Equipartition of two measures by a 4-fan

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Abstract

We show that, given two probability measures in the plane, there exists a 4-fan that simultaneously equipartitions them. In other words, there is a point and four halflines emanating form it such that each of the four sectors have measure 1/4 in both measures.

1 Introduction

This is continuation of our previous work [BM] whose terminology and notations are used here without much change. A point x in the plane and four halflines, $\ell_1, \ell_2, \ell_3, \ell_4$, starting from x form a 4-fan. The halflines are in clockwise order around x. They determine four angular sectors $\sigma_1, \dots, \sigma_4$ with σ_i between ℓ_i and ℓ_{i+1}. Assume λ and μ are nice probability measures on $\mathbf{R}^2$, say, none of them is positive on any line. We showed in [BM], that there is a 4-fan with $\lambda(\sigma_i) = \mu(\sigma_i) = \frac{1}{5}$ for $i = 1, 2, 3$. (Then, of course, $\lambda(\sigma_4) = \mu(\sigma_4) = \frac{2}{5}$.) Our main result here is

Theorem 1.1 *Assume λ and μ are absolutely continuous (with respect to the Lebesgue measure) Borel probability measures on $\mathbf{R}^2$. Then there is a 4-fan simultaneously equipartitioning both, i.e., $\lambda(\sigma_i) = \mu(\sigma_i) = \frac{1}{4}$ for $i = 1, 2, 3, 4$.*

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It will be more convenient to lift the measures and the 4-fans from $\mathbf{R}^2$ to the 2-sphere S^2. So let S^2 be the unit sphere of $\mathbf{R}^3$ and let $\mathbf{R}^2$ be embedded in $\mathbf{R}^3$ as the horizontal plane tangent to S^2 (at the South Pole). Denote by π the central projection from the lower hemisphere to the embedded $\mathbf{R}^2$. Clearly, π^{-1} lifts any Borel measure on $\mathbf{R}^2$ to a Borel measure on the lower hemisphere of S^2. A 4-fan in $\mathbf{R}^2$ is lifted to a 4-fan in S^2 in a natural way: a spherical 4-fan is a point $x \in S^2$ and four great halfcircles $\ell_1, \dots, \ell_4$ starting at x (and ending at $-x$) that are ordered clockwise when viewed from the origin. The angular sector between ℓ_i and ℓ_{i+1} is σ_i. It is clear that a spherical 4-fan is projected by π to a 4-fan in $\mathbf{R}^2$. We will prove Theorem 1.1 in a slightly stronger form:

Theorem 1.2 *Assume λ and μ are absolutely continuous (with respect to the Lebesgue measure) Borel probability measures on S^2. Then there is a spherical 4-fan simultaneously equipartitioning both, i.e., $\lambda(\sigma_i) = \mu(\sigma_i) = \frac{1}{4}$ for $i = 1, 2, 3, 4$.*

In fact, this theorem holds under the weaker assumption that neither λ nor μ is positive on a great circle. This follows from a simple compactness argument.

We will call a measure nice if it is a Borel measure, a probability measure, it is absolutely continuous with respect to the Lebesgue measure, and it is positive on every nonvoid open set. We will prove Theorem 1.2 assuming that both measures are nice. This will suffice for the general case of by the same compactness argument. By the same token it is enough prove the theorem for a dense set of nice measures, and we will assume, in case of need, that our measures satisfy certain extra properties.

2 Equivariant topology

Write $V = \{(x, y) \in S^2 \times S^2 \colon x \perp y\}$; V is the Stiefel manifold of orthogonal 2-frames in $\mathbf{R}^3$, which is homeomorphic to $SO(3)$ (see [Br]). Of this, we will only need that V is 3-dimensional and orientable. To every $(x, y) \in V$ we assign the 4-fan $\psi(x, y) = (x; \ell_1, \dots, \ell_4)$ as follows: y is the midpoint of the great halfcircle ℓ_1, and ℓ_2, ℓ_3, ℓ_4 are defined by the condition $\lambda(\sigma_i) = \frac{1}{4}$ for all i. Write X_4 or X for the set of all 4-fans equipartitioning the measure λ. Then ψ is a map $V \to X_4$.

