A case when the union of polytopes is convex

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Abstract

We present a necessary and sufficient condition for the union of a finite number of convex polytopes in $\mathbb{R}^d$ to be convex. This generalises two theorems on convexity of the union of convex polytopes due to Bemporad et al.

1 Introduction

A *convex polytope* or simply *polytope* is the convex hull of a finite set of points in Euclidean space $\mathbb{R}^d$. Bemporad, Fukuda and Torrisi [BFT01] studied various necessary and sufficient conditions for the union of several polytopes in $\mathbb{R}^d$ to be convex. In particular, it was shown that for two polytopes P_1 and P_2 in $\mathbb{R}^d$, their union $P_1 \cup P_2$ is convex if and only if the line segment $[v_1, v_2]$ is contained in $P_1 \cup P_2$ for each vertex v_1 of P_1 and each vertex v_2 of P_2. The main objective of the present paper is to give a natural extension of this theorem to the general case of several polytopes.

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Assume that P_i is a polytope in $\mathbb{R}^d$ whose vertex set is X_i, for $i \in [n]$. Here $[n]$ is a shorthand for the set $\{1, 2, \dots, n\}$. Define $P = \bigcup_1^n P_i$, $X = \cup_1^n X_i$ and $Q = \operatorname{conv} X$. Clearly, $P \subset Q$ always, and P is convex if and only if it coincides with Q. Obviously, if P is convex, then $\operatorname{conv} S \subset P$ for every $S \subset X$. Our main theorem is a converse to this simple statement.

Theorem 1. *Assume $d \geq 1$, $n \geq 2$. Then $P = Q$ if and only if $\operatorname{conv} S \subset P$ for each $S \subset X$ with $|S| \leq d + 1$ and $|S \cap X_i| \leq 1$ for each $i \in [n]$.*

It should be noted that Bemporad et al. gave another weaker version (Theorem 5 in [BFT01]) of the theorem in which the last condition "$|S \cap X_i| \leq 1$ for each $i \in [n]$" is replaced by the weaker condition "$|S| \leq n$". It appears that our stronger theorem is substantially harder to prove.

Given sets $A_1, \dots, A_n$ a *transversal* is a set $\{a_1, \dots, a_n\}$ with $a_i \in A_i$ for all i. We are going to use the following theorem from [Bá81] which is sometimes called the Colourful Carathéodory theorem.

Theorem 2. *Given points $a, v \in \mathbb{R}^d$ and sets $A_i \subset \mathbb{R}^d$, $i \in [d]$, with $a \in \cap_1^d \operatorname{conv} A_i$, there exists a transversal $\{a_1, \dots, a_d\}$ of the A_i such that*

$$a \in \operatorname{conv}\{a_1, \dots, a_d, v\}.$$

2 Preparations

Carathéodory's theorem (see [DGK63]) says that the convex hull of $S \subset \mathbb{R}^d$ is the union of simplices $\operatorname{conv} T$ with $T \subset S$ and $|T| \leq d+1$. We will call such a simplex *colourful* if its vertices constitute a transversal of a subsystem of the X_i ($i \in [n]$). In this terminology what we want to prove is the following: P is convex if it contains every colourful simplex.

The statement is invariant under nondegenerate affine transformations, so we may apply any such transformation even during the proof.

We will need a following simple lemma:

Lemma 3. *Assume $T \subset \mathbb{R}^d$ is a polytope with non-empty interior and $E \subset T$ is an ellipsoid. Assume $b_1, \dots, b_s$ are the common points of E and ∂T and* *the outer unit normal to E at b_i is u_i. If $0 \notin \operatorname{int}\operatorname{conv}\{u_1, \ldots, u_s\}$, then there is another ellipsoid $E' \subset T$, arbitrarily close to E, with $\operatorname{Vol} E' > \operatorname{Vol} E$.*

Proof. The statement is invariant under nondegenerate affine transformations so we may assume that E is just B_r, the ball of radius r centered at the origin. If $B_r \cap \partial T = \emptyset$, then any B_ρ with ρ slightly larger than r will do for E'.

