) is an \( S_2 \) sequence, or
- \( t > 3 \), \( \{v_{t-2}, v_{t-1}, v_t\} \in S(D) \), \( v_{t-1} \in D \), and \( v_1, \ldots, v_{t-3} \) is an \( S_2 \) sequence.

An \( S_2 \) sequence \( s \) is called maximal if there are no vertices \( x \) (or \( x, y, z \)) that do not already occur in \( s \), such that \( s, x \) or \( s, x, y, z, s \) or \( s, x, y, z \).
form an $S_2$ sequence. An $S_2$ sequence $s = v_1, v_2,\ldots, v_t$ is called maximal from $v_1$ if there are no vertices $x$ (or $x, y, z$) that do not already occur in $s$, such that $s, x$ or $x, y, z, s$ form an $S_2$ sequence. Clearly, if $s = v_1, v_2,\ldots, v_t$ is an $S_2$ sequence, then $s^- = v_t, v_{t-1},\ldots, v_1$ is an $S_2$ sequence, as well; $s$ is maximal if and only if $s$ is maximal from $v_1$ and $s^-$ is maximal from $v_t$. If $v_1$ is adjacent to $v_t$ we will consider that $s$ and $s^-$ are equivalent. For $S \in \mathcal{S}(D)$, let $C(S, D)$ collect all vertices from $C(D)$ that are neighbors of vertices from $S$. For a collection $S$ of stars, let $C(S, D) = \bigcup_{S \in S} C(S, D)$. Let $\mathcal{D}_4(\Gamma) = \{D \in \mathcal{D}_3 \mid \forall D' \in \mathcal{D}_3(|C(S_3(D'), D')| \leq |C(S_3(D), D')|)\}$. So, in the following, let $D \in \mathcal{D}_4(\Gamma)$.

For every $D \in \mathcal{D}_4(\Gamma)$ we denote by $s_2(D)$ the number of maximal inequivalent $S_2$ sequences in $\Gamma$, and $\mathcal{D}_5(\Gamma) = \{D \in \mathcal{D}_4(\Gamma) : \forall D' \in \mathcal{D}_4(\Gamma) (s_2(D) \leq s_2(D'))\}$. In the following, we consider $D \in \mathcal{D}_5(\Gamma)$.

According to our priorities, for the discussion of any maximal $S_2$ sequence $s$ starting with $x$, it sufficient to distinguish three different cases:

**non-C case** $s$ contains no vertex from $C(D)$ at all.

**single-C case** $s$ contains exactly one vertex from $C(D)$, which is $x$.

**double-C case** $s$ contains exactly two vertices from $C(D)$, which are the first vertex $x$ in $s$ and the second vertex $y$ in $s$.

In order to prove our bound, the first of the three cases does not harm, since it implies a better ratio (of three) on the $S_2$ path. Of particular danger to our counting are those $C(D)$-vertices that are not close to big stars.

In the following, we abbreviate $C_2(D) = C(D) \setminus \left( \bigcup_{S \in S, j \geq 3} C(S, D) \right)$.

We start discussing the single-C case:

**Lemma 6** Consider a maximal $S_2$ sequence $s$ starting with $x$ such that $s$ contains exactly one vertex from $C(D)$, which is $x$. Assume that $x \notin N(S)$ for any $S \in S_{\geq 3}(D)$. Let $z$ be the last vertex of $s$ and let $N(z, \notin s)$ collect all neighbors of $z$ that are not already in $s$. Then, the following case may occur (and only this):

- $N(z, \notin s) = \emptyset$; since $\delta(z) \geq 2$, this means that the sequence $s$ stops because all neighbors of $z$ are already listed in $s$.

A similar statement is true for the double-C case.

We will say that a vertex $x \in C_2(D)$ has a private $S_2$ star if there exists a maximal $S_2$ sequence starting with $x$ containing this star, which does not belong to a $S_2$ sequence starting with $x' \in C_2(D)$ with $x' \neq x$.

**Lemma 7** Every maximal $S_2$ sequence starting with $x \in C_2(D)$ contains more
than one $S_2$ star.

A similar Lemma is true for the double-C case.