To evaluate the quality of a 4-fan $(x; \ell_1, \dots, \ell_4) \in X$ we define a map $\varphi\colon X \to \triangle^3$ (where $\triangle^3$ is the 3-dimensional standard simplex: $\triangle^3 = \{x \in \mathbf{R}^4 : x_i \geq 0, \sum_{i=1}^4 x_i = 1\}$) as $\varphi(x; \ell_1, \dots, \ell_4) = (\mu(\sigma_1), \dots, \mu(\sigma_4))$. All we want to show is that the map $\varphi \circ \psi\colon V \to \triangle^3$ takes the value $c = (1/4, 1/4, 1/4, 1/4)$, the center of the simplex. Assume it does not. Write π for the central projection $\triangle^3 \setminus \{c\} \to T$, where T is the sphere of radius $1/\sqrt{2}$ centered at c in the affine hull of $\triangle^3$. Thus T, the target space, is topologically an S^2. (The special choice of the radius $1/\sqrt{2}$ will result in simpler notation later.)

We are going to use equivariant topology. The cyclic groups Z_4 and Z_2 act on $V, X, \triangle^3$ (and, of course, on $T \subset \operatorname{aff} \triangle^3$). The following describes the actions of the generator of Z_4 on V, X, and $\triangle^3$, denoted by ω, ω^*, and ω_0, respectively, and the actions of the generator of Z_2 on these spaces, denoted by ν, ν^*, and ν_0. For $(x,y) \in V$ with $y = y^1$, we have

$$\begin{array}{ll} \omega(x,y^i) = (x,y^{i+1}), & \nu(x,y) = (-x,y), \\ \omega^*(x;\ell_1,\dots,\ell_4) = (x,\ell_2,\ell_3,\ell_4,\ell_1) & \nu^*(x;\ell_1,\ell_2,\ell_3,\ell_4) = (x;\ell_4,\ell_3,\ell_2,\ell_1), \\ \omega_0(t_1,t_2,t_3,t_4) = (t_2,t_3,t_4,t_1), & \nu_0(t_1,t_2,t_3,t_4) = (t_4,t_3,t_2,t_1). \end{array}$$

It is very easy to see that $\psi \circ \omega = \omega^* \circ \psi$, $\varphi \circ \omega^* = \omega_0 \circ \varphi$, $\pi \circ \omega_0 = \omega_0 \circ \pi$, and the same with ν, ν^*, ν_0. This shows that the composite map $f = \pi \circ \varphi \circ \psi: V \to T$ is a Z_4-map and a Z_2-map as well: $\omega_0 \circ f = f \circ \omega$ and $\nu_0 \circ f = f \circ \nu$.

Let $G = Z_2 \times Z_2$ (direct product). This G acts on both V and T with generators corresponding to ω^2 and ν and to ω_0^2 and ν_0. Thus $f: V \to T$ is a G-map. We will show that such a map does not exist if f, (or rather φ) comes form two measures.

For later reference we record the actions of G on V and T. We rename them as g_1, g_2 corresponding to ω^2, ν, and h_1, h_2 to ω_0^2, ν_0. Then $g_3 = g_1 g_2$ and $h_3 = h_1 h_2$ correspond to the the third element of G, and with $y = y^1$

$$\begin{array}{ll} g_1(x,y) = (x,y^3) & h_1(t_1,t_2,t_3,t_4) = (t_3,t_4,t_1,t_2) \\ g_2(x,y) = (-x,y) & h_2(t_1,t_2,t_3,t_4) = (t_4,t_3,t_2,t_1). \end{array}$$

Remark. There is no obstruction to the existence of a G-map $f: V \to T$. An example of such a map is constructed in [BM]. In order to prove Theorem 1.2 we will need to use the fact f comes from two measures.

