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Hard Ball Systems Are Fully Hyperbolic

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Abstract. We consider the system of N (≥ 2) elastically colliding hard balls with masses $m_1, \dots, m_N$, radius r, moving uniformly in the flat torus $\mathbb{T}_L^\nu = \mathbb{R}^\nu / L \cdot \mathbb{Z}^\nu$, $\nu \geq 2$. It is proved here that the relevant Lyapunov exponents of the flow do not vanish for almost every $(N+1)$-tuple $(m_1, \dots, m_N; L)$ of the outer geometric parameters.

1. Introduction.

The proper mathematical formulation of Ludwig Boltzmann's ergodic hypothesis, which has incited so much interest and discussion in the last hundred years, is still not clear. For systems of elastic hard balls on a torus, however, Yakov Sinai, in 1963, [Sin(1963)] gave a stronger, and at the same time mathematically rigorous version of Boltzmann's hypothesis: *the system of an arbitrarily fixed number N of identical elastic hard balls moving in the ν-torus $\mathbb{T}^\nu = \mathbb{R}^\nu/\mathbb{Z}^\nu$ ($\nu \geq 2$) is ergodic — of course, on the submanifold of the phase space specified by the trivial conservation laws.* Boltzmann used his ergodic hypothesis when laying down the foundations of statistical physics, and its various forms are still intensively used in modern statistical physics. The importance of Sinai's hypothesis for the theory of dynamical systems is stressed by the fact that the interaction of elastic hard balls defines the only physical system of an arbitrary number of particles in arbitrary dimension whose dynamical behaviour has been so far at least guessed — except for the completely integrable system of harmonic oscillators. (As to the history of Boltzmann's hypothesis, see the recent work [Sz(1996)].)

Sinai's hypothesis was partially based on the physical arguments of Krylov's 1942 thesis (cf. [K(1979)] and its afterword written by Ya. G. Sinai, [Sin(1979)]),

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where Krylov discovered that hard ball collisions provide effects analogous to the hyperbolic behaviour of geodesic flows on compact manifolds of constant negative curvature, exploited so beautifully in the then recent works of Hedlund, [He(1939)] and Hopf, [Ho(1939)].

The aim of the present paper is to establish that **hard ball systems are, indeed, fully hyperbolic**, i. e. all relevant Lyapunov exponents of these systems are non-zero almost everywhere. Our claim holds for typical $(N+1)$-tuples $(m_1, \dots, m_N; L) \in \mathbb{R}_+^{N+1}$ of the outer geometric data of the system, that is — in contrast to earlier results — we do not require that particles have identical masses, though, on the other hand, we have to exclude a countable union of proper submanifolds of $(N+1)$-tuples $(m_1, \dots, m_N; L)$. (Which set is, in fact, very likely to be empty.)

Full hyperbolicity combined with Katok-Strelcyn theory (see [K-S(1986)]) immediately provides that **the ergodic components of these systems are of positive measure**. Consequently, there are at most countably many of them, and, moreover, on each of them the system is K-mixing. Our methods so far do not give the expected global ergodicity of the systems considered (an additional hypothesis to provide that is formulated in Section 6).

The equality of the *radii* of the balls is not essential, but, for simplicity, it will be assumed throughout. For certain values of the radii — just think of the case when they are large — the phase space of our system decomposes into a finite union of different connected components, and these connected parts certainly belong to different ergodic components. Now, according to the wisdom of the ergodic hypothesis, these connected components are expected to be just *the* ergodic components of the system, and on each of them the system should also possess the Kolmogorov-mixing property.

Let us first specify the model and formulate our result.

Assume that, in general, a system of $N(\geq 2)$ balls, identified as $1, 2, \dots, N$, of masses $m_1, \dots, m_N$ and of radius $r > 0$ are given in $\mathbb{T}_L^\nu = \mathbb{R}^\nu / L \cdot \mathbb{Z}^\nu$, the ν-dimensional cubic torus with sides L, $(\nu \geq 2)$. Denote the phase point of the ith ball by $(q_i, v_i) \in \mathbb{T}_L^\nu \times \mathbb{R}^\nu$. A priori, the configuration space $\tilde{\mathbf{Q}}$ of the N balls is a subset of $\mathbb{T}_L^{N\cdot\nu}$: from $\mathbb{T}_L^{N\cdot\nu}$ we cut out $\binom{N}{2}$ cylindric scatterers:

$$\tilde{C}_{i,j} = \left\{ Q = (q_1, \dots, q_N) \in \mathbb{T}_L^{N\cdot\nu} : \mid q_i - q_j \mid < 2r \right\}, \tag{1.1}$$

$1 \leq i < j \leq N$, or in other words $\widetilde{\mathbf{Q}} := \mathbb{T}_L^{N\nu} \setminus \bigcup_{1 \leq i < j \leq N} \tilde{C}_{i,j}$. The energy $H = \frac{1}{2}\sum_1^N m_i v_i^2$ and the total momentum $P = \sum_1^N m_i v_i$ are first integrals of the motion. Thus, without loss of generality, we can assume that $H = \frac{1}{2}$, $P = 0$. (If $P \neq 0$, then the system has an additional conditionally periodic or periodic motion.) Now, for these values of H and P, we define our dynamical system.

Remark. It is clear that not actual values of r and L, but merely their ratio L/r is elevant for our model. Therefore, throughout this paper we fix the value of $r > 0$, and will only consider (later in sections 3–4) the size L of the torus as a variable.

In earlier works (cf. [S-Sz(1995)-II] and the references therein), where the masses were identical, one could and — to obtain ergodicity — had to fix a center of mass.

For different masses — as observed in [S-W(1989)] — this is generically not possible and we should follow a different approach.

The equivalence relation Ψ over $\widetilde{\mathbf{Q}}$, defined by $Q \sim_\Psi Q^*$ *if and only if there exists an* $a \in \mathbb{T}_L^\nu$ *such that for every* $i \in [1, N]$ $\quad q_i^* = q_i + a$, allows us to introduce $\mathbf{Q} := \widetilde{\mathbf{Q}}/\Psi$ (the equivalence relation means, in other words, that for the internal coordinates $q_i - q_j = q_i^* - q_j^*$ holds for every $i, j \in [1, N]$). The set $\mathbf{Q}$, a compact, flat Riemannian manifold with boundary will actually be the configuration space of our system, whereas its phase space will be the ellipsoid bundle $\mathbf{M} := \mathbf{Q} \times \mathcal{E}$, where $\mathcal{E}$ denotes the ellipsoid $\sum_{i=1}^N m_i (dq_i)^2 = 1, \quad \sum_{i=1}^N m_i dq_i = 0$. Clearly, $d := \dim \mathbf{Q} = N\nu - \nu$, and $\dim \mathcal{E} = d - 1$. The well-known Liouville measure μ is invariant with respect to the evolution $S^{\mathbb{R}} := \{S^t : t \in \mathbb{R}\}$ of our dynamical system defined by elastic collisions of the balls of masses $m_1, \dots, m_N$ and their uniform free motion. Here we have two remarks:

(i) the collision laws for a pair of balls with different masses are well-known from mechanics and will also be reproduced in the equation (3.9);
(ii) the dynamics can, indeed, be defined for $\mu-$a. e. phase point, see the corresponding references in Section 2.

The dynamical system $(\mathbf{M}, S^{\mathbb{R}}, \mu)_{\vec{m},L}$ is called *the standard billiard ball system with the outer geometric parameters* $(\vec{m}; L) \in \mathbb{R}_+^{N+1}$.

Denote by $R_0 := R_0(N, \nu, L)$ the interval of those values of $r > 0$, for which the interior $\mathrm{Int}\mathbf{M} = \mathrm{Int}\mathbf{Q} \times \mathcal{E}$ of the phase space of the standard billiard ball flow is connected.

The basic result of our paper is the following

Main Theorem. *For $N \geq 2, \nu \geq 2$ and $r \in R_0$, none of the relevant Lyapunov exponents of the standard billiard ball system $(\mathbf{M}, \{S^{\mathbb{R}}\}, \mu)_{\vec{m},L}$ vanishes — apart from a countable union of proper analytic submanifolds of the outer geometric parameters $(\vec{m}; L) \in \mathbb{R}_+^{N+1}$.*

An interesting consequence of this theorem is the following

Corollary. *For the good set of typical outer geometric parameters $(\vec{m}; L)$ the ergodic components of the system have positive measure and on each of them the standard billiard ball flow has the K-property.*

Note 1. In general, if $r \notin R_0$, then $\mathrm{Int}\mathbf{M}$ decomposes into a finite number of connected components. Our results extend to these cases, too.

Note 2. As it will be seen in Section 2 (see Lemma 2.1) our system is isomorphic to a *semi-dispersing billiard*, and, what is more, is a semi-dispersing billiard in a weakly generalized sense, and, consequently, we can use the theory of semi-dispersing billiards. As it has been proved in recent manuscripts by N. I. Chernov and C. Haskell, [C-H(1996)] on one hand, and by D. Ornstein and B. Weiss, [O-W(1996)] on the other hand, the K-mixing property of a semi-dispersing billiard flow on a positive ergodic component actually implies its Bernoulli property, as well.

Note 3. As to the basic results (and history) concerning the Boltzmann and Sinai hypotheses we refer to the recent survey [Sz(1996)].

The basic notion in the theory of semi-dispersing billiards is that of the *sufficiency* of a phase point or, equivalently, of its orbit. The conceptual importance of sufficiency can be explained as follows (for a technical introduction and our prerequisites, see Section 2): In a suitably small neighbourhood of a (typical) phase point of a dispersing billiard the system is *hyperbolic*, i. e. its relevant Lyapunov exponents are not zero. For a semi-dispersing billiard the same property holds for sufficient points only! Physically speaking, a phase point is sufficient if its trajectory encounters in its history all possible degrees of freedom of the system.

Our proof of the full hyperbolicity is the first major step in the basic strategy for establishing global ergodicity of semi-dispersing billiards as it was suggested in our series of works with A. Krámli (see the references), and, in fact, some elements of that approach are also used in this paper. In the sense of our strategy for proving global ergodicity, initiated in [K-S-Sz(1991)] and explained in the introductions of [K-S-Sz(1992)] and of [Sim(1992)-I] , there are two fundamental parts in the demonstration of hyperbolicity, once a combinatorial property, called *richness* of the symbolic collision sequence of a trajectory, had been suitably defined:

(1) The "geometric–algebraic considerations" on the codimension of the manifolds describing the non–sufficient trajectory segments with a combinatorially rich symbolic collision structure;
(2) Proof of the fact that the set of phase points with a combinatorially non–rich collision sequence has measure zero.

In contrast to earlier proofs of ergodic properties of hard ball systems our method for obtaining full hyperbolicity will not be inductive, more precisely, we do not use the hyperbolicity of smaller systems. This has become possible since our Theorem 5.1 (settling step (2) here) which is a variant of the so-called *weak ball-avoiding theorems*, can now be proven without any inductive assumption. Nevertheless, the proof of our crucial Key Lemma 4.1, which copes with part (1), is inductive and, indeed, one of the main reasons to use *varying masses* is just that by choosing the mass of one particle to be equal to zero permits us to use an inductive assumption on the smaller system (but about its algebraic behaviour, only). As just mentioned, to prove our theorem we have introduced varying masses, and can only claim hyperbolicity for typical $(N+1)$-tuples $(m_1, \dots, m_N; L) \in \mathbb{R}_+^{N+1}$ of outer geometric parameters. Furthermore, we will, in Section 3, *complexify the dynamics*. The complexification is required, on one hand, by the fact that our arguments in Section 4 use some algebraic tools that assume the ground field to be algebraically closed. Another advantage of the complexification is that, in the inductive derivation of Key Lemma 4.1, one does not have to worry about the sheer existence of orbit segments with a prescribed symbolic collision sequence: they do exist, thanks to the algebraic closedness of the complex field.

The paper is organized as follows: Section 2 is devoted to prerequisites. Section 3 then describes the complexified dynamics, while in Section 4 we establish Key Lemma 4.1 — both for the complexified and the real dynamics — settling, in particular, part (1) of the strategy. Section 5 provides the demonstration of our Main Theorem through the afore mentioned Theorem 5.1, and, finally, Section 6 contains some comments and remarks.

2. Prerequisites

Semi-dispersing billiards and hard ball systems

Our approach is based on a simple observation:

Lemma 2.1. *The standard billiard ball flow with mass vector $\vec{m} \in \mathbb{R}_+^N$ is isomorphic to a semi-dispersing billiard.*

Because of the importance of the statement we sketch the proof.

Proof. Introduce new coordinates in the phase space as follows: For $(Q, V) \in \mathbf{M}$ let $\hat{q}_i = \sqrt{m_i} q_i, \quad \hat{v}_i = \sqrt{m_i} v_i$, where thus $\hat{q}_i \in \sqrt{m_i}\mathbb{T}_L^\nu$ and $\sum_{i=1}^N \hat{v}_i^2 = 1$. Moreover, the vector $\hat{V} = (\hat{v}_1, \dots, \hat{v}_N)$ necessarily belongs to the hyperplane $\sum_{i=1}^N \sqrt{m_i}\hat{v}_i = 0$. Denote $\mathbb{T}_{\vec{m},L} = \prod_{i=1}^N (\sqrt{m_i}\ \mathbb{T}_L^\nu)$ and

$$\widehat{C}_{i,j} = \left\{ \hat{Q} = (\hat{q}_1, \dots, \hat{q}_N) \in \mathbb{T}_{\vec{m},L} : \quad \left\| \frac{\hat{q}_i}{\sqrt{m_i}} - \frac{\hat{q}_j}{\sqrt{m_j}} \right\| < 2r \right\} \tag{2.2}$$

and $\tilde{\hat{\mathbf{Q}}} = \mathbb{T}_{\vec{m},L} \setminus \cup_{1 \le i < j \le N} \hat{C}_{i,j}$. The equivalence relation $\hat{\Psi}_{\vec{m}}$ over $\mathbb{T}_{\vec{m},L}$ is as follows: $\hat{Q} \sim_{\hat{\Psi}_{\vec{m}}} \hat{Q}^*$ if and only if there exists an $a \in \mathbb{T}_L^\nu$ such that for every $i \in [1, N]$ $\hat{q}_i^* = \hat{q}_i + \sqrt{m_i} a$ allows us to define $\hat{\mathbf{Q}} = \tilde{\hat{\mathbf{Q}}} / \hat{\Psi}_{\vec{m}}$. Let $\hat{\mathbf{M}}_{\vec{m},L} = \hat{\mathbf{Q}} \times \mathbb{S}^{d-1}$ be the unit tangent bundle of $\hat{\mathbf{Q}}$, and denote by $d\hat{\mu}$ the probability measure $const \cdot d\hat{Q} \cdot d\hat{V}$, where $d\hat{V}$ is the surface measure on the $(d-1)-$sphere $\mathbb{S}^{d-1}$, and $d\hat{Q}$ is the Lebesgue-measure on $\mathbb{T}_{\vec{m},L}$. Now the standard billiard ball system $(\mathbf{M}, S^{\mathbb{R}}, \mu)_{\vec{m},L}$ with mass vector $\vec{m}$ and the billiard system $(\hat{\mathbf{M}}_{\vec{m},L}, \hat{S}^{\mathbb{R}}, \hat{\mu})$ are isomorphic.