Lemma 8 If $s$ is an $S_2$ sequence that starts with $x \in C_2(D)$ (or with $e = \{x, y\} \subseteq C_2(D)$), then any $S_2$ star in $s$ is private for $x$ (or for $e$).

We denote by $k_j$ the number of $S \in S(D)$ such that $|S| = j + 1$. By the previous lemmas we obtain the following result.

Lemma 9 If $|C_2(D)| = r$, then $3r \leq 2k_2$.

This is the final cornerstone to establish a contradiction to the existence of a counterexample by a rather straightforward counting argument.

4 Concluding Remarks

Having established some lower bounds, some natural questions prevail:

- We do not have many examples of maximum degree three that attain the bound we have proved for general graphs in Theorem 2. Possibly, one could show a larger lower bound in these cases.
- Reed could establish a better bound in relation with the domination (or enclaveless) parameter for graphs with minimum degree three. Are similar achievements possible for the differential?
- So far, we could only make use of Theorem 1 for purposes of parameterized complexity. Is there any way to employ Theorem 2 for this purpose to obtain better run times of our parameterized algorithms (mentioned elsewhere)?

References


A Omitted Proofs and Further Comments on the Preliminaries Section

In the proof of Proposition 1, we need a new definition. Given $X \subseteq V$ and $u \in X$, we will say that $v \in V \setminus X$ is an $X$-external private neighbor ($X$-epn) of $u$ if $N(v) \cap X = \{u\}$.

**Proof of Proposition 1** For every $S = \{X_1, \ldots, X_k\}$ satisfying $SP(\Gamma, S)$, if we consider the set $D$ which collects all centers $v_1, \ldots, v_k$ of the stars $X_1, \ldots, X_k$, we have

$$\partial(\Gamma) = |B(D)| - |D| \geq \sum_{j=1}^{k} (|N(v_j) \cap X_j| - |v_j|) = \sum_{j=1}^{k} (|X_j| - 2),$$

therefore, $\partial(\Gamma) \geq \max\{\sum_{S \in \mathcal{S}} (|S| - 2) : SP(\Gamma, S)\}$.

On the other hand, if we take $D = \{v_1, \ldots, v_k\} \subseteq V$, with minimum cardinality, such that $\partial(D) = \partial(\Gamma)$, we know that every vertex $v \in D$ has, at least, two $D$-external private neighbors. Then, the family of sets

$$\left\{ \{v_j\} \cup \left( N(v_j) \setminus \bigcup_{i=1}^{j-1} N(v_i) \right) : j = 1, \ldots, k \right\}$$

is a big star packing of $\Gamma$ and

$$\partial(D) = \sum_{j=1}^{k} \left( \left| N(v_j) \setminus \bigcup_{i=1}^{j-1} N(v_i) \right| - 1 \right)$$

$$= \sum_{j=1}^{k} \left( |\{v_j\}| \cup \left( N(v_j) \setminus \bigcup_{i=1}^{j-1} N(v_i) \right) \right| - 2 \right)$$

$$\leq \max\{\sum_{S \in \mathcal{S}} (|S| - 2) : SP(\Gamma, S)\}. \quad \square$$

**Proof of Proposition 2** This bound is attained in any tree $T = (V, E)$ with maximum degree $\Delta$ such that, if $v$ is a vertex of maximum degree, $\langle V \setminus \{v\} \rangle$ is a disconnected graph with $\Delta$ connected components, two of them are paths $P_2$ and the rest are $P_5$; moreover, we require that the order of $T$ is divisible by five. This family can be obtained from Theorem 13 in [3], taking $\Delta = \frac{n}{5} - 1$, and it looks as follows:
Proof of Lemma 1 It is clear that $\partial(\Gamma) \geq \partial(\Gamma_1) + \partial(\Gamma_2)$. Let $D$ be a set such that $\partial(D) = \partial(\Gamma)$. We know that $D = D_1 \cup D_2$ where $D_i \subseteq V_i$. If $u_2 \in B(D) \cap B(D_1)$, then $\partial(D_1) \leq \partial(\Gamma_1) + 1$ and, as $u_2$ is a critical vertex of $\Gamma_2$, $|B(S_2) \cap (V_2 \setminus \{u_2\})| - |S_2| \leq \partial(\Gamma_2) - 1$, therefore $\partial(S) \leq \partial(\Gamma_1) + \partial(\Gamma_2)$. If $u_2 \notin B(D) \cap B(S_1)$, then $\partial(S_1) \leq \partial(\Gamma_1)$. Since $u_1$ is a critical vertex of $\Gamma_1$, it belongs to $S_1 \cup B(S_1)$, so $|B(S_2) \setminus \{u_1\}| - |S_2| \leq \partial(\Gamma_2)$, therefore $\partial(S) \leq \partial(\Gamma_1) + \partial(\Gamma_2)$.