Now assume that the set $B_r \cap \partial T = \{b_1, \ldots, b_s\}$ is nonvoid. It is clear that $u_i = b_i/r$, and T has a facet with outer normal u_i at distance r from the origin (for all i), and all other facets are farther away.

Suppose $0 \notin \operatorname{int}\operatorname{conv}\{u_1, \ldots, u_s\}$ and set $C = \operatorname{conv}\{b_1, \ldots, b_s\}$. Then $0 \notin \operatorname{int} C$ as well, and there is a unit vector u such that the hyperplane $u \cdot x = 0$ separates C and 0, that is, $u \cdot x \le 0$ for every $x \in C$.

If the separation is strict, that is, $u \cdot x < 0$ for all $x \in C$, then, for sufficiently small $\varepsilon > 0$, the point εu is farther than r from each facet of T. In this case the ball $\varepsilon u + B_r$ is disjoint from the boundary of T if $\varepsilon > 0$ is small enough. Then $\varepsilon u + B_\rho$ will do for E' for all ρ slightly larger then r.

If the separation is not strict, then 0 is in the relative interior of the convex hull of a subset of $\{b_1, \ldots, b_s\}$. Say $0 \in \operatorname{relint}\{b_1, \ldots, b_j\}$. Then, for all small enough $\varepsilon > 0$, the ball $\varepsilon u + B_r$ touches all facets of T that contain b_i, $i \in [j]$ and is disjoint from all other facets. The set $\operatorname{conv}\{(\varepsilon u + B_r) \cup B_r\}$ is contained in T, and contains an ellipsoid E', arbitrarily close to B_r and of larger volume than B_r. $\square$

The following fact can be proved easily by induction on n.

Lemma 4. *If Q and P_i, $i \in [n]$ are polytopes in $\mathbb{R}^d$, then the closure of $Q \setminus \bigcup_1^n P_i$ can be written as a finite union of simplices.*

3 Proof of the main theorem

The statement is true for $d = 1$ and any $n \ge 2$. We use induction on d so assume $d \ge 2$ and the statement is true in dimension $d - 1$.

Assume the contrary: suppose polytopes $P_1, \ldots, P_n$ in $\mathbb{R}^d$ form a counterexample to the theorem with minimal n. The induction hypothesis implies that Q is full dimensional. Further, let F be a facet of Q. Then, in the hyperplane containing F, the induction hypothesis can be used to show that $F \subset P$. This implies that $\partial Q \subset P$.

The set $G = Q \setminus P$ is open. By Lemma 4, its closure $\operatorname{cl} G$ can be written as a finite union of full dimensional simplices F_s. Let V denote the set of vertices of the F_s. Choose a unit vector $u \in \mathbb{R}^d$ so that

$$\min\{u \cdot x \, : x \in \operatorname{cl} G\} = \min\{u \cdot x \, : x \in V\}$$

is reached on a unique vertex $a \in V$. Assume that a coincides with the origin (otherwise apply a suitable affine transformation). Write $H(t) = \{x \in \mathbb{R}^d : u \cdot x = t\}$. Clearly, $H(t) \cap Q = H(t) \cap P$ for $t \leq 0$. Let $t_1 = \min\{u \cdot x \, : x \in V, \, x \neq 0\}$. Then $t_1 > 0$ and for $t \in (0, t_1)$

$$H(t) \cap \operatorname{cl} G = H(t) \cap \cup F_s$$

where the union is taken over all simplices F_s with $0 \in F_s$. No polytope P_i contains the origin in its interior. But $0 \in P_i$ for some $i \in [n]$ since otherwise $0 \notin P$.