Indeed, we can reduce the question to the case of one-dimensional particles because for both models the velocity components perpendicular to the normal of impact remain unchanged. The claimed isomorphy for one-dimensional particles, however, is well-known, and for its simple proof we can refer, for instance, to Section 4 of Chapter 5 in [C-F-S(1981)]. Thus, the point is that in the isomorphic flow $\left(\hat{\mathbf{M}}_{\vec{m},L}, \hat{S}^{\mathbb{R}}, \hat{\mu}\right)$ the velocity transformations at collisions become orthogonal reflections across the tangent hyperplane of the boundary of $\hat{\mathbf{Q}}$, see also (3.9) for the mentioned velocity transformation.

The fact that the billiard system $(\hat{\mathbf{M}}_{\vec{m},L}, \hat{S}^{\mathbb{R}}, \hat{\mu})$ is semi-dispersing, is obvious, because the scattering bodies in $\hat{\mathbf{Q}}$ are cylinders built on ellipsoid bases (therefore this system is a cylindric billiard as introduced in [Sz(1993)]). $\square$

For convenience and brevity, we will throughout use the concepts and notations, related to semi-dispersing billiards and hard ball systems, of the papers [K-S-Sz(1990)] and [Sim(1992)-I-II], respectively, and will only point out where and how different masses play a role.

Remark 2.3 By slightly generalizing the notion of semi-dispersing billiards with allowing Riemannian metrics different from the usual ones (see (2.2) of [K-S-Sz(1990)]), we could have immediately identified the standard billiard ball system

$(\mathbf{M}, S^{\mathbb{R}}, \mu)_{\vec{m},L}$ with mass vector $\vec{m}$ with a (generalized) semi-dispersing billiard. (In the coordinates of the proof of Lemma 2.1, the Riemannian metric is

$$(d\rho)^2 = \sum_{i=1}^{N} \left(\|d\hat{q}_i\|^2 + \|ds\|^2 \right)$$

where $\|ds\|^2$ is the square of the natural Riemannian metric on the unit sphere $\mathbb{S}^{d-1}$.) Then the standard billiard ball system $(\mathbf{M}, S^{\mathbb{R}}, \mu)_{\vec{m},L}$ itself will be a (generalized) semi-dispersing billiard. The advantage is that — as it is easy to see — the results of [Ch-S(1987)] and [K-S-Sz(1990)] remain valid for this class, too. For simplifying our exposition therefore, we will omit the change of coordinates of Lemma 2.1 and will be using the notion of semi-dispersing billiards in this slightly more general sense. In this sense the model of the Introduction is a semi-dispersing billiard.

An often used abbreviation is the shorthand $S^{[a,b]}x$ for the trajectory segment $\{S^t x \,:\, a \le t \le b\}$. The natural projections from $\mathbf{M}$ onto its factor spaces are denoted, as usual, by $\pi : \mathbf{M} \to \mathbf{Q}$ and $p : \mathbf{M} \to \mathcal{S}^{N\cdot\nu-\nu-1}$ or, sometimes, we simply write $\pi(x) = Q(x) = Q$ and $p(x) = V(x) = V$ for $x = (Q, V) \in \mathbf{M}$. Any $t \in [a, b]$ with $S^t x \in \partial M$ is called *a collision moment or collision time.*

As it is pointed out in previous works on billiards, the dynamics can only be defined for trajectories where the moments of collisions do not accumulate in any finite time interval (cf. Condition 2.1 of [K-S-Sz(1990)]). An important consequence of Theorem 5.3 of [V(1979)] is that — for semi-dispersing billiards! — there are *no trajectories at all with a finite accumulation point of collision moments* (see also [G(1981)] and [B-F-K(1995)]).

As a result, for an arbitrary non-singular orbit segment $S^{[a,b]}x$ of the standard billiard ball flow, there is a uniquely defined maximal sequence $a \le t_1 < t_2 < \cdots < t_n \le b :\ n \ge 0$ of collision times and a uniquely defined sequence $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ of "colliding pairs", i. e. $\sigma_k = \{i_k, j_k\}$ whenever $Q(t_k) = \pi(S^{t_k}x) \in \partial\tilde{C}_{i_k,j_k}$. The sequence $\Sigma := \Sigma(S^{[a,b]}x) := (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is called the *symbolic collision sequence* of the trajectory segment $S^{[a,b]}x$.

Definition 2.4. *We say that the symbolic collision sequence $\Sigma = (\sigma_1, \ldots, \sigma_n)$ is connected if the collision graph of this sequence:*

$$\mathcal{G}_\Sigma := (\mathcal{V} = \{1, 2, \ldots, N\}, \mathcal{E}_\Sigma := \{\{i_k, j_k\} :\ \textit{where } \sigma_k = \{i_k, j_k\},\ 1 \le k \le n\})$$

is connected.

Definition 2.5. *We say that the symbolic collision sequence $\Sigma = (\sigma_1, \ldots, \sigma_n)$ is C-rich, with C being a natural number, if it can be decomposed into at least C consecutive, disjoint collision sequences in such a way that each of them is connected.*

Neutral Subspaces, Advance and Sufficiency

Consider a *non–singular* trajectory segment $S^{[a,b]}x$. Suppose that a and b are *not moments of collision.* Before defining the neutral linear space of this trajectory segment, we note that the tangent space of the configuration space $\mathbf{Q}$ at interior points can be identified with the common linear space

$$\mathcal{Z} = \left\{ (w_1, w_2, \dots, w_N) \in (\mathbb{R}^\nu)^N : \sum_{i=1}^N m_i w_i = 0 \right\}. \tag{2.6}$$

Definition 2.7. *The neutral space $\mathcal{N}_0(S^{[a,b]}x)$ of the trajectory segment $S^{[a,b]}x$ at time zero ($a < 0 < b$) is defined by the following formula:*

$$\begin{aligned}\mathcal{N}_0(S^{[a,b]}x) = \big\{ W \in \mathcal{Z} : \exists(\delta > 0) \ s.\ t. \ \forall \alpha \in (-\delta, \delta) \\ p\left(S^a\left(Q(x) + \alpha W, V(x)\right)\right) = p(S^a x) \, \& \\ p\left(S^b\left(Q(x) + \alpha W, V(x)\right)\right) = p(S^b x) \big\}.\end{aligned}$$

It is known (see (3) in Section 3 of [S-Ch (1987)]) that $\mathcal{N}_0(S^{[a,b]}x)$ is a linear subspace of $\mathcal{Z}$ indeed, and $V(x) \in \mathcal{N}_0(S^{[a,b]}x)$. The neutral space $\mathcal{N}_t(S^{[a,b]}x)$ of the segment $S^{[a,b]}x$ at time $t \in [a, b]$ is defined as follows:

$$\mathcal{N}_t(S^{[a,b]}x) = \mathcal{N}_0\left(S^{[a-t,b-t]}(S^t x)\right). \tag{2.8}$$

It is clear that the neutral space $\mathcal{N}_t(S^{[a,b]}x)$ can be canonically identified with $\mathcal{N}_0(S^{[a,b]}x)$ by the usual identification of the tangent spaces of $\mathbf{Q}$ along the trajectory $S^{(-\infty,\infty)}x$ (see, for instance, Section 2 of [K-S-Sz(1990)]).

Our next definition is that of the *advance*. Consider a non-singular orbit segment $S^{[a,b]}x$ with symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ $(n \geq 1)$ as at the beginning of the present section. For $x = (Q, V) \in \mathbf{M}$ and $W \in \mathcal{Z}$, $\|W\|$ sufficiently small, denote $T_W(Q, V) := (Q + W, V)$.

Definition 2.9. *For any $1 \leq k \leq n$ and $t \in [a, b]$, the advance*

$$\alpha(\sigma_k) : \ \mathcal{N}_t(S^{[a,b]}x) \to \mathbb{R}$$

is the unique linear extension of the linear functional defined in a sufficiently small neighbourhood of the origin of $\mathcal{N}_t(S^{[a,b]}x)$ in the following way:

$$\alpha(\sigma_k)(W) := t_k(x) - t_k(S^{-t} T_W S^t x).$$

It is now time to bring up the basic notion of *sufficiency* of a trajectory (segment). This is the utmost important necessary condition for the proof of the fundamental theorem for semi–dispersing billiards, see Condition (ii) of Theorem 3.6 and Definition 2.12 in [K-S-Sz(1990)] .

Definition 2.10.

(1) *The non–singular trajectory segment $S^{[a,b]}x$ (a and b are supposed not to be moments of collision) is said to be sufficient if and only if the dimension of $\mathcal{N}_t(S^{[a,b]}x)$ ($t \in [a, b]$) is minimal, i.e. dim $\mathcal{N}_t(S^{[a,b]}x) = 1$.*

(2) *The trajectory segment $S^{[a,b]}x$ containing exactly one singularity is said to be sufficient if and only if both branches of this trajectory segment are sufficient.*

For the notion of trajectory branches see, for example, the end of Section 2 in [Sim(1992)-I].

Definition 2.11. *The phase point $x \in \mathbf{M}$ with at most one singularity is said to be sufficient if and only if its whole trajectory $S^{(-\infty,\infty)}x$ is sufficient, which means, by definition, that some of its bounded segments $S^{[a,b]}x$ is sufficient.*

In the case of an orbit $S^{(-\infty,\infty)}x$ with exactly one singularity, sufficiency requires that both branches of $S^{(-\infty,\infty)}x$ be sufficient.

Connecting Path Formula for particles with different masses

The Connecting Path Formula, abbreviated as CPF, was discovered for particles with identical masses in [Sim(1992)-II] . Its goal was to give an explicit description (a linear coordinatization) of the neutral linear space $\mathcal{N}_0(S^{[-T,0]}x_0)$ in the language of the "advances" of the occuring collisions by using, as coefficients, linear expressions of the (pre-collision and post-collision) velocity differences of the colliding paticles. Since it relied upon the conservation of the momentum, it has been natural to expect that the CPF can be generalized for particles with different masses as well. The case is, indeed, this, and next we give this generalization for particles with different masses. Since its structure is the same as that of the CPF for identical masses, our exposition follows closely the structure of [Sim(1992)-II] .

Consider a phase point $x_0 \in \mathbf{M}$ whose trajectory segment $S^{[-T,0]}x_0$ is not singular, $T > 0$. In the forthcoming discussion the phase point x_0 and the positive number T will be fixed. All the velocities $v_i(t) \in \mathbb{R}^\nu$ $i \in \{1, 2, \dots, N\}$, $-T \le t \le 0$ appearing in the considerations are velocities of certain balls at specified moments t and always with the starting phase point x_0. ($v_i(t)$ is the velocity of the i-th ball at time t.) We suppose that the moments 0 and $-T$ are not moments of collision. We label the balls by the natural numbers $1, 2, \dots, N$ (so the set $\{1, 2, \dots, N\}$ is always the vertex set of the collision graph) and we denote by $e_1, e_2, \dots, e_n$ the collisions of the trajectory segment $S^{[-T,0]}x_0$ (i.e. the edges of the collision graph) so that the time order of these collisions is just the opposite of the order given by the indices. A few more definitions and notations:

1. $t_i = t(e_i)$ denotes the time of the collision e_i, so $0 > t_1 > t_2 > \dots > t_n > -T$.
2. If $t \in \mathbb{R}$ is not a moment of collision ($-T \le t \le 0$), then

$$\Delta q_i(t) : \mathcal{N}_0(S^{[-T,0]}x_0) \to \mathbb{R}^\nu$$

is a linear mapping assigning to every element $W \in \mathcal{N}_0(S^{[-T,0]}x_0)$ the displacement of the i-th ball at time t, provided that the configuration displacement at time zero is given by W. Originally, this linear mapping is only defined for vectors $W \in \mathcal{N}_0(S^{[-T,0]}x_0)$ close enough to the origin, but it can be uniquely extended to the whole space $\mathcal{N}_0(S^{[-T,0]}x_0)$ by preserving linearity.

3. $\alpha(e_i)$ denotes the advance of the collision e_i, thus

$$\alpha(e_i) : \mathcal{N}_0(S^{[-T,0]}x_0) \to \mathbb{R}$$

is a linear mapping ($i = 1, 2, \dots, n$).

4. The integers $1 = k(1) < k(2) < \dots < k(l_0) \le n$ are defined by the requirement that for every j ($1 \le j \le l_0$) the graph $\{e_1, e_2, \dots, e_{k(j)}\}$ consists of $N - j$ connected components (on the vertex set $\{1, 2, \dots, N\}$, as always!) while

the graph $\{e_1, e_2, \dots, e_{k(j)-1}\}$ consists of $N - j + 1$ connected components and, moreover, we require that the number of connected components of the whole graph $\{e_1, e_2, \dots, e_n\}$ be equal to $N - l_0$. It is clear from this definition that the graph

$$\mathcal{T} = \{e_{k(1)}, e_{k(2)}, \dots, e_{k(l_0)}\}$$

does not contain any loop, especially $l_0 \le N - 1$.

Here we make two remarks commenting the above notions.

Remark 2.12. We often do not indicate the variable $W \in \mathcal{N}_0(S^{[-T,0]}x_0)$ of the linear mappings $\Delta q_i(t)$ and $\alpha(e_i)$, for we will not be dealing with specific neutral tangent vectors W but, instead, we think of W as a typical (running) element of $\mathcal{N}_0(S^{[-T,0]}x_0)$ and $\Delta q_i(t)$, $\alpha(e_i)$ as linear mappings defined on $\mathcal{N}_0(S^{[-T,0]}x_0)$ in order to obtain an appropriate description of the neutral space $\mathcal{N}_0(S^{[-T,0]}x_0)$.

Remark 2.13. If $W \in \mathcal{N}_0(S^{[-T,0]}x_0)$ has the property $\Delta q_i(0)[W] = \lambda v_i(0)$ for some $\lambda \in \mathbb{R}$ and for all $i \in \{1, 2, \dots, N\}$ (here $v_i(0)$ is the velocity of the i-th ball at time zero), then $\alpha(e_k)[W] = \lambda$ for all $k = 1, 2, \dots, n$. This particular W corresponds to the direction of the flow. In the sequel we shall often refer to this remark.

Let us fix two *distinct* balls α, $\omega \in \{1, 2, \dots, N\}$ that *are in the same connected component* of the collision graph $\mathcal{G}_n = \{e_1, e_2, \dots, e_n\}$. The CPF expresses the relative displacement $\Delta q_\alpha(0) - \Delta q_\omega(0)$ in terms of the advances $\alpha(e_i)$ and the relative velocities occuring at these collisions e_i. In order to be able to formulate the CPF we need to define some graph-theoretic notions concerning the pair of vertices (α, ω).