Complementing the proof of Proposition 3, we display the graph obtained by the construction:

More generally, by Lemma 1, one can construct a larger family of graphs with $11k$ vertices attaining the bound $3k$ taking, as in the proof of Proposition 3, $\Gamma_k = (V_k, E_k)$ such that $V_k = \bigcup_{i=1}^{k} V^i$ and $E_k = \bigcup_{i=1}^{k} E^i \cup M$, where

$$M \subseteq \{v_1v_2, ..., v_1v_k, v_2v_3, ..., v_2v_k, ..., v_{k-1}v_k\}$$

such that $\Gamma_k$ is a connected graph.

We conjecture that any graph of minimum degree two with at least nine vertices that is different from the cycle $C_{11}$ and from a $C_{11}$ with one chord...
and that attains the claimed bound on the partial is a graph from this more general graph family.

B Omitted Proofs for the Main Results

Proof of Theorem 1 According to Proposition 1,

\[
\partial(\Gamma) = \max_{\{S_2, \ldots, S_\Delta\}-\text{packing}} \sum_{d=2}^\Delta (d-1)k_d,
\]

where \( k_d \) is the number of stars \( S_d \) in the packing. We suppose that \( S \) is the set of vertices which are the centers of the stars in a packing \( P \) giving the differential of \( \Gamma \) with minimum size.

We are going to find the maximum value for \( |C(S)| \). For any vertex \( v \in S \) which is a center of an \( S_d \) star \( X \) where \( d \geq 3 \), we consider the subgraph induced by \( X \). Moreover, slightly abusing notation, let \( B(\{v\}) \) denote \( X \setminus \{v\} \), and \( C(\{v\}) \) are the \( C \)-vertices that are neighbors of \( B(\{v\}) \). Let us note that it is possible that a vertex \( u \) belongs to two different sets \( C(\{v_1\}) \) and \( C(\{v_2\}) \), but it does not matter because we are looking for the maximum cardinality of \( C(S) \). Since the maximum number of neighbors that every vertex in \( B(\{v\}) \) has in \( C(\{v\}) \) is two, and \( C(\{v\}) \) has only \( K_1 \) and \( K_2 \) components, the maximum cardinality of \( C(\{v\}) \) is attained in the following case:

![Diagram of graph](image)

We cannot have more than \( d - 2 \) vertices in \( B(\{v\}) \) having two private neighbors in \( C(\{v\}) \) because, taking all these vertices, we would obtain a bigger differential. None of the grey vertices in \( C(\{v\}) \) can have a neighbor
in $C(\{v\})$ because, in such a case, taking this vertex and black vertices, we would also obtain a bigger differential. The vertices we obtain in $C(\{v\})$ for any star $S_d$ is $4|B(\{v\})| - 6 = 4d - 6$.

If $v \in S$ is a center of a star $S_d$, then the set $C(\{v\})$ cannot have more than two vertices because, in such a case, we can choose one or two vertices in $\{v\} \cup B(\{v\}) \cup C(\{v\})$ giving a bigger differential.

In consequence, $n = |S| + |B(S)| + |C(S)| \leq |S| + |B(S)| + 4|B(S)| - 6|S| = 5\partial(\Gamma)$.

**Proof of Lemma 2** If there exists a subgraph of $\Gamma[D]$ that is a path on three vertices and $v$ is the center of such a path, then we have $\partial(D \cup \{v\}) > \partial(D)$, a contradiction.