3 An equivariant subdivision of V

We are going to construct an equivariant subdivision of V. It is going to consist of four 3-dimensional domains $D, g_1 D, g_2 D, g_3 D$ and their boundaries. The boundary of D is made up of four 2-dimensional cells $P, g_2 P$ and $Q, g_1 Q$.

The construction starts with choosing a great circle that halves the measure λ. We fix the coordinate system so that this great circle coincides with $\{x \in S^2: x_1 = 0\}$. Thus $\lambda(H) = 1/2$ where H is the hemisphere $\{x \in S^2: x_1 \le 0\}$. We may assume that $\mu(H) = \mu_0 < 1/2$, as otherwise we change the orientation of the first axis or, if $\mu(H) = 1/2$, we change the measure μ a little and use a limiting argument in the end.

For $x \in S^2 \setminus \{\pm e_3\}$ let $y^1(x)$ be the (unique) unit vector orthogonal to x with the maximum x_3-coordinate among the unit $y \perp x$. Further, for $i = 2,3,4$ $y^i(x)$ is defined by $\omega^{i-1}(x, y^1(x)) = (x, y^i(x))$. It is easy to check that $y^1(x) = y^1(-x)$ and $y^3(x) = y^3(-x)$, $y^2(x) = y^4(-x)$ and $y^4(x) = y^2(-x)$. Also, for $x \in S^2$ with $x_1 = 0$, we have $y^3(x) = -y^1(x)$.

We need one more piece of notation before we can construct the equvariant subdivision of V. Assume $v_1, v_2, v_3 \in S^2$ are distinct and orthogonal to $x \in S^2$. Then $\langle v_1, v_2, v_3 \rangle$ denotes the closed arc of the great circle orthogonal to x lying between v_1 and v_3 and containing v_2. Define

$$D = \{(x,y) \in V: x_1 \ge 0 \text{ and } y \in \langle y^4(x), y^1(x), y^2(x) \rangle\}.$$

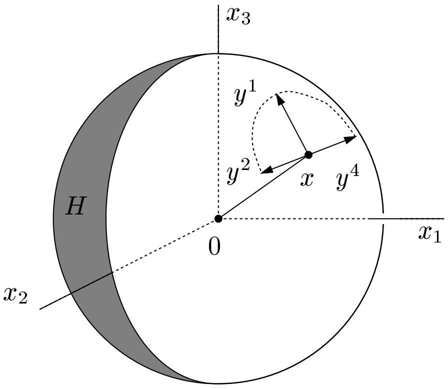

Figure 1: The domain D.

This definition is illustrated in Fig. 1 (which is made for λ being the uniform measure on S^2). Further, clearly

$$\begin{aligned}
g_1D = &\ \{(x,y) \in V\colon x_1 \geq 0 \text{ and } y \in \langle y^2(x), y^3(x), y^4(x)\rangle\}\\
g_2D = &\ \{(x,y) \in V\colon x_1 \leq 0 \text{ and } y \in \langle y^4(-x), y^1(-x), y^2(-x)\rangle\}\\
g_3D = &\ \{(x,y) \in V\colon x_1 \leq 0 \text{ and } y \in \langle y^2(-x), y^3(-x), y^1(-x)\rangle\}
\end{aligned}$$

Thus D, g_1D, g_2D, g_3D have pairwise disjoint interiors and their union covers V.

Remark. Note that when $\lambda = \lambda_0$, the uniform probability measure on S^2, then $D = \{(x,y) \in V\colon x_1 \geq 0,\ y_3 \geq 0\}$. In this case one can show directly that D is homeomorphic to B^3, the unit ball of $\mathbf{R}^3$ with e_1 and $-e_1$ identified and e_2 and $-e_2$ identified. This also holds for all nice measures λ on S^2 as well because the homotopy $\lambda_t = (1-t)\lambda_0 + t\lambda$ induces a homotopy between $D(\lambda_0)$ and $D(\lambda)$.