Definition 2.14. *Since the graph $\mathcal{T} = \{e_{k(1)}, e_{k(2)}, \dots, e_{k(l_0)}\}$ contains no loop and the vertices α, ω belong to the same connected component of $\mathcal{T}$, there is a unique path $\Pi(\alpha, \omega) = \{f_1, f_2, \dots, f_h\}$ in the graph $\mathcal{T}$ connecting the vertices α and ω. The edges $f_i \in \mathcal{T}$ $(i = 1, 2, \dots, h)$ are listed up successively along this path $\Pi(\alpha, \omega)$ starting from α and ending at ω. The vertices of the path $\Pi(\alpha, \omega)$ are denoted by $\alpha = B_0, B_1, B_2, \dots, B_h = \omega$ indexed along this path going from α to ω, so the edge f_i connects the vertices B_{i-1} and B_i $(i = 1, 2, \dots, h)$.*

When trying to compute $\Delta q_\alpha(0) - \Delta q_\omega(0)$ by using the advances $\alpha(e_i)$ and the relative velocities at these collisions, it turns out that not only the collisions f_i $(i = 1, 2, \dots, h)$ make an impact on $\Delta q_\alpha(0) - \Delta q_\omega(0)$, but some other adjacent edges too. This motivates the following definition:

Definition 2.15. *Let $i \in \{1, 2, \dots, h-1\}$ be an integer. We define the set $\mathcal{A}_i$ of adjacent edges at the vertex B_i as follows:*

$$\begin{aligned}\mathcal{A}_i = \big\{e_j :\ & j \in \{1, 2, \dots, n\} \ \&\ (t(e_j) - t(f_i)) \cdot (t(e_j) - t(f_{i+1})) < 0 \ \&\\ & B_i \text{ is a vertex of } e_j\big\}.\end{aligned}$$

We adopt a similar definition to the sets $\mathcal{A}_0$, $\mathcal{A}_h$ of adjacent edges at the vertices B_0 and B_h, respectively:

Definition 2.16.

$$\mathcal{A}_0 = \{e_j : 1 \le j \le n \ \&\ t(e_j) > t(f_1) \ \&\ B_0 \text{ is a vertex of } e_j\};$$
$$\mathcal{A}_h = \{e_j : 1 \le j \le n \ \&\ t(e_j) > t(f_h) \ \&\ B_h \text{ is a vertex of } e_j\}.$$

We note that the sets $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_h$ are not necessarily mutually disjoint.

Finally, we need to define the "contribution" of the collision e_j to $\Delta q_\alpha(0) - \Delta q_\omega(0)$ which is composed from the relative velocities just before and after the moment $t(e_j)$ of the collision e_j.

Definition 2.17. *For $i \in \{1, 2, \ldots, h\}$ the "contribution" $\Gamma(f_i)$ of the edge $f_i \in \Pi(\alpha, \omega)$ is given by the formula*

$$\Gamma(f_i) = \begin{cases} v^-_{B_{i-1}}(t(f_i)) - v^-_{B_i}(t(f_i)), \\ \text{if } t(f_{i-1}) < t(f_i) \ \&\ t(f_{i+1}) < t(f_i); \\ \\ v^+_{B_{i-1}}(t(f_i)) - v^+_{B_i}(t(f_i)), \\ \text{if } t(f_{i-1}) > t(f_i) \ \&\ t(f_{i+1}) > t(f_i); \\ \\ \dfrac{1}{m_{B_{i-1}} + m_{B_i}} \big[m_{B_{i-1}} \big(v^-_{B_{i-1}}(t(f_i)) - v^-_{B_i}(t(f_i)) \big) \\ + m_{B_i} \big(v^+_{B_{i-1}}(t(f_i)) - v^+_{B_i}(t(f_i)) \big) \big], \\ \text{if } t(f_{i+1}) < t(f_i) < t(f_{i-1}) \\ \\ \dfrac{1}{m_{B_{i-1}} + m_{B_i}} \big[m_{B_{i-1}} \big(v^+_{B_{i-1}}(t(f_i)) - v^+_{B_i}(t(f_i)) \big) \\ + m_{B_i} \big(v^-_{B_{i-1}}(t(f_i)) - v^-_{B_i}(t(f_i)) \big) \big], \\ \text{if } t(f_{i-1}) < t(f_i) < t(f_{i+1}). \end{cases}$$

Here $v^-_{B_i}(t(f_i))$ denotes the velocity of the B_i-th particle just before the collision f_i (occuring at time $t(f_i)$) and, similarly, $v^+_{B_i}(t(f_i))$ is the velocity of the same particle just after the mentioned collision. We also note that, by convention, $t(f_0) = 0 > t(f_1)$ and $t(f_{h+1}) = 0 > t(f_h)$. Apparently, the time order plays an important role in this definition.

Definition 2.18. *For $i \in \{0, 1, 2, \ldots, h\}$ the "contribution" $\Gamma_i(e_j)$ of an edge $e_j \in \mathcal{A}_i$ is defined as follows:*

$$\begin{aligned} \Gamma_i(e_j) = \ & \mathrm{sign}\big(t(f_i) - t(f_{i+1})\big) \frac{m_C}{m_{B_i} + m_C} \cdot \\ & \cdot \Big[\big(v^+_{B_i}(t(e_j)) - v^+_C(t(e_j)) \big) - \big(v^-_{B_i}(t(e_j)) - v^-_C(t(e_j)) \big) \Big] \end{aligned}$$

where C is the vertex of e_j different from B_i.

Here again we adopt the convention of $t(f_0) = 0 > t(e_j)$ ($e_j \in \mathcal{A}_0$) and $t(f_{h+1}) = 0 > t(e_j)$ ($e_j \in \mathcal{A}_h$). We note that, by the definition of the set $\mathcal{A}_i$, exactly one of the two possibilities $t(f_{i+1}) < t(e_j) < t(f_i)$ and $t(f_i) < t(e_j) < t(f_{i+1})$ occurs. The subscript i of Γ is only needed because an edge $e_j \in \mathcal{A}_{i_1} \cap \mathcal{A}_{i_2}$ ($i_1 < i_2$) has two contributions at the vertices B_{i_1} and B_{i_2} which are just the endpoints of e_j.

We are now in the position of formulating the Connecting Path Formula:

Proposition 2.19. *Using all definitions and notations above, the following sum is an expression for $\Delta q_\alpha(0) - \Delta q_\omega(0)$ in terms of the advances and relative velocities of collisions:*

$$\Delta q_\alpha(0) - \Delta q_\omega(0) = \sum_{i=1}^{h} \alpha(f_i)\Gamma(f_i) + \sum_{i=0}^{h} \sum_{e_j \in \mathcal{A}_i} \alpha(e_j)\Gamma_i(e_j).$$

The proof of the proposition follows the proof of Simányi's CPF (Lemma 2.9 [Sim(1992)-II]) with the only difference that Lemma 2.8 of [Sim(1992)-II] is replaced here by the following

Lemma 2.20. *If e is a collision at time t between the particles B and C, then*

$$v_B^+(t) - v_C^-(t) = \frac{m_C}{m_B + m_C}\left(v_B^+(t) - v_C^+(t)\right) + \frac{m_B}{m_B + m_C}\left(v_B^-(t) - v_C^-(t)\right)$$

and

$$v_B^+(t) - v_B^-(t) = \frac{m_C}{m_B + m_C}\left[\left(v_B^+(t) - v_C^+(t)\right) - \left(v_B^-(t) - v_C^-(t)\right)\right]$$

Proof. The lemma is an easy consequence of the conservation of momentum for the collision e. $\square$

Remark 2.21. In Section 3, we will complexify the system, and also allow complex masses, in particular. It is easy to see that the CPF of Proposition 2.19 and the whole discussion of this subsection will still hold for complex masses if we assume, in addition, that for every $e = \{B, C\}$, occuring in the symbolic collision sequence of $S^{[-T,0]}x_0$, $m_B + m_C \neq 0$.

3. The Complexified Billiard Map Partial Linearity of the Dynamics

The aim of this section is to properly understand the algebraic relationship between the kinetic data of the billiard flow measured at different moments of time.

From now on we shall investigate orbit segments $S^{[0,T]}x_0$ $(T > 0)$ of the standard billiard ball flow $\left(\tilde{\mathbf{M}}, \{S^t\}, \mu, \vec{m}, L\right)$. We note here that $\tilde{\mathbf{M}} = \tilde{\mathbf{Q}} \times \mathcal{E}$, where the configuration space $\tilde{\mathbf{Q}}$ was defined right after (1.1) and $\mathcal{E}$ is the velocity sphere

$$\mathcal{E} = \left\{ (v_1, \ldots, v_N) \in \mathbb{R}^{\nu N} \,\middle|\, \sum_{i=1}^{N} m_i v_i = 0 \;\&\; \sum_{i=1}^{N} m_i \|v_i\|^2 = 1 \right\}$$

introduced in Section 1. Also note that in the geometric-algebraic considerations of the upcoming sections we do not use the equivalence relation Ψ of the introduction. Later on even the conditions $\sum_{i=1}^N m_i v_i = 0$ and $\sum_{i=1}^N m_i \|v_i\|^2 = 1$ will be dropped, cf. Remark 3.14.

The symbolic collision sequence of $S^{[0,T]}x_0$ is denoted by

$$\Sigma\left(S^{[0,T]}x_0\right) = (\sigma_1, \dots, \sigma_n).$$

The symbol $v_i^k = v_i^+(t_k) \in \mathbb{R}^\nu$ shall denote the velocity $\dot{q}_i$ of the i-th ball right *after* the k-th collision $\sigma_k = \{i_k, j_k\}$ $(1 \le i_k < j_k \le N)$ occuring at time t_k, $k = 1, \dots, n$. Of course, there is no need to deal with the velocities right *before* collisions, since $v_i^-(t_k) = v_i^+(t_{k-1}) = v_i^{k-1}$. As usual, $q_i^k \in \mathbb{T}_L^\nu = \mathbb{R}^\nu / L \cdot \mathbb{Z}^\nu$ denotes the position of the center of the i-th ball at the moment t_k of the k-th collision.

Let us now fix the symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ $(n \ge 1)$, and explore the algebraic relationship between the data

$$\left\{q_i^{k-1},\, v_i^{k-1} \,|\, i = 1, \dots, N\right\}$$

and

$$\left\{q_i^k,\, v_i^k \,|\, i = 1, \dots, N\right\},$$

$k = 1, \dots, n$. By definition, the data $\left\{q_i^0,\, v_i^0 \,|\, i = 1, \dots, N\right\}$ correspond to the initial (non-collision) phase point x_0. We also set $t_0 = 0$.

Since we would like to carry out arithmetic operations on these data, the periodic positions $q_i^k \in \mathbb{T}_L^\nu$ are not suitable for this purpose. Therefore, instead of studying the genuine orbit segments $S^{[0,T]}x_0$, we will deal with their Euclidean liftings.

Proposition 3.1. *Let $S^{[0,T]}x_0 = \{q_i(t),\, v_i(t) \,|\, 0 \le t \le T\}$ be an orbit segment as above, and assume that certain pre-images (Euclidean liftings) $\tilde{q}_i(0) = \tilde{q}_i^0 \in \mathbb{R}^\nu$ of the positions $q_i(0) \in \mathbb{T}_L^\nu = \mathbb{R}^\nu / L \cdot \mathbb{Z}^\nu$ are given. Then there is a uniquely defined, continuous, Euclidean lifting $\{\tilde{q}_i(t) \in \mathbb{R}^\nu \,|\, 0 \le t \le T\}$ of the given orbit segment that is an extension of the initial lifting $\left\{\tilde{q}_i^0 \,|\, i = 1, \dots, N\right\}$. Moreover, for every collision σ_k there exists a uniquely defined integer vector $a_k \in \mathbb{Z}^\nu$ – named the adjustment vector of σ_k – such that*

$$\left\|\tilde{q}_{i_k}^k - \tilde{q}_{j_k}^k - L \cdot a_k\right\|^2 - 4r^2 = 0. \tag{3.2}$$

We shall call the orbit segment $\tilde{\omega} = \left\{\tilde{q}_i^k,\, v_i^k \,|\, i = 1, \dots, N;\, k = 0, \dots, n\right\}$ a lifted orbit segment with the system of adjustment vectors $\mathcal{A} = (a_1, \dots, a_n) \in \mathbb{Z}^{n\nu}$.

The proof of this proposition is straightforward.

The next result establishes the already mentioned polynomial relationships between the kinetic data $\left\{\tilde{q}_i^{k-1},\, v_i^{k-1} \,|\, i = 1, \dots, N\right\}$ and $\left\{\tilde{q}_i^k,\, v_i^k \,|\, i = 1, \dots, N\right\}$, $k = 1, \dots, n$.

Proposition 3.3. *Using all notations and notions from above, one has the following polynomial relations between the kinetic data at σ_{k-1} and σ_k, $k = 1, \dots, n$. In order to simplify the notations, we write down these equations as if σ_k were $\{1, 2\}$:*

$$v_i^k = v_i^{k-1}, \quad i \notin \{1, 2\}, \tag{3.4}$$

$$(3.5)\qquad m_1 v_1^k + m_2 v_2^k = m_1 v_1^{k-1} + m_2 v_2^{k-1}$$

(conservation of the momentum),

$$(3.6)\quad v_1^k - v_2^k = v_1^{k-1} - v_2^{k-1} - \frac{1}{2r^2}\left\langle v_1^{k-1} - v_2^{k-1} \tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k \right\rangle \cdot \left(\tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k\right)$$

(reflection of the relative velocity determined by the elastic collision),

$$(3.7)\qquad \tilde{q}_i^k = \tilde{q}_i^{k-1} + \tau_k v_i^{k-1}, \quad i = 1, \dots, N,$$

where the time slot $\tau_k = t_k - t_{k-1}$ $(t_0 := 0)$ *is determined by the quadratic equation*

$$(3.8)\qquad \left\| \tilde{q}_1^{k-1} - \tilde{q}_2^{k-1} + \tau_k \left(v_1^{k-1} - v_2^{k-1}\right) - L \cdot a_k \right\|^2 = 4r^2.$$

The proof of this proposition is also obvious.

Remark. Observe that the new velocities v_1^k and v_2^k can be computed from (3.5)-(3.6) as follows:

(3.9)
$$\begin{aligned} v_1^k &= v_1^{k-1} - \frac{m_2}{2r^2(m_1+m_2)}\left\langle v_1^{k-1} - v_2^{k-1}; \tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k \right\rangle \cdot \left(\tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k\right), \\ v_2^k &= v_2^{k-1} + \frac{m_1}{2r^2(m_1+m_2)}\left\langle v_1^{k-1} - v_2^{k-1}; \tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k \right\rangle \cdot \left(\tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k\right). \end{aligned}$$

We also note that the equations above even extend analytically to the case when one of the two masses, say m_1, is equal to zero:

$$(3.10)\qquad \begin{aligned} v_1^k &= v_1^{k-1} - \frac{1}{2r^2}\left\langle v_1^{k-1} - v_2^{k-1}; \tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k \right\rangle \cdot \left(\tilde{q}_1^k - \tilde{q}_2^k - L \cdot a_k\right), \\ v_2^k &= v_2^{k-1}. \end{aligned}$$

We call the attention of the reader to the fact that in our understanding the symbol $\langle\, .\, ;\, .\, \rangle$ denotes the Euclidean inner product of ν-dimensional real vectors and $\|\, .\, \|$ is the corresponding norm.

The Complexification of the Billiard Map

Given the pair $(\Sigma, \mathcal{A}) = (\sigma_1, \dots, \sigma_n; a_1, \dots, a_n)$, the equations (3.4)-(3.8) make it possible to iteratively compute the kinetic data $\left\{\tilde{q}_i^k,\, v_i^k \,|\, i = 1, \dots, N\right\}$ by using the preceding data $\left\{\tilde{q}_i^{k-1},\, v_i^{k-1} \,|\, i = 1, \dots, N\right\}$. Throughout these computations one only uses the field operations and square roots. (The latter one is merely used when computing τ_k as the root of the quadratic equation (3.8). Obviously, since – at the moment – we are dealing with genuine, real orbit segments, in this dynamically realistic situation the equations (3.8) have two distinct, positive real roots, and τ_k is the smaller one.) Therefore, the data $\left\{\tilde{q}_i^k,\, v_i^k \,|\, i=1,\dots,N\right\}$ can be expressed as certain algebraic functions of the initial data $\left\{\tilde{q}_i^0,\, v_i^0 \,|\, i=1,\dots,N\right\}$, and the arising algebraic functions merely contain field operations, square roots, the radius r of the balls, the size $L>0$ of the torus and, finally, the masses m_i as constants.