**Proof of Lemma 3** Assume $b \in B(D)$ is a vertex which has more than one neighbor in $C(D)$. By definition, $b$ belongs to some star $S \in S(D)$ with center $v \in D$. If $|S| \geq 4$, then we have $\partial(D \cup \{b\}) \geq \partial(D)$ and $|D \cup \{b\}| > |D|$, a contradiction to $D \in D_2(\Gamma)$. If $|S| = 3$, then we have $\partial((D \setminus \{v\}) \cup \{b\}) > \partial(D)$, again a contradiction.

**Proof of Lemma 4** If there exists a graph $\Gamma = (V,E)$ such that $|V| = n \leq 8$, $\delta(\Gamma) \geq 2$ and $\partial(\Gamma) < \frac{3n}{11}$, we could find a set $D \subseteq V$ such that $D \in D_2(\Gamma)$ and $\partial(D) < \frac{3n}{11}$, and we could consider the big star packing $S(D)$. If $n \leq 7$, then $\partial(D) = 1$ and, in consequence, every vertex has degree two, that is, $\Gamma$ is a cycle. But, the cycles $C_n$ with order $n \leq 7$ such that $\partial(C_n) < \frac{3n}{11}$ are $C_4$ and $C_5$.

Now, we suppose $n = 8$. If the packing $S(D)$ has only a $S_3$ star, by Lemma 3, the number of vertices in $C(D)$ is, at most, three, then $n \leq 7$, a
contradiction. Therefore, the packing has exactly two $S_2$ stars $\{b_1, v_1, b_2\}$ and $\{b_3, v_2, b_4\}$, and there are two vertices $c_1, c_2$ in $C(D)$. If $c_1$ and $c_2$ are adjacent to the same vertex, for instance, $b_1$, taking $D' = (D \setminus \{v_1\}) \cup \{b_1\}$ we would have $\partial(D') > \partial(D)$, a contradiction. Since $\delta(\Gamma) \geq 2$, we have to study some cases:

**Case 1.** If $c_1 \sim c_2$, as both vertices must be connected with some vertices in $B(D)$, we have two subcases.

**Case 1.1.** If $c_1 \sim b_1$ and $c_2 \sim b_2$ we have a cycle with 5 vertices. Since $b_3$ and $b_4$ must be connected to that cycle, the only way to connect them, not to obtain a bigger differential, is to connect them with two adjacent vertices in the cycle. In such a case we have a graph like the fourth figure in the lemma.

**Case 1.2.** If $c_1 \sim b_1$ and $c_2 \sim b_3$, we have a path $(b_2, v_1, b_1, c_1, c_2, b_3, v_2, b_4)$. If $b_2$ was connected to $b_1, c_1, b_3$ or $v_2$, the differential would be bigger than two. If $b_2$ was connected to $b_4$, we would have a cycle and the only extra edge would be a diameter of this cycle. If $b_2$ was connected to $c_2$, we would have a cycle with 5 vertices and, as in case 1.1, $b_4$ must be connected to $c_1$ or $b_2$, producing a graph like the fourth figure in the lemma.

**Case 2.** If $c_1$ and $c_2$ are not adjacent, each one must be connected with two vertices in $B(D)$, then we have two subcases.

**Case 2.1.** If $c_1 \sim b_1$, $c_1 \sim b_2$, $c_2 \sim b_3$ and $c_2 \sim b_4$, we have two cycles with 4 vertices. Since these two cycles must be connected by, at least, an edge, the differential would be bigger than 2.

**Case 2.2.** If $c_1 \sim b_1$, $c_1 \sim b_3$, $c_2 \sim b_2$ and $c_2 \sim b_4$, we have a cycle with eight vertices, and the only extra edge would be a diameter of this cycle, producing a graph like the fourth figure in the lemma. \(\square\)

**Proof of Lemma 5** Consider $S \in S(D)$ with $|S| \geq 4$ such that $x \in S \setminus D$ is neighbor of a $K_2$-component $\{y, z\}$ in $\Gamma[C(D)]$. Assuming $x \in N(y)$, $D' = D \cup \{y\}$ satisfies $\partial(D) = \partial(D')$ but $|D'| > |D|$, hence violating $D \in D_2(\Gamma)$. \(\square\)

Notice that the proof destroys any $K_2$-component adjacent to a $S$ in the packing with $|S| \geq 4$, since according to our list of priorities, increasing the cardinality of the partial set is more important.