It is easy to see that ∂D consists of four pieces; the first two are

$$\begin{aligned}
P = &\ \{(x,y) \in V\colon x_1 = 0, x_2 \leq 0, \text{ and } y \in \langle y^4(x), y^1(x), y^2(x)\rangle\}\\
Q = &\ \{(x,y) \in V\colon x_1 \geq 0 \text{ and } y = y^4(x)\},
\end{aligned}$$

and the other two pieces are g_2P, g_1Q.

Note that if $x \in S^2$ (with $x_1 = 0, x_2 < 0$) tends to e_3, then $y^i(x)$ tends to a limit which we denote by y^i $(i = 1,2,3,4)$. Of course, $y^1 = e_2$ and $y^3 = -e_2$. We also define $y^i(-e_3) = y^i$. Write $v = (e_3, y^4)$ and set $v_i = g_i v$. We define 1-dimensional cells E and F of the subdivision: $E = \{(e_3, y)\colon y \in \langle y^4, y^1, y^2\rangle\}$ and $F = \{(x, y^4(x))\colon x_1 = 0, x_2 \leq 0\}$ with E oriented from v to v_1 and F from v_2 to v; see Fig. 2.

One can now check that the boundaries of P and Q can be written as $\partial P = E + g_1F^* + g_3E + F$, and $\partial Q = F^* + g_2E + g_3F^* + E^*$, where E^* and F^* denote E and F equipped with opposite orientation. Then $\partial g_2 P$ and $\partial g_1 Q$ can be computed, and the condition $\partial\partial D = 0$ implies that $\partial D = P + Q + g_2P^* + g_1Q^*$, where, again, P^* and Q^* are P and Q with opposite orientation.

The map $f\colon V \to T$ is an equivariant G-map, and its restrictions $f|_D$ to D and $f|_{\partial D}$ to ∂D, respectively, are also equivariant in the sense that, for $i = 1,2$, $fg_i(u) = h_i f(u)$ whenever $u, g_i(u) \in D$, or $u, g_i(u) \in \partial D$. More interesting

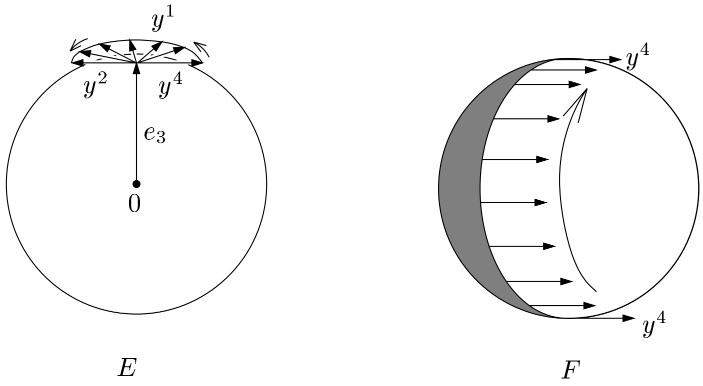

Figure 2: The edges E and F.

(but equally trivial) is the fact that if $f: D \to T$ is an equivariant map in the same sense, then it can be extended, and in a unique way, to an equivariant $f: V \to T$. Simple as it may be, the following observation (cf. Lemma 1 on page 28 in [Mi]) is crucial.

Lemma 3.1 *The restriction $f|_{\partial D}: \partial D \to T$ is of degree zero.*

Proof. Note first that V, which is $SO(3)$, is 3-dimensional and orientable [Br], so $D \subset V$ (which is also 3-dimensional and connected) is also orientable.