Since these algebraic functions make full sense over the complex field $\mathbb{C}$ and, after all, our proof of the Theorem requires the complexification, we are now going to complexify the whole system by considering the kinetic variables, the size L, and the masses as complex ones and retaining the polynomial equations (3.4)-(3.8). However, due to the ambiguity of selecting a root of (3.8) out of the two, it proves to be important to explore first the algebraic frame of the relations (3.4)-(3.8) connecting the studied variables. This is what we do now.

The Field Extension Associated With the Pair $(\Sigma, \mathcal{A})$

We are going to define the commutative function field $\mathbb{K}=\mathbb{K}(\Sigma;\mathcal{A})$ generated by all functions

$$\left\{(\tilde{q}_i^k)_j,\, (v_i^k)_j \,|\, i=1,\dots,N;\; k=0,\dots,n;\; j=1,\dots,\nu\right\}$$

of the lifted orbit segments corresponding to the given parameters

$$(\Sigma,\mathcal{A})=(\sigma_1,\dots,\sigma_n;a_1,\dots,a_n)$$

in such a way that the field $\mathbb{K}(\Sigma;\mathcal{A})$ incorporates all algebraic relations among these variables that are consequences of the equations (3.4)-(3.8). (Here the subscript j denotes the j-th component of a ν-vector.) In our setup the ground field of allowed constants (coefficients) is, by definition, the complex field $\mathbb{C}$. The precise definition of $\mathbb{K}_n=\mathbb{K}(\Sigma;\mathcal{A})$ is

Definition 3.11. *For $n=0$ the field $\mathbb{K}_0=\mathbb{K}(\emptyset;\emptyset)$ is the transcendental extension $\mathbb{C}(\mathcal{B})$ of the coefficient field $\mathbb{C}$ by the algebraically independent formal variables*

$$\mathcal{B}=\left\{(\tilde{q}_i^0)_j,\, (v_i^0)_j,\, m_i,\, L \,|\, i=1,\dots,N;\; j=1,\dots,\nu\right\}.$$

Suppose now that $n>0$ and the commutative field $\mathbb{K}_{n-1}=\mathbb{K}(\Sigma';\mathcal{A}')$ has already been defined, where $\Sigma'=(\sigma_1,\dots,\sigma_{n-1})$, $\mathcal{A}'=(a_1,\dots,a_{n-1})$. Then we consider the quadratic equation $b_n\tau_n^2+c_n\tau_n+d_n=0$ in (3.8) with $k=n$ as a polynomial equation defining a new field element τ_n to be adjoined to the field $\mathbb{K}_{n-1}=\mathbb{K}(\Sigma';\mathcal{A}')$. (Recall that the coefficients b_n, c_n, d_n come from the field $\mathbb{K}_{n-1}$.)

There are two possibilities:

(i) The quadratic polynomial $b_nx^2+c_nx+d_n$ is reducible over the field $\mathbb{K}_{n-1}$. Then we take $\mathbb{K}_n=\mathbb{K}_{n-1}$.

(ii) The polynomial $b_nx^2+c_nx+d_n$ is irreducible over the field $\mathbb{K}_{n-1}$. Then we define $\mathbb{K}_n=\mathbb{K}(\Sigma;\mathcal{A})$ as the extension of $\mathbb{K}_{n-1}=\mathbb{K}(\Sigma';\mathcal{A}')$ by the root τ_n of this irreducible polynomial.

Remark 3.12. The importance of the field $\mathbb{K}_n$ is underscored by the fact that this field encodes all algebraic relations among the kinetic data that follow from the polynomial equations (3.4)-(3.8). Furthermore, Proposition 3.3 gives us an iterative computation rule for successively obtaining the kinetic variables $\left\{\tilde{q}_i^k,\, v_i^k \,|\, i=1,\dots,N\right\}$, $k=0,\dots,n$. The field $\mathbb{K}_n=\mathbb{K}(\Sigma,\mathcal{A})$ is, after all, the algebraic frame of such computations. We note that – everywhere in this paper – the symbols $\langle\, .\,;\,.\,\rangle$ and $\|\,.\,\|^2$ do not refer to an Hermitian inner product but, rather, $\langle x;y\rangle=\sum_{j=1}^{\nu}x_jy_j$ and $\|x\|^2=\sum_{j=1}^{\nu}x_j^2$, so that these expressions retain their polynomial form.

Remark 3.13. Later on it will be necessary to express each kinetic variable $\left(\tilde{q}_i^k\right)_j$ and $\left(v_i^k\right)_j$ as an algebraic function of the initial variables of $\mathcal{B}$. This raises, however, an important question: Which one of the two roots τ_k of (3.8) should be considered during these computations? This is no problem if the polynomial $b_kx^2+c_kx+d_k$, defining τ_k, is irreducible over the field $\mathbb{K}_{k-1}$, because the two roots of this polynomial are algebraically equivalent over $\mathbb{K}_{k-1}$, and any of them can be chosen as τ_k. However, if the polynomial $b_kx^2+c_kx+d_k$ is reducible over the field $\mathbb{K}_{k-1}$, then it is necessary to make a decision and assign to one of the two roots (both in the field $\mathbb{K}_{k-1}$ now) the role of τ_k. This is what we do. The result is the field $\mathbb{K}_n=\mathbb{K}(\Sigma;\mathcal{A})$ endowed with a distinguished n-tuple $(\tau_1,\dots,\tau_n)$ of its elements. The field $\mathbb{K}_n$ with the distinguished n-tuple $(\tau_1,\dots,\tau_n)=\vec{\tau}$ is denoted by $\mathbb{K}(\Sigma;\mathcal{A};\vec{\tau})$. Then the algebraic object $\mathbb{K}(\Sigma;\mathcal{A};\vec{\tau})$ completely controls the whole process of the iterative computation of the kinetic variables.

Note that in the genuine, real case the two roots of $b_kx^2+c_kx+d_k$ are distinct, positive real numbers, and – by the nature of the billiard dynamics – the chosen value is always the smaller one.

Finally, we mention here a simple case when the polynomial $b_kx^2+c_kx+d_k$ is reducible over the field $\mathbb{K}_{k-1}$. Namely, this takes place whenever $\sigma_{k-1}=\sigma_k$ and $a_{k-1}=a_k$. In that case the value $\tau_k\equiv 0$ is clearly a solution of the above polynomial $b_kx^2+c_kx+d_k$, and, being so, that polynomial is reducible over the field $\mathbb{K}_{k-1}$.

Remark 3.14. The reader may wonder why we did not postulate the algebraic dependencies $\sum_{i=1}^{N}m_iv_i^k=0$, $\sum_{i=1}^{N}m_i\|v_i^k\|^2=1$? The answer is the following: The definition of the neutral linear space together with its characterization via the Connecting Path Formula (CPF, see Section 2 of this paper or Lemma 2.9 of [Sim(1992)-II]), i. e. the partial linearity of the billiard flow, are invariant under

(1) all uniform velocity translations (adding the same vector to all velocities) and

(2) time rescalings.

Remark 3.15. All fields occuring in this paper are only defined up to an isomorphism over the coefficient field $\mathbb{C}$. Therefore, the statements like "the field $\mathbb{K}_2$ is an extension of $\mathbb{K}_1$" should be understood as follows: $\mathbb{K}_1$ is a subfield of $\mathbb{K}_2$ after the natural identification of the generating variables bearing the same name. For the necessary notions, properties, and results from the theory of field extensions and Galois theory, the reader is kindly recommended to look up, for instance, the book by I. Stewart, [St(1973)].

Remark 3.16. As said before (see Definition 3.11), the collection $\mathcal{B}$ of elements of the field $\mathbb{K}(\Sigma;\mathcal{A};\vec{\tau})$ is a base of transcendence in that field over the subfield $\mathbb{C}$, and the degree of the extension $\mathbb{K}(\Sigma;\mathcal{A};\vec{\tau}):\mathbb{C}(\mathcal{B})=\mathbb{K}_n:\mathbb{K}_0$ is a power of two. Moreover, each of the following sets is a generator for the field $\mathbb{K}_n$:

(a) $\mathbb{C}\cup\mathcal{B}\cup\{\tau_1,\dots,\tau_n\}$;

(b) $\mathbb{C}\cup\{m_1,\dots,m_N,L\}\cup\{(\tilde{q}_i^n)_j,\,(v_i^n)_j|\,i=1,\dots,N;\,j=1,\dots,\nu\}\cup\{\tau_1,\dots,\tau_n\}$;

(c) $\mathbb{C}\cup\{m_1,\dots,m_N,L\}\cup\big\{(\tilde{q}_i^k)_j,\,(v_i^k)_j|\,i=1,\dots,N;\,j=1,\dots,\nu;\,k=0,\dots,n\big\}$.

Remark 3.17. The above procedure of constructing field extensions is closely related to the classical theory of geometric constructions by a ruler and compass, and this is not for surprise: the billiard trajectory is constructed by intersecting a straight line with a sphere and then reflecting it across the tangent hyperplane of the sphere. This is a sort of classical geometric construction in ν dimensions. For the details see, for instance, §57 and §60 of [VDW (1955)], or Chapter 5 of [St(1973)].

Let us fix the pair $(\Sigma;\mathcal{A})$ and the n-tuple $\vec{\tau}=(\tau_1,\dots,\tau_n)$ of elements of the field $\mathbb{K}(\Sigma;\mathcal{A})$. We are now defining a $(2\nu+1)N+1$-dimensional complex analytic manifold $\tilde{\Omega}=\tilde{\Omega}(\Sigma,\mathcal{A},\vec{\tau})$ and certain holomorphic functions $\tilde{q}_i^k,\,v_i^k:\tilde{\Omega}\to\mathbb{C}^\nu$, $m_i:\tilde{\Omega}\to\mathbb{C}$, $L:\tilde{\Omega}\to\mathbb{C}$, $\tau_k:\tilde{\Omega}\to\mathbb{C}$, $i=1,\dots,N$, $k=0,\dots,n$, (τ_0 is not defined). First we define a domain $D=D(\Sigma,\mathcal{A},\vec{\tau})\subset\mathbb{C}^{(2\nu+1)N+1}$ as the set of all complex $(2\nu+1)N+1$-tuples

$$\big\{(\tilde{q}_i^0)_j,\,(v_i^0)_j,\,m_i,\,L|\,i=1,\dots,N;\,j=1,\dots,\nu\big\}$$

for which

(a) the leading coefficient $\left\|v_{i_k}^{k-1}-v_{j_k}^{k-1}\right\|^2$ and the discriminant of the quadratic equation (3.8) is never zero, $k=1,\dots,n$, (The latter condition is equivalent to

$$\left\langle v_{i_k}^{k-1}-v_{j_k}^{k-1};\,\tilde{q}_{i_k}^k-\tilde{q}_{j_k}^k-L\cdot a_k\right\rangle\neq 0,$$

see also Remark 3.29), and

(b) $m_{i_k}+m_{j_k}\neq 0$ for $k=1,\dots,n$,

provided that we carry out the iterative computation of the kinetic variables

$$\big\{(\tilde{q}_i^k)_j,\,(v_i^k)_j|\,i=1,\dots,N;\,j=1,\dots,\nu;\,k=0,\dots,n\big\}$$

as governed by the polynomial equations (3.4)-(3.8) and by the fixed selection of elements $\vec{\tau}$.

When inverting the dynamics, a useful remark is that

$$\left\|v_{i_k}^{k-1}-v_{j_k}^{k-1}\right\|^2=\left\|v_{i_k}^{k}-v_{j_k}^{k}\right\|^2,$$

and

$$\left\langle v_{i_k}^{k-1}-v_{j_k}^{k-1};\,\tilde{q}_{i_k}^k-\tilde{q}_{j_k}^k-L\cdot a_k\right\rangle=-\left\langle v_{i_k}^{k}-v_{j_k}^{k};\,\tilde{q}_{i_k}^k-\tilde{q}_{j_k}^k-L\cdot a_k\right\rangle.$$

Remark 3.18. Note that we require the validity of (a) for any branch of taking the square roots when computing the τ_k's with irreducible defining polynomials (3.8).

Lemma 3.19. *The complement set $\mathbb{C}^{(2\nu+1)N+1} \setminus D$ is a proper, closed, algebraic subset of $\mathbb{C}^{(2\nu+1)N+1}$, especially a finite union of complex analytic submanifolds with codimension at least one, so the open set $D \subset \mathbb{C}^{(2\nu+1)N+1}$ is connected and dense.*

Proof. We have the polynomial equations (3.4)-(3.8) making it possible to set up an algorithm for iteratively computing the kinetic variables measured at different times. The point is not just this iterative algorithm, but also its invertibility (time reversibility). The inverse process has similar algebraic properties, for it just means time reversal. One can easily prove by an induction on the length n of Σ that the iteratively defined dynamics determines a several-to-several mapping with maximum rank between a nonempty Zariski open set of the kinetic variables with superscript 1 and a nonempty Zariski open set of the kinetic variables measured right after the last reflection (with superscript n). (A Zariski open set is the complement of a closed algebraic set defined as the simultaneous zero set of finitely many polynomials.) Then one can go ahead with the induction from n to $n+1$, because all possible obstructions to extending the process only occur on proper algebraic submanifolds of the Zariski open set of kinetic variables with superscript n.

Definition 3.20. *We define $\tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ as the set of all complex $(2\nu(n+1)N + N + n + 1)$-tuples*

$$\omega = \left(\left(\tilde{q}_i^k\right)_j, (v_i^k)_j, m_i, L, \tau_l \middle| i = 1, \dots, N;\ k = 0, \dots, n;\ j = 1, \dots, \nu;\ l = 1, \dots, n \right)$$

for which these coordinates are interrelated by the equations (3.4)-(3.8) and the vector

$$\vec{x}(\omega) = \left(\left(\tilde{q}_i^0\right)_j, (v_i^0)_j, m_i, L \middle| i = 1, \dots, N;\ j = 1, \dots, \nu \right)$$

of initial data belongs to the set $D(\Sigma, \mathcal{A}, \vec{\tau})$ defined above.

It is clear now that $\tilde{q}_i^k, v_i^k : \tilde{\Omega} \to \mathbb{C}^\nu$, $m_i : \tilde{\Omega} \to \mathbb{C}$, $L : \tilde{\Omega} \to \mathbb{C}$, and $\tau_k : \tilde{\Omega} \to \mathbb{C}$ $(i = 1, \dots, N,\ k = 0, \dots, n)$ are holomorphic functions on $\tilde{\Omega} = \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$. The complex analytic manifold $\tilde{\Omega}$ endowed with the above holomorphic functions can justifiably considered as the complexification of the $(\Sigma, \mathcal{A}, \vec{\tau})$-iterated billiard map.

Remark. It is very likely that the manifold $\tilde{\Omega}$ is connected. Nevertheless, in our proof we do not need this connectedness, and are not going to pursue the goal of proving it.