We summarize simple properties of $S_2$ sequences in the following lemma.

**Lemma 10** Let $s$ be an $S_2$ sequence.

(i) If $s$ contains two vertices $x, y$ from $C(D)$, then they are adjacent. Furthermore, without loss of generality, we can also assume that they are adjacent within $s$ in further discussions.

(ii) $s$ does not contain three vertices $x, y, z$ from $C(D)$.
(iii) If \( s \) is maximal and contains some vertex from \( C(D) \), then the first (and
the last) vertex is not neighbor of any vertex of another \( S_2 \) sequence.

**Proof of Lemma 10** Let \( s \) be an \( S_2 \) sequence that contains (at least) two
vertices \( x, y \) from \( C(D) \). Assume \( s = s_1, x, s_2, y, s_3, \) such that neither \( s_1 \)
or \( s_3 \) contains vertices from \( C(D) \). Let \( s_2 = (u_1, u_2, u_3, \ldots, u_3) \), such that
\( \{u_{3j+1}, u_{3j+2}, u_{3j+3}\} \in S(D) \) with \( u_{3j+2} \in D \). If \( x \) and \( y \) are not adjacent (in
\( \Gamma \)), then
\[
D' = (D \setminus \{u_{3j+2} \mid 0 \leq j < t\}) \cup \{u_{3j+1} \mid 0 \leq j < t\}
\]
satisfies \( \partial(D) = \partial(D') \) and \( |D'| = |D| \), but now \( u_{3t}, y \) are neighbored vertices
in \( C(D') \), hence violating \( D \in \mathcal{D}_3(\Gamma) \), since \( k_3(D') > k_3(D) \). If \( x \) and \( y \)
are adjacent in \( \Gamma \) but not in \( s \), the same shifting argument shows the second
sentence. If there is another vertex \( z \in C(D) \) in \( s_3 \), then a further shift would
even create three vertices from \( C(D) \) in a row, contradicting Lemma 2.

We now prove the third assertion. Consider a maximal \( S_2 \) sequence \( s \) that
contains some vertex from \( C(D) \). Clearly, if the first (or the last) vertex are
not neighbor of any \( C(D) \)-vertex (not yet in \( s \)), nor of any vertex of some
\( S_2 \) star that is not the center of that star, since otherwise we could extend \( s \)
to a longer \( S_2 \) sequence, contradicting the maximality of \( s \). If, e.g., the first
vertex \( x \) of \( s \) is neighbor of the center of some \( S_2 \) star \( S \) (that is not in \( s \)), then
we could shift the \( C \)-vertex in \( s \) to its very beginning (as formally explained
before), leading to some differential set \( D' \) (instead of \( D \)), so that we could
replace \( S \) by \( S \cup \{x\} \), testifying that \( \partial(D') > \partial(D) \), which contradicts the
choice of \( D \in \mathcal{D}_1(\Gamma) \).

If we have a maximal \( S_2 \) sequence \((x, b_1, v_1, b_2, b_3, v_3, b_4, \ldots)\) starting with
\( x \in C_2(D) \), we can suppose that this \( S_2 \) sequences cannot be extended from
the other side of \( x \) because, if there exists a \( S_2 \) star \( \{b'_2, v'_2, b'_1\} \) such that
\((b'_2, v'_2, b'_1, x, b_1, v_1, b_2, b_3, v_3, b_4, \ldots) \) is a \( S_2 \) sequence, we could consider
\( D' = (D \setminus \{v'_2\}) \cup \{b'_1\} \) as the set given, because it satisfies the priorities, and the
vertex of \( C_2(D') \) would be on the left of this new \( S_2 \) sequence. The same
happens if we have two adjacent vertices in \( C_2(D) \).