By changing the measures a little if necessary we may assume $f: D \to T$ is a differentiable map, and $t \in T$ is a regular value of f; that is, its inverse image consists of finitely many pairwise disjoint 1-dimensional manifolds. Each such manifold is either a closed Jordan curve (which does not contribute to the degree of $f|_{\partial D}$) or a Jordan curve connecting two boundary points. Since D is orientable, each such Jordan curve adds $+1$ at one endpoint, and -1 at the other endpoint, to the degree count. Thus the degree is indeed zero. □

Now for Theorem 1.2 it suffices to show

Theorem 3.2 *If $f: D \to T$ comes form two nice measures, then the degree of $f|_{\partial D}$ is odd.*

We will prove this in the last section.

4 The winding number

Define $b = (1/2, 0, 1/2, 0) \in T$ and $b^* = (0, 1/2, 0, 1/2)$, and let L denote the line in $\mathrm{aff}(\triangle^3)$ passing through b and b^*; see Fig. 3. (This is where the special choice of the radius of T is used: b and b^* are both in $\triangle^3$ and T.) Assume $\alpha: S^1 \to \triangle^3$ (where S^1 is the unit circle) is a map that avoids L. Then the winding number $w(\alpha)$ of α is just the number of times the image $\alpha(s)$ winds around L as s runs through S^1. This number is taken with orientation; so we choose the broken polygonal path through the vertices (in this

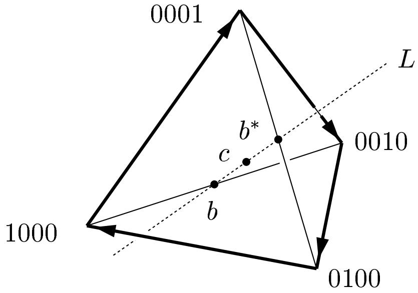

Figure 3: The winding number with respect to L.

order) $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,0,0,0)$ of $\triangle^3$ to have winding number $+1$.

Let $L^\perp$ be the plane, in the affine hull of $\triangle^3$, orthogonal to L and passing through the center $c \in \triangle^3$. Clearly, $w(\alpha)$ is the same as the ordinary winding number (around c) of the mapping $p \circ \alpha\colon S^1 \to L^\perp$, where p denotes orthogonal projection to $L^\perp$.

Let B be a homeomorphic copy of the unit disk in $\mathbf{R}^2$; so ∂B is an S^1. Assume that $\alpha\colon B \to T$ is a map such that the restriction $\alpha|_{\partial B}$ avoids L. Suppose α is differentiable and that $t \in T$ is a regular point (in the sense of Sard's lemma, see [Mi] or [Ll]). Then the signed number of solutions to $\alpha(x) = t$ is denoted by $N(\alpha, t)$, or by $N(\alpha, t, B)$ if we want to specify which B the solutions come from. $N(\alpha, t)$ is just the sum of the signs of the Jacobian of α at the solutions $\alpha(x) = t$. It is well-known and actually easy to check that, under the given conditions,

$$N(\alpha, b, B) - N(\alpha, b^*, B) = w(\alpha|_{\partial B}) \tag{1}$$

provided b and b^* are regular values of α.

We want to use this statement for $\alpha = f|_{P+Q}$. We may assume that $f|_{\partial(P+Q)}$ avoids b and b^*, for otherwise modify the second measure a little (see Remark 4.4 at the end of this section).

Lemma 4.1 *$w(f|_{\partial(P+Q)})$ is odd.*

Proof. Consider $P + Q$ which is a piece of ∂D. Its boundary can be expressed as $\partial(P + Q) = g_1F^* + g_3F^* + g_3E + g_2E$. This is just the sum of two cycles $C_1 + C_2$ where $C_1 = \{(x, y^2(x))\colon x_1 = 0\} = g_1F^* + g_3F^*$ and $C_2 = \{(x, y)\colon x = -e_3\} = g_2E + g_3E$. Clearly, $w(f|_{\partial(P+Q)}) = w(f|_{C_1}) + w(f|_{C_2})$.