We clean the picture further. Fix $t \in (0, t_1)$ very small (to be specified soon) and set, again for $0 \in F_s$ in the union,

$$Z = H(t) \cap \operatorname{cl} G = H(t) \cap \cup F_s.$$

Z is the union of $(d-1)$-dimensional simplices in $H(t)$ which is a copy of R^{d-1}. Choose an ellipsoid $E \subset Z$ of maximum $(d-1)$-dimensional volume. Such an ellipsoid clearly exists and has finitely many points $z \in P$ on its boundary. The segment $[0, z] \subset P$ since otherwise the interior of some simplex F_s with $0 \in F_s$ contains a point from the segment, but then the whole segment is contained in $\operatorname{int} F_s$. Then $[0, \gamma z] \subset P_i$ for some $i \in [n]$ with a suitable $\gamma > 0$. So if t is chosen small enough, then $[0, z] \subset P_i$. Here z is determined by P_i uniquely, we set $b_i = z$ for concreteness.

Assume, for simpler writing, that the set of indices i with b_i on the boundary of E is just $[k]$. Then $1 \leq k \leq n$. Write h_i for the halfspace (in $H(t)$) which contains E and whose boundary hyperplane contains b_i. Then $T = \bigcap_{i \in [k]} h_i$ is a polytope, $E \subset T$ and E is at a positive distance from all P_i, $i \notin [k]$.

Let u_i be the outer unit normal to E at b_i ($i \in [k]$). We claim that

$$0 \in \operatorname{int}\operatorname{conv}\{u_1, \dots, u_k\}.$$

Indeed, if this were not case, then Lemma 3 implies the existence of another ellipsoid $E' \subset T$ arbitrarily close to E with $\operatorname{Vol} E' > \operatorname{Vol} E$. Such an ellipsoid is contained in Z and has larger volume than E, contradicting the choice of E.

The claim shows that $d \le k$ (otherwise $\operatorname{int}\operatorname{conv}\{u_1, \dots, u_k\}$ is empty). Thus $d \le k \le n$. Note that we are finished with the case $n < d$.

Now we apply a nondegenerate linear transformation (to all polytopes P_i, P and Q) that keeps the hyperplane $H(0)$ fixed and moves the ellipsoid E to a ball B in $H(t)$ whose center, b, is orthogonal to $H(0)$. We keep the same notation, so the images of P, Q, P_i, and the points b_i will go under the same name. This should cause no confusion as we won't return to their preimages.

We write $C = \operatorname{pos} B$, this is a closed circular cone whose axis is the halfline $L(b) = \{\lambda b : \lambda \ge 0\}$. The cone C is separated from each P_i ($i \in [k]$) and the (unique) separating hyperplane is tangent to C along the halfline $L(b_i)$. Moreover, $C \cap P_i$ is contained in $L(b_i)$. It is also clear that $b \in \operatorname{int} C$. We define $C^* = C \cap \{x : u \cdot x \le t\}$ and note that $\operatorname{int} C^* \cap P = \emptyset$.

Claim 5. *For distinct $i, j \in [k]$, the rays $L(b_i)$ and $L(b_j)$ are distinct.*

Proof. Assume, on the contrary that $L(b_1) = L(b_2)$, say. Set $P_0 = \operatorname{conv}(X_1 \cup X_2)$. Since P_1 and P_2 are separated by the same hyperplane from C, P_0 is separated from C by that hyperplane. It is not hard to check now that $P_0, P_3, \dots, P_n$ is another counterexample to the theorem with $n-1$ polytopes, contrary to the minimality of n. □

We claim now that $k < n$. Indeed, the halfline $L(b)$ intersects ∂Q at the point b^*, say. As C and P_i are separated for all $i \in [k]$, $b^* \notin P_i$. Now

$$b^* \in \partial Q \subset P = \cup_1^n P_i$$

so b^* is contained in one of the polytopes P_i, $i > k$. Assume, for concreteness, that $b^* \in P_n$. (Note that we are finished with the case $n \le d$ now.)