The Complex Neutral Space $\mathcal{N}(\omega)$

Fix a base point $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$. The tangent space $\mathcal{T}_\omega \tilde{\Omega}$ of $\tilde{\Omega}$ at ω consists of all complex vectors

$$x = \left(\delta \tilde{q}_i^0,\, \delta v_i^0,\, \delta m_i,\, \delta L \,|\, i = 1, \dots, N\right) \in \mathbb{C}^{(2\nu+1)N},$$

thus $\mathcal{T}_\omega \tilde{\Omega}$ can be naturally identified with the complex vector space $\mathbb{C}^{(2\nu+1)N}$. Set

$$(3.21) \qquad \begin{aligned} \mathcal{N}(\omega) = \Big\{ x = \left(\delta \tilde{q}_i^0,\, \delta v_i^0,\, \delta m_i,\, \delta L \,|\, i = 1, \dots, N\right) \in \mathcal{T}_\omega \tilde{\Omega} \Big| \\ D_x(v_i^k) = D_x(m_i) = D_x(L) = 0 \text{ for } i = 1, \dots, N;\ k = 0, \dots, n \Big\}, \end{aligned}$$

where $D_x(\,.\,)$ denotes the directional differentiation in the direction of x. Clearly, the complex linear subspace $\mathcal{N}(\omega)$ of $\mathbb{C}^{(2\nu+1)N+1}$ is the proper complex analogue of the neutral space of a genuine, real orbit segment.

Remark. Since δv_i^0, δm_i, and δL must be equal to zero for a neutral vector $x \in \mathcal{N}(\omega)$, we shall simply write $x = (\delta \tilde{q}_1^0, \dots, \delta \tilde{q}_N^0)$ instead of indicating the zero entries.

The proof of the following proposition is completely analogous to that one for the real case, and therefore we omit it.

Proposition 3.22. *For every tangent vector $x \in \mathcal{N}(\omega)$ ($\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$) and $1 \le k \le n$ we have*

$$(3.23) \qquad D_x\left(\tilde{q}_{i_k}^{k-1} - \tilde{q}_{j_k}^{k-1}\right) = \left(\alpha_k(x) - \alpha_{k-1}(x)\right) \cdot \left(v_{i_k}^{k-1} - v_{j_k}^{k-1}\right),$$

$$(3.24) \qquad D_x(t_k) = -\alpha_k(x),$$

and

$$(3.25) \qquad D_x\left(\tau_k\right) = D_x(t_k - t_{k-1}) = \alpha_{k-1}(x) - \alpha_k(x),$$

where, by definition, $t_0 = 0 = \alpha_0(x)$. The functions $\alpha_k : \mathcal{N}(\omega) \to \mathbb{C}$ ($k = 1, \dots, n$) are linear functionals. The name of the functional α_k is the advance of the k-th collision $\sigma_k = \{i_k, j_k\}$, see also Section 2.

This result pregnantly shows that, indeed, the vector space $\mathcal{N}(\omega)$ describes the linearity of the $(\Sigma, \mathcal{A}, \vec{\tau})$-iterated billiard map.

Remark 3.26. It is a matter of simple computation to convince ourselves that all assertions of Section 2 of the present paper and [Sim(1992)-II] pertaining to the neutral space $\mathcal{N}(\omega)$ remain valid for the complexified dynamics. Here we only point out the two most important statements out of those results:

(1) The vector space $\mathcal{N}(\omega)$ measures the ambiguity in determining the orbit segment ω purely by its velocity history $\left\{v_i^k(\omega) : i = 1, \dots, N;\ k = 0, \dots, n\right\}$ and the outer geometric data $(\vec{m}, L)$. In other words, this means that if, locally in the phase space $\tilde{\Omega}$ (in a small open set), two phase points ω_1 and ω_2 have the same

velocities ($v_i^k(\omega_1) = v_i^k(\omega_2)$, $i = 1, \dots, N$; $k = 0, \dots, n$) and geometric parameters $(\vec{m}, L)$, then the initial data of ω_1 and ω_2 can only differ by a spatial translation by a vector from the space $\mathcal{N}(\omega_1) = \mathcal{N}(\omega_2)$, and this statement is obviously reversible.

(2) The reflection laws for the neutral vectors are exactly the same as for velocities.

(3) The Connecting Path Formula (CPF, Lemma 2.9 of [Sim(1992)-II], or Proposition 2.19 in this paper) is applicable, and it enables us to compute the neutral space $\mathcal{N}(\omega)$ via solving a homogeneous system of linear equations (now over the field $\mathbb{C}$)

$$\sum_{k=1}^{n} \alpha_k \Gamma_{ik} = 0, \; i = 1, \dots, n + P_\Sigma - N, \tag{3.27}$$

where each equation in (3.27) is a ν-dimensional complex vector equation, the coefficients $\Gamma_{ik} = \Gamma_{ik}(\omega) \in \mathbb{C}^\nu$ are certain linear combinations of relative velocities $v_i^k(\omega) - v_j^k(\omega)$ (the coefficients of those linear combinations are just fractional linear expressions of the masses, see definitions 2.17-2.18) and, finally, P_Σ denotes the number of connected components of the collision graph of Σ. For a bit more detailed explanation of (3.27), see the beginning part of Remark 4.6.

Remark 3.28. We note here that the ordering of the moments of collisions $t(f_i)$ (which playes a significant role in definitions 2.17-2.18) is no longer meaningful over the unordered field $\mathbb{C}$. However, the use of the corresponding inequalities in 2.17-2.18 is purely technical/notational. Those inequalities only serve to introduce the combinatorial ordering of the collisions f_i (which are σ's) given by the indexes of the symbolic collision sequence $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$.

Remark 3.29. As a straightforward consequence of the equation (3.6), we obtain that $v_i^{k-1} = v_i^k$ for $i = 1, \dots, N$ (i. e. the k-th reflection does not change the compound velocity) if and only if

$$\left\langle v_{i_k}^{k-1} - v_{j_k}^{k-1}; \tilde{q}_{i_k}^k - \tilde{q}_{j_k}^k - L \cdot a_k \right\rangle = 0,$$

and this is just the case of a tangential collision. Easy calculation shows that the mentioned tangentiality occurs if and only if the two roots τ_k of (3.8) coincide.

It is obvious that $\dim_{\mathbb{C}} \mathcal{N}(\omega)$ is at least $\nu + 1$ (as long as not all velocities are the same), because the flow direction and the uniform spatial translations are necessarily contained by $\mathcal{N}(\omega)$.

Definition 3.30. (Definition of sufficiency). *The orbit segment $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ is said to be sufficient if and only if $\dim_{\mathbb{C}} \mathcal{N}(\omega) = \nu + 1$.*

Finally, the main result of the next section (Key Lemma 4.1) will use the following

Definition 3.31. *We say that the triple $(\Sigma, \mathcal{A}, \vec{\tau})$ has Property (A) iff the following assertion holds:*

For every pair of indices $1 \le k < l \le n$ for which $\sigma_k = \sigma_l$, $a_k = a_l$, and $\sigma_j \cap \sigma_k = \emptyset$ for $j = k+1, \dots, l-1$, we have that $\tau_l \not\equiv -\sum_{j=k+1}^{l-1} \tau_j$ in the algebraic selection

$\vec{\tau}$ of the time slots. (Note that in this case the quadratic polynomial $b_l x^2 + c_l x + d_l$ is reducible over the field $\mathbb{K}_{l-1}$ because the time slot $\tau_l \equiv -\sum_{j=k+1}^{l-1} \tau_j \in \mathbb{K}_{l-1}$ is automatically a solution of (3.8).)

Remark 3.32. It is obvious that the schemes $(\Sigma, \mathcal{A}, \vec{\tau})$ corresponding to genuine, real billiard trajectory segments enjoy Property (A). Indeed, in those cases even the equations $\sigma_k = \sigma_l$, $a_k = a_l$, and $\sigma_j \cap \sigma_k = \emptyset$ $(j = k+1, \dots, l-1)$ cannot hold simultaneously. Concerning the necessity for introducing Property (A), see Example I at the end of the next section.

Remark 3.33. It is very likely that — for combinatorial schemes satisfying Property (A) — the quadratic polynomial $b_l x^2 + c_l x + d_l$ is reducible over the field $\mathbb{K}_{l-1}$ (i. e. $\mathbb{K}_{l-1} = \mathbb{K}_l$) only in the case described above in 3.31. However, we have no clue how to prove that statement and, fortunately, our proof of hyperbolicity does not require that result.

4. The Notion of Richness. Richness Implies Sufficiency

The result of this section is

Key Lemma 4.1. *There exists a positive number $C(N)$ (depending merely on the number of balls $N \geq 2$) with the following property: If a symbolic sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ is $C(N)$-rich and the triple $(\Sigma, \mathcal{A}, \vec{\tau})$ has Property (A) (see Definition 3.31), then $\dim_{\mathbb{C}} \mathcal{N}(\omega) = \nu + 1$ (i. e. the trajectory is sufficient) for almost every $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$. The real version of this result is also valid. (For the definition of a C-rich symbolic sequence, see 2.5.)*

Remark 1. Plainly, the exceptional zero measure subset of $\tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ mentioned in this lemma must actually be a countable union of proper, analytic submanifolds.

As mentioned in the Introduction, the aim of sections 3 and 4, in particular that of Key Lemma 4.1, is to settle part (1) of the strategy detailed there. The methods are mainly algebraic, and for an easier separation of our dynamical vs. algebraic arguments, we are now going to describe *what are the only properties of the dynamics (and nothing more) that we are going to use.*

(1) The dynamical properties of an orbit segment $S^{[0,T]} x_0$ with a symbolic collision sequence Σ are completely characterized by the equations (3.4)–(3.8) if we, in addition, specify τ_k as the smaller of the two, necessarily positive, roots of the equation (3.8). In the inductive steps of lemmas 4.8, 4.13, and 4.14 we also use the remark to Proposition 3.3 saying that the collision laws (3.9), themselves consequences of (3.5)–(3.6), extend analytically to the case of zero mass, cf. (3.10).

(2) Since our goal is to show that orbits with a rich collision structure are typically sufficient or, equivalently, their neutral subspaces have typically minimal possible dimension, we also need an algebraic characterization of the neutral subspace. This is attained through the CPF (see Proposition 2.19 and the system of linear equations (3.27)), and our arguments also use some simple inference about the CPF and the neutral subspaces from [Sim(1992)-II] .

(3) As we have seen in Section 3, the algebraic formalisms of points (1) and (2) can be complexified, and our forthcoming discussion is based upon this complexification.

Remark 2. In the course of proving the key lemma we will see that the thresholds $C(N)$ $(N \geq 2)$ can actually be defined by the recursion $C(2) = 1$, $C(N) = \frac{N}{2} \cdot \max\{C(N-1), 3\}$ $(N \geq 3)$, that is, $C(2) = 1$, $C(N) = \dfrac{3N!}{2^{N-1}}$ for $N \geq 3$. We note that, as it has been proved in [K-S-Sz(1991)], for $N = 3$ the value $C(3) = 2$ already does the job.

Proof of Key Lemma 4.1. All the rest of this section is devoted to the inductive prof of the key lemma, and it will be split into several lemmas.

First of all, we prove the original, complex version of the key lemma by an induction on the number N. Once it is completed, we will make the necessary remark by showing that the validity of the key lemma carries over to the real case.

Lemma 4.2. *Consider the open, connected and dense subset D of $\mathbb{C}^{(2\nu+1)N+1}$ formed by all possible complex $(2\nu+1)N+1$-tuples*

$$\vec{x} = \vec{x}(\omega) = \Big(\big(\tilde{q}_i^0(\omega)\big)_j, \big(v_i^0(\omega)\big)_j, m_i(\omega), L(\omega)\Big|\, i = 1, \dots, N;\ j = 1, \dots, \nu\Big), \qquad \Big(\omega \in \tilde{\Omega}\Big)$$

of initial data of the complex $(\Sigma, \mathcal{A}, \vec{\tau})$-dynamics, see the definition of $\tilde{\Omega}\,(\Sigma, \mathcal{A}, \vec{\tau})$ in the previous section. We claim that there are finitely many complex polynomials $P_1(\vec{x}), \dots, P_s(\vec{x})$ $(\vec{x} \in \mathbb{C}^{(2\nu+1)N+1})$, canonically determined by the discrete parameters $(\Sigma, \mathcal{A}, \vec{\tau})$, such that for every $\vec{x} \in D\,(\Sigma, \mathcal{A}, \vec{\tau})$ the following implication holds true:

$$\begin{aligned} \text{(4.3)} \qquad & \Big(\exists\, \omega \in \tilde{\Omega}\,(\Sigma, \mathcal{A}, \vec{\tau}) \text{ such that } \vec{x} = \vec{x}(\omega)\,\&\, \dim_{\mathbb{C}}\mathcal{N}(\omega) > \nu + 1\Big) \\ & \qquad\qquad \Longrightarrow P_1(\vec{x}) = P_2(\vec{x}) = \dots = P_s(\vec{x}) = 0. \end{aligned}$$

Remark. The left-hand-side in the above equivalence precisely says that *some* branch of the multiple-valued $(\Sigma, \mathcal{A}, \vec{\tau})$-dynamics with initial data $\vec{x}$ is not sufficient, see also Remark 3.18.

Proof. This proof is a typical application of the Connecting Path Formula (see Proposition 2.19 here or Lemma 2.9 of [Sim(1992)-II]) and Proposition 3.4 of [Sim(1992)-II]. Namely, Proposition 3.4 of [Sim(1992)-II] asserts that

$$\text{(4.4)} \qquad \dim_{\mathbb{C}}\mathcal{N}(\omega) = \nu \cdot P_\Sigma + \dim_{\mathbb{C}}\{\alpha_1, \dots, \alpha_n\},$$

where, as said before, P_Σ denotes the number of connected components of the collision graph of Σ, and $\{\alpha_1, \dots, \alpha_n\}$ is a shorthand for the complex linear space of all possible n-tuples $(\alpha_1(x), \dots, \alpha_n(x))$ of advances, $x \in \mathcal{N}(\omega)$.

Remark 4.5. It is worth noting here that our present formula (4.4) differs from its counterpart in Proposition 3.4 of [Sim(1992)-II] by an additional term ν. This is, however, due to the fact that in the present approach we no longer have the reduction equation $\sum_{i=1}^{N} m_i \delta\tilde{q}_i = 0$. It follows easily from (4.4) that the sufficiency is equivalent to $\dim_{\mathbb{C}}\{\alpha_1, \dots, \alpha_n\} = 1$, i. e. sufficiency means that all advances are equal to the same functional. Note that $\dim_{\mathbb{C}}\{\alpha_1, \dots, \alpha_n\} = 1$ obviously implies $P_\Sigma = 1$.

A simple, but important consequence of the Connecting Path Formula is that the linear space $\{\alpha_1, \dots, \alpha_n\}$ is the solution set of a homogeneous system of linear equations (3.27).

Remark 4.6. Note that, when applying the mentioned lemma, the left-hand-side of the Connecting Path Formula has to be written as the relative velocity of the colliding balls multiplied by the advance of that collision. It follows immediately from the exposition of [Sim(1992)-II] that the equations of this sort (arising from all CPF's) are the *only* constraints on the advance functionals. The reason why this is true is that the fulfillment of all CPF's precisely means that the relative displacement (variation of position) of every pair of particles right before their new collision is parallel to the relative incoming velocity of these particles (see also (3.23)), and this fact guarantees that the variation of the newly formed relative outgoing velocity will also be zero, just as required in (3.21).

Also note that the number $n + P_\Sigma - N$ of equations in (3.27) is equal to the number of all collisions $\sigma_k = \{i_k, j_k\}$ for which i_k and j_k are in the same connected component of the collision graph of $\{\sigma_1, \sigma_2, \dots, \sigma_{k-1}\}$.