Claim 4.2 $w(f|_{C_1}) = 0$.

Proof. To see this note that, for $(x, y) \in C_1$, with $\psi(x, y) = (x; \ell_1, \ldots, \ell_4)$, the union of the first and last sectors is the hemisphere $H = \{x \in S^2\colon x_1 \le 0\}$. So by the choice of the coordinate system (at the start of section 3), $\mu(\sigma_1) + \mu(\sigma_4) = \mu(H) = \mu_0 < \frac{1}{2}$. Thus all values of φ on C_1, and consequently all values of f on C_1 are separated by a plane from L. (This proof also shows that $f|_{C_1}$ avoids the line L.)

Claim 4.3 *$w(f|_{C_2})$ is odd.*

Proof. The cycle C_2 is an S^1, the tangent unit circle to S^2 at $-e_3$. Thus the composite map $p \circ f: C_2 \to L^\perp$ is just an $S^1 \to S^1$ map. This is a Z_2 map since $h_1 \circ f = f \circ g_1$ and, as it is very easy to check, the projection $p: T \to L^\perp$ also commutes with h_1, the corresponding Z_2-action. Now one form of Borsuk's theorem (1-dimensional version) says that every Z_2-map $S^1 \to S^1$ has odd degree. This implies the Claim.

The two claims prove Lemma 4.1. $\square$

Remark. Alternatively, one can show that $w(f|_{C_2}) \equiv 1 \bmod 4$, which is even more than needed for Lemma 4.1. To prove this one observes first that $p \circ f: S^1 \to S^1$ is a Z_4-map whose action is ω and then use a special case of a theorem from [KZ] (see also [BSS] for a simple proof). The special case says that the degree of an $S^1 \to S^1$ Z_q-map is 1 mod q.

Remark 4.4 Here is a sketch of the proof that $f|_{\partial(P+Q)}$ avoids b and b^*, provided that μ is chosen suitably. As $\partial(P+Q) = C_1 + C_2$ and $f|_{C_1}$ avoids L, it is enough to work with $f|_{C_2}$. So what we have to achieve by modifying μ is that $\varphi \circ \psi(-e_3, y) = (t_1, \ldots, t_4) \notin L$ for any $(-e_3, y) \in V$. Points on L are characterized by

$$t_1 = t_3, \ t_2 = t_4 \tag{2}$$

We assume that λ is the normalized Lebesgue measure; the general case goes much the same way. Let $y(s) = (\cos s, \sin s, 0)$, and $\ell(s)$ be the great half-circle with endpoints $e_3, -e_3$ and midpoint $y(s)$. Write $F(s)$ for the μ-measure of the sector between $\ell(0)$ and $\ell(s)$. $F(s)$ is a strictly increasing function with $F(0) = 0, F(2\pi) = 1$. Write $\varphi \circ \psi(-e_3, y(s)) = t(s) = (t_1(s), \ldots, t_4(s))$. It is readily seen that, for $i = 1, 2, 3, 4$,

$$t_i(s) = F(s + i\tfrac{\pi}{2}) - F(s + (i-1)\tfrac{\pi}{2})$$

with the obvious meaning when the argument of F is larger than 2π. If $t(0) \in L$, then by adding a small mass to μ near $y(\pi - \varepsilon)$ and subtracting the same little mass near $y(\pi + \varepsilon)$ we can reach $t(0) \notin L$ (here ε is small). So we may assume that $t(s) \notin L$ in a small vicinity of $s = 0$.