We can now apply Theorem 2: $0 \in \operatorname{conv} X_i$ for each $i \in [k]$. The last vector $v \in \mathbb{R}^d$ can be anything; it will often but not always come from X_n. As k may be larger than d we will consider *partial transversals* of the system X_i. A partial transversal, or I-transversal, of this system is $\{x_i \in X_i : i \in I\}$, here I can be any subset of $[k]$. Theorem 2 says now the following:

Lemma 6. *For every $I \subset [k]$ with $|I| = d$ and every $v \in \mathbb{R}^d$ there exists an I-transversal $\{x_i \in X_i : i \in I\}$ such that*

$$0 \in \operatorname{conv}(\{x_i : i \in I\} \cup \{v\}).$$

We have to distinguish some cases.

Case 1. There is a colourful simplex whose interior contains the origin.

By the conditions of Theorem 1 this colourful simplex is contained in P, so a full neighbourhood of 0 lies in P, contradicting $\operatorname{int} C^* \cap P = \emptyset$.

Thus 0 does not lie in the interior of any colourful simplex. Then by Lemma 6, for every $x_n \in X_n$ and for every $I \subset [k]$, $|I| = d$, 0 is on the boundary of the convex hull of an I-transversal and x_n. Thus 0 is contained in the *relative* interior of the convex hull of a partial transversal plus possibly x_n.

Case 2. When 0 is not contained in the convex hull of any I-transversal, with $|I| \le d$, of the X_i, $i \in [k]$. Then $k = d$ since otherwise Lemma 6 can be applied to $X_1, \dots, X_d$ with v equal to an arbitrary $x_{d+1} \in X_{d+1}$ and the convex hull of the transversal and x_{d+1} either contains 0 in its interior (which is impossible since Case 1 is excluded by now), or 0 is contained in the convex hull of some partial transversal from $[k]$ which is impossible in Case 2.

So $k = d$. Choose a point $x_n \in X_n$. There are finitely many transversals $\{x_1, \dots, x_d\}$ such that the set $F = \operatorname{conv}\{x_1, \dots, x_d, x_n\}$ contains the origin. Write $\mathcal{F}$ for the collection of such sets.

Claim 7. *The union of these sets contains a small neighbourhood of the origin.*

Proof. Write W for the set of unit vectors w such that, for all small $\varepsilon \in (0, \varepsilon_0]$, εw is not contained in the affine hull of any d points from X. (All unit vectors, except those in a finite union of hyperplanes, satisfy this requirement if ε_0 is chosen small enough.) Now apply Lemma 6 to the system X_i, $i \in [d]$ and the point $v = x_n - \varepsilon w$ where $w \in W$. The result is a transversal $x_i \in X_i$ ($i \in [d]$) such that

$$0 \in \operatorname{conv}(\{x_i : i \in [d]\} \cup \{x_n - \varepsilon w\}).$$

In other words, there are $\gamma_i \geq 0$ (for $i \in [d] \cup \{n\}$) that sum to one with

$$0 = \gamma_1 x_1 + \dots \gamma_d x_d + \gamma_n(x_n - \varepsilon w).$$

We claim that all $\gamma_i > 0$ here. First, $\gamma_n = 0$ would show that 0 is in the convex hull of $x_1, \dots, x_d$, contrary to the assumptions of Case 2. So $\gamma_n > 0$ (for every small positive ε), and

$$\gamma_n \varepsilon w = \gamma_1 x_1 + \dots \gamma_d x_d + \gamma_n x_n.$$

If some $\gamma_i = 0$ here, then $\gamma_n \varepsilon w$ is in the affine hull of d or fewer vectors from X, contrary to the choice of w. Thus all $\gamma_i > 0$.

The last equation shows that the simplex $F = \operatorname{conv}\{x_1, \dots, x_d, x_n\}$ contains the segment $[0, \delta(w)w]$ for some small $\delta(w) > 0$. Let $r(F)$ be the distance from the origin to the union of the facets of F not containing the origin. Thus if F contains $[0, \delta(w)w]$, then it contains $[0, r(F)w]$ as well.