**Proof of Lemma 6** Assume that \( N(z, \notin s) \neq \emptyset \). Consider \( v \in N(z, \notin s) \).
Due to the previous Lemma and by our assumption, \( v \notin C(D) \). Moreover,
\( v \notin D \), since otherwise we could move the \( C(D) \)-vertex along \( s \) so that then
\( z \in C(D') \) for some \( D' \) with \( \partial(D) < \partial(D') \), contradicting our assumptions.

\( v \in S \) for any \( S \in S_{\geq 3}(D) \) is excluded due to our priorities: we could
move the \( C(D) \)-vertex along \( s \) so that then \( z \in C(D') \) for some \( D' \) with
\( \partial(D) = \partial(D') \) and \( |D| = |D'| \) and \( k_2(D) = k_2(D') \), but \( D' \) has one \( C(D')-\)
vertex more than $D$ in the neighborhood of a big star, contradicting our assumptions.

Hence, $v \in S$ for some $S \in S_2(D)$, and $v \notin D$. But then, we could continue the $S_2$ path $s$, contradicting the maximality of $s$. $\Box$

With a similar proof, we can deal with the double-$C$ case:

**Lemma 11** Consider a maximal $S_2$ sequence $s$ starting with $x$ such that $s$ contains exactly two vertices from $C(D)$, which are the first vertex $x$ in $s$ and the second vertex $y$ in $s$. Assume that $x \notin N(S)$ for any $S \in S_{\geq 3}(D)$. Let $z$ be the last vertex of $s$ and let $N(z, \notin s)$ collect all neighbors of $z$ that are not already in $s$. Then, the following case may occur (and only this):

- $N(z, \notin s) = \emptyset$; since $\delta(z) \geq 2$, this means that the sequence $s$ stops because all neighbors of $z$ are already listed in $s$.

So, in both cases, maximal $S_2$ sequences must “stop at itself”. Since we are dealing with graphs without double edges or loops, this immediately implies that any maximal $S_2$ sequence that contains a $C(D)$-vertex that is not neighbor of a big star must have minimum length four.

**Proof of Lemma 7** We consider a maximal $S_2$ sequence $s$ starting with $x$. If $s = (x, b_1, v_1, b_2)$, by Lemma 6, $b_2$ is adjacent to $b_1$ or to $x$. If $b_2$ is adjacent to $b_1$, we can take $D' = (D \setminus \{v_1\}) \cup \{b_1\}$ to obtain $\partial(D') > \partial(D)$, a contradiction. Therefore, $b_2$ is adjacent to $x$ (and $s = (x, b_1, v_1, b_2)$ is equivalent to $(x, b_2, v_1, b_1)$). Since $n \geq 9$, this cycle must be connected to a vertex $u$ in the rest of the graph. Since no vertex in this cycle with four vertices is a critical vertex in the cycle, we can remove the vertices in such a cycle to get, if $u \in D$, a contradiction with $D \in D_1(\Gamma)$, if $u \in C(D)$, a contradiction with $D \in D_3(\Gamma)$, if $u \in B(D) \cap S_{\geq 3}(D)$, a contradiction with $D \in D_4(\Gamma)$, if $u \in B(D) \cap S_2(D)$ and $u \sim b_1$ or $u \sim b_2$, a contradiction with the maximality of $s$, if $u \in B(D) \cap S_2(D)$ and $u \sim v_1$, a contradiction with $D \in D_5(\Gamma)$.

Therefore, $s$ has, at least, two $S_2$ stars. $\Box$

We will say that an edge $e = \{x, y\}$, with $x, y \in C_2(D)$, has a private $S_2$ star if there exists a maximal $S_2$ sequence starting with $e$ containing this star, which does not belong to a $S_2$ sequence starting with $e' = \{x', y'\} \neq e$, with $x', y' \in C_2(D)$.

**Lemma 12** Every maximal $S_2$ sequence starting with an edge $e = \{x, y\}$, with $x, y \in C_2(D)$, contains more than two $S_2$ stars.