By symmetry, it suffices to work with $s \in [0, \frac{\pi}{2}]$. We can assume next that there are only finitely many solutions to $t_1(s) = t_3(s)$ by approximating μ with a measure for which $F(s)$ is a polynomial for $s \in [0, 2\pi]$. Now adding a small mass η near $y(\varepsilon)$ and subtracting the same mass near $y(\pi + \varepsilon)$ produces a new (nice) measure μ_η. (Here ε is even smaller than before.) The set of solutions to $t_1(s) = t_3(s)$ (for $s \in [0, \frac{\pi}{2}]$) is the same for μ and μ_η. Let s_0 be a fixed solution to $t_1(s) = t_3(s)$. Then $t_2(s_0) = t_4(s_0)$ can only hold for a single value of η. Thus there are only finitely many η for which (2) can have a solution for the new measure μ_η, and so, for all small enough η, the new measure μ_η approximates μ well, and the corresponding $f|_{\partial(P+Q)}$ avoids b, b^*.

5 Proof of Theorem 3.2

By Hopf's theorem (see [Mi] or [Ll]) the degree of $f|_{\partial D}$ only depends on its homotopy class. We have seen that $\partial D = P + Q + g_2P^* + g_1Q^*$. Fix now our map f on $\partial(P+Q)$ so that it avoids L. Then the degree of $f|_{\partial D}$ is determined by the homotopy classes of $f|_P$ and $f|_Q$. By choosing these homotopy classes suitably we may assume that b, b^* are regular values of the map $f\colon \partial D \to T$.

Set $p = N(f, b, P)$, $q = N(f, b, Q)$ and $p^* = N(f, b^*, P)$, $q^* = N(f, b^*, Q)$. Observe that, for $u \in P \cup g_2P^*$, $f(u) = b$ if and only if $f(g_2(u)) = b^*$. Similarly, for $u \in Q \cup g_1Q^*$, $f(u) = b$ if and only if $f(g_1(u)) = b$, and $f(u) = b^*$ if and only if $f(g_1(u)) = b^*$: checking the first statement, say, goes via $f(g_2(u)) = h_2(f(u)) = h_2(b) = b^*$. One has to see that the sign of the solution u_0 to $f(u) = b$ is the same as the sign of the solution $g_2(u_0)$ to $f(u) = b^*$, and similarly for the other cases. We will do so in the remark at the end of this section.

These observations show that $p = N(f, b^*, g_2P^*)$, $p^* = N(f, b, g_2P^*)$ and $q = N(f, b, g_1Q^*)$ $q^* = N(f, b^*, g_1Q^*)$.

We count the degree of $f|_{\partial D}$ in two ways: at b it is $p + q + p^* + q$ and at b^* it is $p^* + q^* + p + q^*$. This shows that $q = q^*$.

Now formula (1) says that

$$w(f|_{\partial(P+Q)}) = p + q - p^* - q^* = p - p^*,$$

which is odd by Lemma 4.1. Consequently, $\deg f|_{\partial D} = p+q+p^*+q = p+p^*+2q$ is odd, finishing the proof of the theorem. □

Remark. It suffices to work out the details for the solutions u_0 to $f(u) = b$ and $g_2(u_0)$ to $f(u) = b^*$. We start with taking the derivative of $f(g_2(u_0)) = h_2(f(u_0))$, and then determinants:

$$\det f'(g_2(u_0)) \det g_2'(u_0) = \det h_2'(f(u_0) \det f'(u_0).$$

So it is enough to see that the sign of $\det g_2'(u_0)$ and of $\det h_2'(f(u_0))$ are equal to one. Or, to put it differently, g_2 and h_2 keep the orientations of ∂D and T, respectively. The mapping g_2 is defined on P and on g_2P^*, and one can check directly that it preserves the orientation. For h_2, we linearly embed $\triangle^3$ in $\mathbf{R}^3$ by placing its vertices $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ at $(1,1,1), (-1,-1,1), (1,-1,-1), (-1,1,-1)$. Under this embedding h_2 is just reflection in the second axis of $\mathbf{R}^3$, so it is a linear map whose matrix is diagonal with entries $-1, 1, -1$. So h_2 keeps the orientation of T (and of $\mathbf{R}^3$).

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