Set $r = \min\{r(F) : F \in \mathcal{F}\}$. Then for each $w \in W$ the segment $[0, rw]$ is contained in some $F \in \mathcal{F}$. This holds then for all unit vectors w as the union of $F \in \mathcal{F}$ is a closed set. □

The Claim shows that the union of the colourful simplices contains a small neighbourhood of the origin. This contradicts, again, the assumption that $\operatorname{int} C^*$ is disjoint from P.

Case 3. When 0 is in the relative interior of the convex hull of some I-transversal with $I \subset [k]$, $|I| \leq d$.

This is very simple if $|I| = d$. Assume the I-transversal is $\{x_1, \dots, x_d\}$. Then the affine hull of $x_1, \dots, x_d$ is a hyperplane, and b and b^* are on the same side of it. As $b^* \in \operatorname{conv} X_n$, there is an $x_n \in X_n$ on the same side, and then the colourful simplex $\operatorname{conv}\{x_1, \dots, x_d, x_n\}$ contains εb for some small $\varepsilon > 0$.

Hence the segment $[0, \varepsilon b]$ is in P and $\operatorname{int} C^* \cap P$ is nonempty, a contradiction again.

Assume $|I| < d$ for all I in Case 3 and consider the family, $\mathcal{T}$, of transversals of the form

$$T = \{x_{i_1}, \ldots, x_{i_d}, x_n\} \text{ with } 0 \in \operatorname{conv} T$$

where $1 \leq i_1 < \cdots < i_d \leq k$. Our target is to show that for some $T \in \mathcal{T}$, $\operatorname{conv} T$ intersects the interior of C.

This would finish the proof as follows: Let x be a common point of $\operatorname{conv} T$ and $\operatorname{int} C$. Both C and $\operatorname{conv} T$ contain the origin, so the segment $(0, x]$ is contained in $\operatorname{int} C \cap \operatorname{conv} T$. But $\operatorname{conv} T$ is a colourful simplex, so it is contained in P, yet $\operatorname{int} C^*$ should be disjoint from P.

For $T \in \mathcal{T}$ let $v(T)$ be the point in $\operatorname{conv} T$ nearest to b, so $v(T)$ is on the boundary of $\operatorname{conv} T$. If b is short enough, then the whole segment $[0, v(T)]$ lies on the boundary of $\operatorname{conv} T$. This can be reached for all $T \in \mathcal{T}$, if we fix $t \in (0, t_1)$ small enough. Our target is to show that $v(T) \in \operatorname{int} C$ for some $T \in \mathcal{T}$.

Set $w(T) = b - v(T)$, then $v(T)$ and $w(T)$ are orthogonal, and $v(T)$ is the orthogonal projection of b onto the $(d-1)$-dimensional subspace whose normal is $w(T)$. Moreover, if $\operatorname{conv} T \cap \operatorname{int} C = \emptyset$, then the vector $w(T)$ separates C and $\operatorname{conv} T$, that is, $x \cdot w(T) \leq 0$ for all $x \in \operatorname{conv} T$ and $x \cdot w(T) \geq 0$ for all $x \in C$. Further, if $x \cdot w(T) = 0$ for some $x \in C$, then x lies on a unique extreme ray of the circular cone C.

Note further, that $[0, v(T)]$ lies on the boundary of $\operatorname{conv} T$. By Carathéodory's theorem, there is an $x_i \in T$ such that $[0, v(T)] \subset \operatorname{conv}(T \setminus \{x_i\})$. Write T/y_i for the transversal $T = T \cup \{y_i\} \setminus \{x_i\}$ where $y_i \in X_i$. Clearly, $T/y_i \in \mathcal{T}$ and $w(T/y_i)$ is not longer than $w(T)$ since $v(T) \in \operatorname{conv}(T/y_i)$.