The determinant $D(M)$ of every minor M of the coefficient matrix of (3.27) is a homogeneous velocity polynomial and, therefore, $D(M)$ is a holomorphic function on $\tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$. Thus, there are finitely many velocity polynomials (actually, determinants) $R_1(\omega), \dots, R_s(\omega)$ such that their simultaneous vanishing is equivalent to the non-sufficiency of the orbit segment ω. By using the equations (3.4)-(3.8) (which define the complex dynamics recursively), each of these velocity polynomials $R_i(\omega)$ can be – essentially uniquely – written as an algebraic function $f_i(\vec{x}(\omega)) = f_i(\vec{x})$ of the initial data $\vec{x}$, and these algebraic functions $f_i(\vec{x})$ only contain (finitely many) field operations and square roots. By using the canonical method of successive elimination of the square roots from the equation $f_i(\vec{x}) = 0$, one obtains a complex polynomial $P_i(\vec{x})$ such that

$$P_i(\vec{x}) = 0 \iff f_i(\vec{x}) = 0.$$

We emphasize here that we understand the relation $f_i(\vec{x}) = 0$ in such a way that $f_i(\vec{x})$ becomes zero on *some* branch of evaluating the multiple-valued algebraic function $f_i(\vec{x})$, cf. Remark 3.18.

The process of eliminating the square roots from an equation $f(\vec{x}) = 0$ (of the above type), however, requires a bit of clarification. The algebraic function f has a natural representing element $\hat{f}$ in the field $\mathbb{K}_n$, see Definition 3.11. Suppose that $\hat{f} \in \mathbb{K}_k$ but $\hat{f} \notin \mathbb{K}_{k-1}$. Then the field element $\hat{f} \in \mathbb{K}_k$ can be written uniquely in the form $\hat{f} = a + b\sqrt{\delta}$, where $a, b, \delta \in \mathbb{K}_{k-1}$, $b \neq 0$, $\sqrt{\delta} \notin \mathbb{K}_{k-1}$, and

$$\delta = \left\langle v_{i_k}^{k-1} - v_{j_k}^{k-1};\ \tilde{q}_{i_k}^{k-1} - \tilde{q}_{j_k}^{k-1} - L \cdot a_k \right\rangle^2 -$$
$$\left\| v_{i_k}^{k-1} - v_{j_k}^{k-1} \right\|^2 \cdot \left(\left\| \tilde{q}_{i_k}^{k-1} - \tilde{q}_{j_k}^{k-1} - L \cdot a_k \right\|^2 - 4r^2 \right)$$

is the discriminant of the quadratic equation (3.8) with the unknown τ_k.

In the first step of the successive elimination of the square roots from the equation $f(\vec{x}) = 0$, we switch from the field element $\hat{f} = a + b\sqrt{\delta} \in \mathbb{K}_k$ (representing f) to the new field element

$$\hat{f}' = \left(a + b\sqrt{\delta}\right) \cdot \left(a - b\sqrt{\delta}\right) = a^2 - b^2\delta \in \mathbb{K}_{k-1},$$

which is a representing element of an algebraic function $f'(\vec{x})$ containing less square roots than $f(\vec{x})$. Corresponding to the switching $\hat{f} \mapsto \hat{f}'$, we also replace the original equation $f(\vec{x}) = 0$ by the new equation $f'(\vec{x}) = 0$. Then we keep going by continuing this process until we arrive at a field element

$$\hat{g}(\vec{x}) = \frac{P(\vec{x})}{Q(\vec{x})} \in \mathbb{K}_0 = \mathbb{C}(\vec{x}),$$

which is a complex rational function $\dfrac{P(\vec{x})}{Q(\vec{x})}$ of the initial variables $\vec{x}$. Of course, here we assume that the polynomials $P(\vec{x})$ and $Q(\vec{x})$ have no non-trivial common divisor. Then the equation $P(\vec{x}) = 0$ just has the meaning that $f(\vec{x})$ becomes zero *on some branch of evaluating the algebraic function f.*

By using some simple facts from Galois theory, we can easily detect an intimate relationship between the rational function $\hat{g}(\vec{x}) \in \mathbb{K}_0$ just constructed and the product α of all conjugate elements of $\hat{f} \in \mathbb{K}_n$. (The conjugates of $\hat{f}$ (over the base field $\mathbb{K}_0$) are sitting in the normal hull $\overline{\mathbb{K}_n} : \mathbb{K}_0$ of the field extension $\mathbb{K}_n : \mathbb{K}_0$. The element $\alpha \in \mathbb{K}_0$ is obviously the constant term in the normalized minimal polynomial of $\hat{f}$ over the field $\mathbb{K}_0$. For the elementary concepts and facts from Galois theory, the reader is kindly referred to the book [St(1973)].) Namely, by establishing a natural matching between all conjugates of $a^2 - b^2\delta$ and the conjugates of the ordered pair $(a + b\sqrt{\delta},\ a - b\sqrt{\delta})$, we can easily see that there exists an integer l such that $\hat{g} = \alpha^{2^l}$. (In the case $l < 0$ this should be understood as $\alpha = (\hat{g})^{2^{-l}}$.) Especially, α and $\hat{g}$ have the same irreducible factors both in the numerator and in the denominator. Another consequence is that the field element $\hat{f}$ is zero if and only if $P(\vec{x}) \equiv 0$.

So far we have seen that for each velocity determinant $R_i(\omega)$ $(i = 1, \ldots, s)$ one can find a *canonically determined* polynomial $P_i(\vec{x})$ with the following property: For every vector $\vec{x} \in D(\Sigma, \mathcal{A}, \vec{\tau})$ the equation $P_i(\vec{x}) = 0$ is equivalent to the equation $f_i(\vec{x}) = 0$ *on some branch of evaluating the algebraic function $f_i(\vec{x})$.* Hence Lemma 4.2 follows. $\square$

Corollary (Dichotomy Corollary). *As another consequence of the above identity $\hat{g} = \alpha^{2^l}$, we obtain that if $P \equiv 0$, then $f(\vec{x}) = 0$ for every $\vec{x} \in \mathbb{C}^{(2\nu+1)N+1}$ and on every branch of evaluating the multiple-valued algebraic function $f(\vec{x})$. This* *also means that if all polynomials* $P_1, \dots, P_s$ *in Lemma 4.2 happen to be identically zero, then no orbit segment* $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ *is sufficient.*

Therefore, thanks to the algebraic feature of the dynamics, we have got a nice dichotomy here: **Either every** $(\Sigma, \mathcal{A}, \vec{\tau})$**-orbit segment is non-sufficient, or almost every such trajectory segment is sufficient.** □

Remark. We note that the reason why we do not have an equivalence in (4.3) is that – for a typical $\vec{x} \in \mathbb{C}^{(2\nu+1)N+1}$ – the branches, on which $f_i(\vec{x}) = 0$, may be different for different values of i.

According to the above lemma, in order to prove Key Lemma 4.1, it is enough to show that at least one of the polynomials $P_i(\vec{x})$ **is nonzero.** This is exactly what we are going to do by an induction on the number $N \geq 2$.

Lemma 4.7. *Pertaining to the polynomials* $P_i(\vec{x})$ *associated with* $R_i(\omega) = f_i(\vec{x}(\omega))$ *in Lemma 4.2, we assert the following:*

If the algebraic function $f_i(\vec{x})$ *is identically zero on a non-empty, open subset* B *of* $\mathbb{R}^{(2\nu+1)N+1}$*, then* f_i *is algebraically trivial, i. e.* $\hat{f}_i = 0$ *or, equivalently,* $P_i(\vec{x}) \equiv 0$.

Proof. Denote the minimal polynomial of $\hat{f}_i$ over the base field $\mathbb{K}_0 = \mathbb{C}(\vec{x})$ by $M(x) = \sum_{j=0}^{k} c_j x^j$, $c_j \in \mathbb{K}_0$. Suppose, on the contrary, that f_i yet vanishes on an open ball B of $\mathbb{R}^{(2\nu+1)N+1}$. Since f_i fulfills the identity $\sum_{j=0}^{k} c_j (f_i)^j \equiv 0$, the vanishing of f_i on the ball B implies that the element c_0 of the rational function field $\mathbb{K}_0$ vanishes on B, as well. Therefore, $c_0 = 0$ in $\mathbb{K}_0$, so $M(x)$ is divisible by x. The irreducibility of M yields $M(x) = x$, i. e. $\hat{f}_i = 0$. Hence the lemma follows. □

The next lemma describes the $(\Sigma, \mathcal{A}, \vec{\tau})$-orbit segments in the case when the N-th particle is "infinitely light" compared to the others, i. e. when $m_N = 0$ but $m_1 \cdot \ldots \cdot m_{N-1} \neq 0$.

Lemma 4.8. *Suppose that* $m_N = 0$ *and* $m_1 \cdot \ldots \cdot m_{N-1} \neq 0$*. Assume further that there exist indices* $1 \leq p < q \leq n$ *such that* $N \in \sigma_p$, $N \in \sigma_q$ *and* $N \notin \sigma_j$ *for* $j = p+1, \dots, q-1$*. Then an orbit segment* $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ *is sufficient (i. e.* $\dim_{\mathbb{C}} \mathcal{N}(\omega) = \nu + 1$*) if*

(1) the $\{1, 2, \dots, N-1\}$*-part*

$$\left\{ \tilde{q}_i^k(\omega),\ v_i^k(\omega),\ m_i(\omega),\ L(\omega) \,|\, i = 1, \dots, N-1;\ k \in I_N \right\}$$

of the orbit segment is sufficient as an orbit segment of the particles $1, \dots, N-1$ *and*

(2) the relative velocities $v_N^p(\omega) - v_{i_p}^p(\omega)$ *and* $v_N^{q-1}(\omega) - v_{i_q}^{q-1}(\omega)$ *are not parallel.*

(Here, as usual, i_p (i_q) is the index of the ball colliding with the N-th particle at σ_p (σ_q), and the index set $I_N \subset \{0, 1, \dots, n\}$ contains 0 and those indices $i > 0$ for which $N \notin \sigma_i$.)

Remark. It is worth noting that, as it follows easily from the conditions, $v_N^p(\omega) = v_N^{q-1}(\omega)$.

Proof. First of all, we observe that the motion of the N-th ball with zero mass has absolutely no effect on the evolution of the $\{1,\dots,N-1\}$-part of the trajectory segment, and this statement is also valid for the time evolution of the $\{1,\dots,N-1\}$-part $\left\{\delta\tilde{q}_i^k \,|\, i=1,\dots,N-1\right\}$ of a neutral vector $\left(\delta\tilde{q}_1^0,\dots,\delta\tilde{q}_N^0\right)\in\mathcal{N}(\omega)$, see also the equations (3.4), (3.7)–(3.8) and (3.10).

Consider now an arbitrary neutral vector $\delta Q=\left(\delta\tilde{q}_1^0,\dots,\delta\tilde{q}_N^0\right)\in\mathcal{N}(\omega)$. According to (1) and the above principle, we can modify the neutral vector δQ by a scalar multiple of the flow direction and by a uniform spatial displacement in such a way that $\delta\tilde{q}_i^0=0$ for $i=1,\dots,N-1$, and then $\delta\tilde{q}_i^k=0$ remains true for the whole orbit segment, $i=1,\dots,N-1$, $k=0,\dots,n$. The neutrality of δQ with respect to the entire orbit segment means, however, that

$$\delta\tilde{q}_N^p=\alpha_p\cdot\left(v_N^p(\omega)-v_{i_p}^p(\omega)\right)\ \text{and}\ \delta\tilde{q}_N^{q-1}=\alpha_q\cdot\left(v_N^{q-1}(\omega)-v_{i_q}^{q-1}(\omega)\right). \tag{4.9}$$

According to our hypothesis (2), the consequence of (4.9) and of the obvious equation $\delta\tilde{q}_N^p=\delta\tilde{q}_N^{q-1}$ is that $\alpha_p=\alpha_q=0$, and thus $\delta Q=0$. Hence the lemma follows. □

Derived Schemes

The crucial part of the inductive proof of Key Lemma 4.1, i. e. the substitution $m_N=0$, requires some preparatory thoughts and lemmas. Assume that a combinatorial-algebraic scheme $(\Sigma,\mathcal{A},\vec{\tau})$ is given for the N-ball system $\{1,2,\dots,N\}$ so that Property (A) holds (see Definition 3.31). In the proof of Key Lemma 4.1 we want to use some combinatorial-algebraic $(N-1)$-schemes $(\Sigma',\mathcal{A}',\vec{\tau}')$ that

(i) govern the time evolution of the $\{1,2,\dots,N-1\}$-part of *some* $(\Sigma,\mathcal{A},\vec{\tau})$-orbit segments with an infinitely light N-th ball, i. e. $m_N=0$ and

(ii) enjoy Property (A).

In the construction of such $(N-1)$-schemes $(\Sigma',\mathcal{A}',\vec{\tau}')$, the so called *derived schemes*, we want to follow the guiding principles below, which also serve as the definition of the *derived schemes:*

(1) First of all, we discard all symbols σ_j, a_j, and τ_j from $(\Sigma,\mathcal{A},\vec{\tau})$ that correspond to a σ_j containing the label N. This also means that we retain the other symbols σ_j, a_j without change: they only get re-indexed, due to the dropping of the other symbols;

(2) As far as the selection of $\vec{\tau}'$ is concerned, we want to retain *all algebraic relations* among the variables that are encoded in the original scheme $(\Sigma,\mathcal{A},\vec{\tau})$, i. e. all algebraic relations that follow from the scheme $(\Sigma,\mathcal{A},\vec{\tau})$ and from $m_N=0$;

(3) We want the derived scheme $(\Sigma',\mathcal{A}',\vec{\tau}')$ to enjoy Property (A), see 3.31.

It is straightforward that, following just the instructions in (1)-(2) above, one can easily construct such schemes $(\Sigma,\mathcal{A},\vec{\tau})$ which, perhaps, do not have Property (A).

Remark 4.10/a. Part (2) of the above definition means the following: Initially, the algebra of $(\Sigma, \mathcal{A}, \vec{\tau})$ consists

(1) of the variables

$$\left\{ \left(\tilde{q}_i^k\right)_j,\, (v_i^k)_j,\, m_i,\, L,\, \tau_l \,\middle|\, i = 1, \ldots, N;\; k = 0, \ldots, n;\; j = 1, \ldots, \nu;\; l = 1, \ldots, n \right\};$$

(2) of the equations (3.4-8), and finally

(3) of the choices of the signs of the square roots in the solutions of the quadratic equations (3.8) whenever the equations are reducible (cf. Remark 3.13).

To obtain the algebra of the derived scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$, we put first $m_N = 0$ and then $\tau_l' = \sum_{j=k+1}^{l} \tau_j$ whenever for some $1 \le k < l \le n$ we have $N \notin \sigma_k \cup \sigma_l$, $N \in \bigcap_{j=k+1}^{l-1} \sigma_j$ (here the indexing of the time slots τ_l' of $(\Sigma', \mathcal{A}', \vec{\tau}')$ is lacunary). After trivial substitutions, the quadratic equation for τ_l leads to the natural quadratic equation for τ_l', and then, finally, we can cancel all the variables $\left\{ (\tilde{q}_N^k)_j, (v_N^k)_j, m_N \,\middle|\; 0 \le k \le n, 1 \le j \le \nu \right\}$, and all the equations they figure in as well. The collection of the remaining variables and equations with the corresponding vector $\vec{\tau}'$ make $(\Sigma', \mathcal{A}', \vec{\tau}')$.