**Proof of Lemma 12** We consider a maximal $S_2$ sequence $s$ starting with $e$. If $s = (x, y, b_1, v_1, b_2)$, by Lemma 6, $b_2$ is adjacent to $x$, $y$ or to $b_1$. If $b_2$ is adjacent
to \(b_1\) (or \(y\)), we can take \(D' = (D \setminus \{v_1\}) \cup \{b_1\}\) (or \(D' = (D \setminus \{y\}) \cup \{b_1\}\)) to obtain \(\partial(D') > \partial(D)\), a contradiction. Therefore, \(b_2\) is adjacent to \(x\) (and \(s = (x, y, b_1, v_1, b_2)\) is equivalent to \((y, x, b_2, v_1, b_1)\)). Since \(n \geq 9\), this cycle must be connected to a vertex \(u\) in the rest of the graph and, as in the previous lemma, we can get a contradiction. Therefore, \(s\) has, at least, two \(S_2\) stars, that is \(s = (x, y, b_1, v_1, b_2, v_3, b_1)\). Since \(x\) has to be adjacent to \(b_1, v_1, b_3\) or \(v_3\), we would be able to improve the differential, we have that \(x\) is adjacent to \(b_2\) or \(b_4\).

If \(x\) is adjacent to \(b_4\), since \(n \geq 9\), this cycle must be connected to a vertex \(u\) in the rest of the graph. Since no vertex in this cycle with eight vertices is a critical vertex in the cycle, as in in the proof of the previous lemma, we can remove the vertices in such a cycle to get a contradiction.

If \(x\) is adjacent to \(b_2\), as we know that \(b_4\) has to be adjacent to a vertex in \(s\), it has to be adjacent to \(v_1\) (in other case we could get a bigger differential).

Then, \((x, y, b_1, v_1, b_2, b_3, v_3, b_4)\) is a cycle with eight vertices and we can also get a contradiction.

\[
\text{Proof of Lemma 8} \quad \text{Let us prove the result for a vertex } x \in C_2(D), \text{ the proof for a edge is similar. Let us see that, if } s = (x, b_1, v_1, b_2, b_3, v_3, b_4, ...) \text{ is a maximal } S_2 \text{ sequence, then } \{b_{2j-1}, v_{2j-1}, b_{2j}\} \text{ is a private } S_2 \text{ star for } x. \text{ By absurdum, taking a } x' \in C_2(D), \text{ with } x' \neq x, \text{ and a maximal } S_2 \text{ sequence } s' \text{ starting with } x', \text{ we can suppose two cases:}
\]

**Case 1.** If \(s' = (x', b'_1, v'_1, b'_2, ..., b'_{2k-1}, v'_{2k-1}, b'_2, b_{2j-1}, v_{2j-1}, b_{2j}, b'_{2k+3}, v'_{2k+4}, ..., )\),

we could take \(D' = (D \setminus \{v'_1, v'_3, ..., v'_{2k-1}, v_1, ..., v_{2j-1}\}) \cup \{b'_1, b'_3, ..., b'_{2k-1}, b_1, ..., b_{2j-1}\}\) to obtain \(\partial(D') > \partial(D)\), a contradiction.

**Case 2.** If \(s' = (x', b'_1, v'_1, b'_2, ..., b'_{2k-1}, v'_{2k-1}, b'_2, b_{2j-1}, v_{2j-1}, b_{2j-1}, b'_{2k+3}, v'_{2k+4}, ..., )\),
In conclusion, such a counterexample does not exist.

To obtain a contradiction with the fact that $D \in D_k(T)$, we could take $D' = (D, \{c_1, c_2, \ldots, c_{2r}, v_{-1}, v_0, v_1, \ldots, v_{2r-1}\})$.

Proof of Lemma 9: If $r_1$ is the number of isolated vertices in $C(D)$ and $r_2$ is the number of $K_2$ components in $C(D)$, we have shown in the previous lemmas that $2r_1 + r_2 \leq 2k_3$, then $3r = 3r_1 + 2r_2 \leq 3k_3$. This allows us to complete the proof of Theorem 2.

Therefore, by Lemma 9 we know $3r \leq 3k_3$, then

$$\frac{3n}{2} \leq \theta(D) + \frac{3r}{2} \leq \frac{3k_3}{2} \leq \theta(D).$$

Hence, we have

$$\left| \mathcal{A} \setminus \mathcal{A}_D \right| \leq \left| \mathcal{B}_D \right|,$$