Choose now $S \in \mathcal{T}$ so that

$$\|w(S)\| = \min\{\|w(T)\| \ : T \in \mathcal{T}\}.$$

Observe that $v(S) \neq 0$ or, equivalently, $w(S) \neq b$. Indeed, there is a partial transversal, $\{x_1, \ldots, x_d\}$ say, containing 0 in its convex hull. Further, there is $x_n \in X_n$ with $u \cdot x_n > 0$ as $u \cdot b^* > 0$ and $b^* \in \operatorname{conv} X_n$ where u is the unit vector from the very beginning of this proof. So $T = \{x_1, \ldots, x_d, x_n\} \in \mathcal{T}$ and $v(T)$ is shorter than b as the distance between b and $[0, x_n]$ is shorter than b.

Claim 8. $v(S) \in \operatorname{int} C$.

Proof. Assume the contrary, then $\operatorname{conv} S$ and C are separated by the hyperplane $H = \{x : x \cdot w(S) = 0\}$.

Assume first that $v(S) \notin C$. Let $x_i \in S$ be a vector with $[0, v(S)] \subset \operatorname{conv}(S \setminus \{x_i\})$. As $C \cap \operatorname{conv} S = \{0\}$ and $b_i \in C$, $b_i \cdot w(S) > 0$. Further, $b_i \in \operatorname{conv} X_i$, so there is a $y_i \in X_i$ with $y_i \cdot w(S) > 0$. But then $\|w(S/y_i)\| < \|w(S)\|$ showing that $v(S) \in C$, or rather, $v(S) \in \partial C$. (For $i = n$ one should take b instead of b_i.)

Now with $v(S) \in \partial C$ the assumption $b_i \cdot w(S) > 0$ leads to the same contradiction. Thus $b_i \cdot w(S) = 0$ so $w(S)$ is the (unique) normal to the cone C at the point b_i. This implies that $\beta b_i = v(S)$ for some positive β. Note that this i is unique, otherwise $\gamma b_j = v(S)$ (for some $\gamma > 0$) and then rays $L(b_i)$ and $L(b_j)$ coincide which is impossible by Claim 5. Assume for concreteness that $i = 1$.

Set now $V = S \setminus \{x_1\}$, then $V \subset H$ and $|V| = d$ and the segment $[0, v(S)] \subset \operatorname{conv} V$. Next, $[0, b_1] \subset H \cap P_1$ since $0, b_1 \in P_1$. The set P_1 is separated from C by H. The set $Y = X_1 \cap H \neq \emptyset$ since $b_1 \in \operatorname{conv} Y$, let $y_1 \in Y$ arbitrary. Note that both V and Y are contained in H, which is a copy of $\mathbb{R}^{d-1}$.

For every vector $a \in (0, v(S)] \cap (0, b_1]$,

$$a \in \operatorname{conv} V \subset \operatorname{conv}(V \cup \{y_1\}).$$

The last set lies in H and contains $d + 1$ elements. A well-known version of Carathéodory's theorem (cf. [DGK63]) implies the existence of $v \in V$ such that $a \in \operatorname{conv}(V \cup \{y_1\} \setminus \{v\})$. The vector v may depend on a but it is the same for infinitely many a while a tends to 0, say $v = x_d \in X_d$. Then

$$[0, a] \subset \operatorname{conv}(S \cup \{y_1\} \setminus \{x_1, x_d\}).$$

for some a on the segment $(0, v(S)]$. This new partial transversal contains $v(S)$ in its convex hull, and contains no point of X_d. As $b_d \cdot w(S) > 0$,

$y_d \cdot w(S) > 0$ for a suitable $y_d \in X_d$. This gives a new transversal $S^* = S \cup \{y_1, y_d\} \setminus \{x_1, x_d\}$ with $w(S^*)$ shorter than $w(S)$. This final contradiction shows that $v(S) \in \operatorname{int} C$. $\square$

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