N. B. Below we show the inheritance of the reducibility $\mathbb{K}_{l-1} = \mathbb{K}_l \Longrightarrow \mathbb{K}_{l-1}' = \mathbb{K}_l'$. We note, however, that this result *is not used* in the proof of Key Lemma 4.1. We merely include it – along with its nice consequence: part (3) of Lemma 4.12 – in order to give the reader a broader view of the subject. The subsequent proof of Lemma 4.14 only uses the weaker part (2) of 4.12 which, in turn, does not need the above mentioned inheritance of reducibilty.

Another interpretation of part (2) of the above definition is the following one: Once we have a complex rational expression $g(z_1, \ldots, z_p)$ of arbitrary elements

$$z_1, \ldots, z_p \in \mathbb{K}_n' = \mathbb{K}(\Sigma', \mathcal{A}', \vec{\tau}')$$

so that the same expression of these variables is zero in the field $\mathbb{K}_n = \mathbb{K}(\Sigma, \mathcal{A}, \vec{\tau})$, then it must also be zero in the field $\mathbb{K}_n'$. An important case (when (2) has special significance) is when $N \notin \sigma_k \cup \sigma_l$ $(1 \le k < l \le n)$, $N \in \bigcap_{j=k+1}^{l-1} \sigma_j$, and $\mathbb{K}_{l-1} = \mathbb{K}_l$. Then the time slot $\tau_l' = \sum_{j=k+1}^{l} \tau_j$ (Here the indexing of the time slots τ_l' of $(\Sigma', \mathcal{A}', \vec{\tau}')$ is lacunary.) should be expressed as a *rational function* of the earlier defined $(\Sigma, \mathcal{A}, \vec{\tau})$-variables (the variables constituting the field $\mathbb{K}_{l-1}$), just as it is instructed by the rational dependence of τ_l on the variables constituting the field $\mathbb{K}_{l-1}$. Therefore, $\tau_l' = \sum_{j=k+1}^{l} \tau_j$ is a rational expression

$$(\heartsuit) \qquad \tau_l' \equiv g\left(\vec{x}; \tilde{q}_N^0, v_N^0, m_N; \tau_1, \ldots, \tau_{l-1}\right)$$

of the initial variables $\left(\vec{x}; \tilde{q}_N^0, v_N^0, m_N\right) \in \mathbb{C}^{(2\nu+1)N+1}$ and the time slots $\tau_1, \ldots, \tau_{l-1}$. Especially, the identity $(\heartsuit)$ is also true if we impose additionally the side condition $m_N = 0$:

$$(\diamondsuit) \qquad \tau_l^{'0} \equiv g\left(\vec{x}; \tilde{q}_N^0, v_N^0, 0; \tau_1^0, \dots, \tau_{l-1}^0\right).$$

However, the $\{1, \dots, N-1\}$-dynamics evolves independently of the data of the N-th ball with zero mass m_N. This means that

$$g\left(\vec{x}; \tilde{q}_N^0(1), v_N^0(1), 0; \tau_1^0(1), \dots, \tau_{l-1}^0(1)\right) = g\left(\vec{x}; \tilde{q}_N^0(2), v_N^0(2), 0; \tau_1^0(2), \dots, \tau_{l-1}^0(2)\right)$$

whenever all cumulative sums $\sum_{j=k_1+1}^{k_2} \tau_j^0(1)$ are equal to the corresponding cumulative sums $\sum_{j=k_1+1}^{k_2} \tau_j^0(2)$ for $1 \le k_1 < k_2 < l$, $N \notin \sigma_{k_1} \cup \sigma_{k_2}$, $N \in \bigcap_{j=k_1+1}^{k_2-1} \sigma_j$. This precisely means that the rational function $g\left(\vec{x}; \tilde{q}_N^0, v_N^0, 0; \tau_1^0, \dots, \tau_{l-1}^0\right)$

(i) does not contain the variables $\left(\tilde{q}_N^0\right)_\beta$, $\left(v_N^0\right)_\beta$ $(1 \le \beta \le \nu)$, and

(ii) only depends through the above cumulative sums $\sum_{j=k_1+1}^{k_2} \tau_j^0 = \tau_{k_2}^{'0}$.

(The indexing of $\tau_j^{'0}$ is still lacunary!) Thus, we get that $g\left(\vec{x}; \tilde{q}_N^0, v_N^0, 0; \tau_1^0, \dots, \tau_{l-1}^0\right)$ is, in fact, a rational expression of $\vec{x}$ and the time slots $\tau_j^{'0}$ preceding $\tau_l^{'0}$. This shows that $\tau_l^{'0} \in \mathbb{K}'_{l-1}$, i. e. $\mathbb{K}'_{l-1} = \mathbb{K}'_l$ follows from the similar reducibility $\mathbb{K}_{l-1} = \mathbb{K}_l$.

The main point is the following one: Once τ_l is the root of a reducible polynomial, it is selected "artificially", following the instructions encoded in the scheme $\vec{\tau}$. Now we want to avoid the selection of the other root when defining the derived scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$! This is why (2) is a requirement, and not a corollary, and this is why we use the wording "the time slot τ'_l should be expressed" in Remark 4.10/a.

The fact that the reducibility of the polynomial defining τ_l is inherited during a substitution $m_N = 0$ followed from a general principle: If some algebraic function in $\mathbb{K}_{l-1}$ (the discriminant of the quadratic polynomial defining τ_l) is the square of another such algebraic expression, then this square relation remains even "more true" if one annihilates a variable, i. e. considers that square relation on a smaller set. It is now the nature of our model that $m_N = 0$ immediately kills all variables involving the motion of the N-th particle, since the considered new discriminant of the quadratic equation defining τ'_l is obviously independent of those variables.

Remark 4.10/b. Note that the construction of such schemes $(\Sigma', \mathcal{A}', \vec{\tau}')$, by just requiring (1)-(2), is far from unique: It is well possible that, originally, for some k we have $|\mathbb{K}_k : \mathbb{K}_{k-1}| = 2$ (i. e. the polynomial (3.8) determining τ_k is irreducible), but after the substitution $m_N = 0$ the corresponding field extension collapses to a trivial, degree-one extension. (The paramount example for this phenomenon is just the set-up of the upcoming Main Lemma 4.11.) If this phenomenon of "reducible splitting" takes place, then we have to make our choice: Which one of the two (now distinguishable!) roots of the reducible (3.8) to include in $\vec{\tau}'$?

The requirement (A) for the derived scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$, however, ties our hands when selecting this root in the scenario of Main Lemma 4.11. Thus, if the conjecture in 3.33 were indeed true, then it would imply the uniqueness of the derived scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$.

The phenomenon of this reducible splitting can also be well represented by the following, simple paradigm: Let $a \in \mathbb{C}$ be a complex parameter (playing the role of m_N). Consider the Riemann-leaf of the function $y = f(z) = \sqrt{z^2 + a}$, i. e. the algebraic variety defined by the equation $y^2 - z^2 - a = 0$. This is irreducible (undecomposable) for $a \neq 0$, and it splits into two components ($y - z = 0$ and $y + z = 0$) for $a = 0$. However, given a sequence $a_k \to 0$ ($a_k \neq 0$), every point (z, y) on the curve $y^2 - z^2 = 0$ turns out to be a limit of a sequence (z_k, y_k) with $y_k^2 - z_k^2 - a_k = 0$.

Definition-notation 4.11/♠. Denote by Φ_p $(1 \leq p \leq n)$ the degeneracy $\left\| v^p_{i_p} - v^p_{j_p} \right\|^2 = 0$ and by Ψ_p $(1 \leq p \leq n)$ the degeneracy defined by $\left\langle v^p_{i_p} - v^p_{j_p};\, \tilde{q}^p_{i_p} - \tilde{q}^p_{j_p} - L \cdot a_p \right\rangle = 0$.

In order to show that there really exist derived schemes (that is, fulfilling not only (1)-(2), but also (3)), we need to prove

Main Lemma 4.11. *Assume that $(\Sigma, \mathcal{A}, \vec{r})$ enjoys Property (A). We claim that*

(I) The codimension-one submanifolds defined either by $m_i = 0$ $(1 \leq i \leq N)$ or by $L = 0$ can not even locally coincide with any degeneracy Φ_p or Ψ_p, $1 \leq p \leq n$;

(II) The degeneracy Φ_p $(1 \leq p < n)$ can not even locally coincide with the singularity Ψ_n;

(III) For any p, $1 \leq p < n$, the degeneracy Ψ_p can not even locally coincide with the singularity Ψ_n, unless $(\sigma_p, a_p) = (\sigma_n, a_n)$ and for any j, $p < j < n$, the relation $\sigma_j \cap \sigma_n \neq \emptyset$ implies $(\sigma_j, a_j) = (\sigma_n, a_n)$;

(IV) If there is no natural number $p < n$ with the property excluded in (III), then there exists a (non-degenerate) orbit segment

$$\omega = \{\tilde{q}^\alpha_i(\omega),\, v^\alpha_i(\omega),\, m_i(\omega),\, L(\omega) \,|\, i = 1, \dots, N;\ \alpha = 0, 1, \dots, n-1\}$$

$$\in \tilde{\Omega}\,(\sigma_1, \dots, \sigma_{n-1}; a_1, \dots, a_{n-1}; \tau_1, \dots, \tau_{n-1}) = \tilde{\Omega}_{n-1}$$

for which the discriminant

$$\delta(\omega) := \left\langle v^{n-1}_{i_n}(\omega) - v^{n-1}_{j_n}(\omega); \tilde{q}^{n-1}_{i_n}(\omega) - \tilde{q}^{n-1}_{j_n}(\omega) - L \cdot a_n \right\rangle^2$$

$$- \left\| v^{n-1}_{i_n}(\omega) - v^{n-1}_{j_n}(\omega) \right\|^2 \cdot \left(\left\| \tilde{q}^{n-1}_{i_n}(\omega) - \tilde{q}^{n-1}_{j_n}(\omega) - L \cdot a_n \right\|^2 - 4r^2 \right)$$

of (3.8) (with $k = n$) is equal to zero and $m_i(\omega) \neq 0$ $(i = 1, \dots, N)$, $L(\omega) \neq 0$;

(V) If $1 \leq k < n$, $\sigma_k = \sigma_n$, $a_k = a_n$, $N \notin \sigma_n$, and
(i) for $k < j < n$ either $\sigma_j \cap \sigma_n = \emptyset$ or $N \in \sigma_j$,
(ii) there exists a j_0 with $k < j_0 < n$ and $\sigma_{j_0} \cap \sigma_n \neq \emptyset$,

then $|\mathbb{K}_n : \mathbb{K}_{n-1}| = 2$ or, equivalently, the polynomial (3.8), defining τ_n, is irreducible over the field $\mathbb{K}_{n-1}$;

(VI) If $1 \leq k < n$, $\sigma_k = \sigma_n$, $a_k \neq a_n$, and for every $k < j < n$ we have $\sigma_j \cap \sigma_n = \emptyset$, then again $|\mathbb{K}_n : \mathbb{K}_{n-1}| = 2$.

Note: By the local coincidence of two codimension-one, algebraic sets we always mean that their intersection contains a non-empty, smooth, codimension-one manifold.

Remark 4.11/♣. Assertion (V) of 4.11 is just taylor suited to ensure that — whenever carrying out the substitution $m_N = 0$ — at every "critical" step with the possibility of selecting the wrong τ'_j (and thus violating Property (A)), we are not forced to do so, because at an irreducible τ_j either branches of the corresponding square root of the discriminant of (3.8) may be selected for continuing the dynamics. Quite similarly, assertion (VI) guarantees the same thing for the substitution $L = 0$.

Proof.

1. First we prove (I). For the sake of brevity, we prove (I) for the case $m_i = 0$. The handling of the submanifold $L = 0$ is completeley analogous. Assume that there is a counterexample, and p (the time of singularity) has the smallest possible value. Push forward the manifold $m_i = 0$ up to the collision σ_{p-1} by the algebraic dynamics: the form of the equation $m_i = 0$ defining this manifold does not change. (This push-forward can be made without difficulty by the minimality of the index p.) In this way we can reduce the proof of (I) to the case $p = 1$. However, the statement is obviously true in that case. □

Corollary. *If for the combinatorial scheme $(\Sigma, \mathcal{A}, \vec{\tau})$ and for some $\omega_0 \in \tilde{\Omega}_{n-1}$ we have $\delta(\omega_0) = 0$, then we can also find an $\omega^* \in \tilde{\Omega}_{n-1}$ such that still $\delta(\omega^*) = 0$ and none of the masses $m_1(\omega^*), \dots, m_N(\omega^*)$ is zero.*

Proof. Indeed, the conditions imply that for a codimension-one algebraic subset Z of $D\left(\tilde{\Omega}_{n-1}\right)$ it is true that for every $x \in Z$ there exists an $\omega \in \tilde{\Omega}_{n-1}$ such that $x(\omega) = x$ and $\delta(\omega) = 0$. By (I), Z is not even locally identical with any submanifold $\{m_i = 0\}$ $(i = 1, \dots, N)$. Consequently, we can select a small perturbation $x^* \in Z$ of $x_0 = x(\omega_0)$ and an $\omega^* \in \tilde{\Omega}_{n-1}$ such that $x(\omega^*) = x^*$, $\delta(\omega^*) = 0$, and none of the masses $m_1(\omega^*), \dots, m_N(\omega^*)$ vanishes. □

2. The implication $(I)\&(II)\&(III) \Longrightarrow (IV)$ is obvious. It is no problem to avoid the additional degeneracies $m_i + m_j = 0$, $1 \le i < j \le N$.

3. We prove next that the existence of an $\omega \in \tilde{\Omega}_{n-1}$ with $\delta(\omega) = 0$ (guaranteed in (IV)) implies $|\mathbb{K}_n : \mathbb{K}_{n-1}| = 2$. This will immediately prove the implications $(IV) \Longrightarrow (V)$ and $(IV) \Longrightarrow (VI)$. (It is easy to check that the conditions in (V) or (VI) actually imply the precondition of (IV).)

Let, therefore, $\omega \in \tilde{\Omega}_{n-1}$ be an orbit segment with $\delta(\omega) = 0$. By re-labeling the balls, we can assume that $\sigma_n = \{1, 2\}$. In a small neighborhood $U_0 \subset \tilde{\Omega}_{n-1}$ of ω the discriminant $\delta(\omega')$ from (IV) $(\omega' \in U_0)$ is an analytic function of ω' or, equivalently, $\delta(\omega')$ is an analytic, algebraic function $f(\vec{x})$ of the initial variables $\vec{x} = \vec{x}(\omega) \in \mathbb{C}^{(2\nu+1)N+1}$.

Consider now the "normal vector of collision"

$$w = w(\omega) = \tilde{q}_1^n(\omega) - \tilde{q}_2^n(\omega) - L \cdot a_n =$$

$$\tilde{q}_1^{n-1}(\omega) - \tilde{q}_2^{n-1}(\omega) - L \cdot a_n + \tau_n(\omega) \cdot \left[v_1^{n-1}(\omega) - v_2^{n-1}(\omega)\right],$$

where $\tau_n(\omega)$ is just the double root of (3.8) with $k = n$. Execute now the small perturbations $\omega(\varepsilon) \in U_0$, ($\varepsilon \in \mathbb{C}$, $|\varepsilon|$ is small) as follows:

$$\tilde{q}_1^{n-1}(\omega(\varepsilon)) = \tilde{q}_1^{n-1}(\omega) + \varepsilon \cdot w;$$

$$\tilde{q}_i^{n-1}(\omega(\varepsilon)) = \tilde{q}_i^{n-1}(\omega), \quad i = 2, \dots, N;$$

$$v_i^{n-1}(\omega(\varepsilon)) = v_i^{n-1}(\omega), \quad i = 1, \dots, N.$$

(Note: If $1 \in \sigma_{n-1}$, then the configuration perturbation defined right above ought to be modified by an appropriate scalar multiple of the flow direction in order to ensure $\left\| \tilde{q}_{i_{n-1}}^{n-1} - \tilde{q}_{j_{n-1}}^{n-1} - L \cdot a_{n-1} \right\|^2 = 4r^2$.) Easy geometric consideration yields that

$$\frac{d}{d\varepsilon}\delta(\omega(\varepsilon))\big|_{\varepsilon=0} = \frac{d}{d\varepsilon} f(\vec{x}(\omega(\varepsilon)))\bigg|_{\varepsilon=0} = -8r^2 \left\| v_1^{n-1}(\omega) - v_2^{n-1}(\omega) \right\|^2 \neq 0.$$

That is, the holomorphic function $g(\varepsilon) := f(\vec{x}(\omega(\varepsilon)))$ of $\varepsilon \in \mathbb{C}$ ($|\varepsilon|$ is small) has the properties $g(0) = 0$, $g'(0) \neq 0$.

Suppose now, indirectly, that the algebraic function $f(\vec{x}) \in \mathbb{K}_{n-1}$ is a square $(h(\vec{x}))^2$ in $\mathbb{K}_{n-1}$. Consider the function $z(\varepsilon) := h(\vec{x}(\omega(\varepsilon)))$ of the single complex variable $\varepsilon \in \mathbb{C}$, $|\varepsilon|$ small. It is clear that $z(\varepsilon)$ is a meromorphic function of ε, being a rational function of the variables $\{\tilde{q}_i^\alpha,\, v_i^\alpha,\, m_i,\, L \,|\, i = 1, \dots, N;\ \alpha < n\}$, and thus being the ratio of two holomorphic functions. This is the point where we use the indirect assumption of $h(\vec{x}) \in \mathbb{K}_{n-1}$. However, the square $[z(\varepsilon)]^2 = g(\varepsilon)$ of the meromorphic function $z(\varepsilon)$, obviously, can not have the properties $g(0) = 0$, $g'(0) \neq 0$.

This proves that, indeed, $|\mathbb{K}_n : \mathbb{K}_{n-1}| = 2$. $\square$

So far we have proved that (II) and (III) together imply all other assertions of Main Lemma 4.11.

4. We prove now (II) and (III) (therefore, the entire Main Lemma 4.11) by using an induction on the distance $n - p$ between the considered collisions σ_p and σ_n.

Base of the induction, $n - p = 1$: This is a straightforward inspection.

Assume now that $n - p \geq 2$, and (II)–(III) have been proved for all smaller values of $n - p$. By re-labeling the balls, we can achieve that $\sigma_n = \{1, 2\}$, $\sigma_p = \{i, j\}$, $1 \leq i < j \leq 4$, and if $j = 4$, then $i = 3$. ("Densely packed" indices.)

4/a: Proof of (II) by using the inductive assumption. We argue by contradiction. Suppose, therefore, that Φ_p locally coincides with the degeneracy Ψ_n defined by the equation $\delta(\omega) = 0$. We apply our favored

Globalization/substitution Argument: By using the quadratic equation defining our considered σ_p-degeneracy Φ_p, we can eliminate one variable, say $(v_i^p)_1$, by expressing it as an algebraic function $f(\vec{y})$ of the other variables

$$\vec{y} = \left\{ (\tilde{q}^p_\alpha)_\beta \, , \, (v^p_\alpha)_\beta \, , \, m_\alpha, \, L \, \middle| \, \alpha = 1, \dots, N; \, 1 \le \beta \le \nu \right\} \setminus \left\{ (v^p_i)_1 \right\}$$

measured at the collision σ_p. The kinetic variables measured at σ_p (i. e. with superscript p) are now serving as the initial, or reference variables in the upcoming globalization/substitution argument. As usual, the algebraic function $f(\vec{y})$ only contains field operations and square roots. Then, by using the recursive feature of the $(\Sigma, \mathcal{A}, \vec{\tau})$-dynamics and the inductive assumption of 4.11, we can express the discriminant $\delta(\omega)$ as another (also essentially unique) algebraic function $g(\vec{y})$ of the variables $\vec{y}$, which are now serving as the initial variables.

Remark 4.11/a. The reason why we can shift the index (time) of the initial variables from 0 to p is as follows: If the triple $(\Sigma, \mathcal{A}, \vec{\tau})$ has Property (A), $(\Sigma_1, \mathcal{A}_1) = (\sigma_{n_1}, \dots, \sigma_n; \, a_{n_1}, \dots, a_n)$ $(1 < n_1 \le n)$ is an end segment of $(\Sigma, \mathcal{A})$, then the Σ_1-restriction of every $(\Sigma, \mathcal{A}, \vec{\tau})$-orbit segment ω is automatically a $(\Sigma_1, \mathcal{A}_1, \tau_1)$-orbit segment with *some* τ_1, so that $(\Sigma_1, \mathcal{A}_1, \tau_1)$ obviously *enjoys Property (A).*

The local coincidence of our two studied degeneracies precisely means that

(*) the algebraic expression $g(\vec{y})$ is identically zero on some non-empty, open set of vectors $\vec{y}$.

Then this becomes a *global* identity (or global coincidence of degeneracies) according to Lemma 4.7. However, this statement remains valid after the one-by-one successive substitutions $m_3 = 0, \dots, m_N = 0$, $L = 0$, provided that it was already valid in the original set-up. We remind the reader that the almost everywhere (non-degenerate) definability of the algebraic dynamics subjected to the side condition $m_3 = 0$ (as for the first step) easily follows from (I).

It is very important to note here that, thanks to the inductive hypothesis of 4.11, we can assume that Property (A) is inherited to the algebraic billiard trajectory segment subjected to the side conditions $m_3 = \cdots = m_N = L = 0$. Indeed, as we mentioned in Remark 4.11/♣, parts (V)–(VI) are just designed to ensure the inheritance of Property (A).

Remark 4.11/b. A natural question arises: What should we mean by the "successive" substitutions $m_3 = 0, \dots, m_N = 0$, $L = 0$? This is an especially intriguing question, since after these substitutions the conditions $m_\alpha + m_\beta \ne 0$ $(1 \le \alpha < \beta \le N)$ are violated for many pairs (α, β).

The resolution of this problem is the following one: The algebraic dynamics does not depend on the actual values of the masses, but only on their ratios m_α / m_β. In terms of these ratios, the conditions $m_\alpha + m_\beta \ne 0$ mean that $m_\alpha / m_\beta \ne -1$. In our understanding of the successive substitutions $m_3 = 0, \dots, m_N = 0$, this means that $m_1/m_2 \ne 0$, $m_2/m_1 \ne 0$, $m_N/m_1 = 0$, and $m_\alpha/m_{\alpha+1} = 0$ for $\alpha = 3, 4, \dots, N-1$.

Remark 4.11/c. In these arguments we implicitly understood that the current working domain $D = D_{p+1,n-1}$ is not the one we mentioned earlier when defining $\tilde{\Omega}\,(\Sigma; \mathcal{A}; \vec{\tau})$ but, instead, the working domain $D_{p+1,n-1}$ describes the orbit segments that are non-degenerate from the $(p+1)$-st collision up to the $(n-1)$-st one (inclusive). The reference time (the upper index of the initial data) is now $p+1$. Of course, we do not want to exclude just the relevant singularities! The inductive hypothesis of 4.11 (i. e. the validity of (II)–(III) for all smaller values of $n-p$) then guarantees that the intersections of the considered degeneracies with $D_{p+1,n-1}$ are nonvoid, and the proof works.

It is obvious that after these successive substitutions $m_3=0,\dots,m_N=0$, $L=0$ we can typically avoid the unwanted degeneracies of the collisions $\sigma_{p+1},\dots,\sigma_{n-1}$, see also the proof of (I).

So far we have proved (by assuming an indirect assumption) that Φ_p and Ψ_n are globally identical even under the side conditions $m_3=\dots=m_N=L=0$. In the case $\sigma_p=\sigma_n$ $(=\{1,2\})$, the identity of Φ_p and Ψ_n under the side conditions $m_3=\dots=m_N=L=0$ is an assertion completely in terms of the now independently evolving two-particle subsystem $\{1,2\}$! We will show that in such a complexified, two-particle algebraic system the tangential degeneracy $\delta(\omega)=0$ of the collision σ_n cannot coincide with the degeneracy $\|v_1^p-v_2^p\|^2=0$, $1\le p\le n-1$. (Now the σ's only denote the collisions of the two-ball system $\{1,2\}$.) The proof will be a simple induction on the positive integer $n-p$. For $n-p=1$ the statement is obviously true. Assume now that $n-p=a>1$, and the assertion has been proven for all smaller values of $n-p$. According to our inductive hypothesis, for a typical element ω of the σ_n-degeneracy $\delta(\omega)=0$ we have that $\left\|v_1^{p+1}(\omega)-v_2^{p+1}(\omega)\right\|^2\neq 0$. On the other hand, the collision σ_{p+1} is known to preserve the quantity $\|v_1-v_2\|^2$ (the velocity reflection is complex orthogonal), and, therefore, we get that $\|v_1^p(\omega)-v_2^p(\omega)\|^2\neq 0$, as well.

The obtained contradiction finishes the discussion of 4/a in the case $\sigma_p=\sigma_n$.

On the other hand, in the case $\sigma_p\neq\sigma_n$ the identity of Φ_p and Ψ_n under the side conditions $m_3=\dots=m_N=L=0$ is impossible for the following reason: In the case $m_3=\dots=m_N=0$ the motion of the particles $\{3,4,\dots,N\}$ has absolutely no effect on the time evolution of the $\{1,2\}$ subsystem. However, the equation defining Φ_p is obviously sensitive of the kinetic data (position and velocity) of the j-th ball, while the equation $\delta(\omega)=0$ is written fully in terms of the subsystem $\{1,2\}$. The obtained contradiction proves (II) in the case $\sigma_p\neq\sigma_n$.

4/b: Proof of (III) by using the inductive hypothesis. Assume now, besides the inductive hypothesis, the precondition of (III). The proof of (III) will be split into three parts:

Case 4/b/1: $\sigma_p\neq\sigma_n$. In this case the globalization/substitution argument from the proof of (II) can be repeated almost word-by-word: The substitution $m_3=\dots=m_N=L=0$ again shows the impossibility of the identity $\Psi_p=\Psi_n$ the same way as in the proof of (II).

Case 4/b/2: $\sigma_p=\sigma_n$ and there is a k, $p<k<n$, such that $\sigma_k\cap\sigma_n\neq\emptyset$ and $\sigma_k\neq\sigma_n$.

Again argue by contradiction, i. e. assume the local, and therefore also global, identity of Ψ_p and Ψ_n. By re-labeling the particles, we can assume that $\sigma_p=\sigma_n=\{1,2\}$ and $\sigma_k=\{2,3\}$.

Apply again the globalization/substitution argument from the proof of (II) with the substitutions $m_4=m_5=\dots=m_N=L=0$. The assumed identity of Ψ_p and Ψ_n now implies that even in the 3-particle system $\{1,2,3\}$ (having a time evolution with Property (A)) the tangential singularities at σ_p and σ_n globally coincide. This 3-particle system $\{1,2,3\}$ also has the property $L=0$, which amounts to $a_l=0$ for $l=1,\dots,n$. Obviously, we can assume that $k=p+1$, i. e. $\sigma_{p+1}=\{2,3\}$. Case 4/b/2 will be taken care of by our

Auxiliary Lemma. *Suppose that the triple $(\Sigma,0,\vec{\tau})$ enjoys Property (A), $N=3$, $\sigma_p=\sigma_n=\{1,2\}$ and, say, $\sigma_{p+1}=\{2,3\}$. Then the tangential singularities at σ_p and σ_n (i. e. Ψ_p and Ψ_n) can not locally (or, equivalently, globally) coincide.*

Proof. Consider first the following, essentially two-dimensional and completely real limiting set-up: $m_1=m_2=1$, $m_3=0$, $\tilde{q}_1^p=0$, $\tilde{q}_2^p=2r\cdot e_1$, $\tilde{q}_3^p=-2r\cdot e_2$, $v_1^p=v_2^p=0$, $v_3^p=e_1+\varepsilon e_2$, where $e_1=(1,0,\dots,0)\in\mathbb{R}^\nu$, $e_2=(0,1,0,\dots,0)\in\mathbb{R}^\nu$, and $0<\varepsilon<<1$ is a small, positive parameter. The structure of the symbolic sequence $\Sigma=(\sigma_1,\dots,\sigma_n)$ is such that between neighboring $\{1,2\}$-collisions we can find alternating sequences of pure $\{1,3\}$ and $\{2,3\}$ blocks. By using Property (A) of $(\Sigma,0,\vec{\tau})$ and the entire inductive hypothesis of 4.11, we can now construct a $(\Sigma,0,\vec{\tau})$ orbit segment

$$\omega=\{\tilde{q}_i^\alpha,\, v_i^\alpha \mid i=1,2,3;\ \alpha=1,\dots,n\}$$

by extending the fixed initial values $m_1=m_2=1$, $m_3=0$, $\tilde{q}_1^p=0$, $\tilde{q}_2^p=2r\cdot e_1$, $\tilde{q}_3^p=-2r\cdot e_2$, $v_1^p=v_2^p=0$, $v_3^p=e_1+\varepsilon e_2$ in such a way that

(i) at every collision $\sigma_j=\{2,3\}$ $(p<j<n)$ always the smaller root τ_j is selected out of the two distinct, real roots;
and

(ii) at every collision $\sigma_j=\{1,3\}$ $(p<j<n)$ always the greater root τ_j is selected out of the two distinct, real roots.
(We note here that, obviously, the above constructed orbit segment ω is a highly degenerate one because of the degenerate $\{1,2\}$-collisions. However, we will transform ω into a nondegenerate segment (except one tangential collision at σ_p) by the upcoming perturbation.)

Let us now define the following, cumulative velocity exchanges of the 3rd ball:

$$\Delta V_1=\sum_{\substack{j\\ p<j<n\\ \sigma_j=\{1,3\}}}\left(v_3^{j-1}-v_3^j\right),\quad \Delta V_2=\sum_{\substack{j\\ p<j<n\\ \sigma_j=\{2,3\}}}\left(v_3^{j-1}-v_3^j\right).$$

An easy geometric inspection shows that $\langle\Delta V_1;e_1\rangle=\eta_1(\varepsilon)\ge 0$ and $\langle\Delta V_2;e_1\rangle=\eta_2(\varepsilon)>0$. (The equation $\eta_1(\varepsilon)=0$ can only hold if $\sigma_j\neq\{1,3\}$ for all j, $p<j<n$.) The point is that at least one of the quantities $\eta_1(\varepsilon)$ and $\eta_2(\varepsilon)$ is nonzero! Therefore, by choosing some *typical* complex perturbations (*typical* in the codimension-one manifold describing the tangential collisions at σ_p) $m_1\approx 1$, $m_2\approx 1$, $m_3\approx 0$, $\tilde{q}_1^p\approx 0$, $\tilde{q}_2^p\approx 2r\cdot e_1$, $\tilde{q}_3^p\approx -2r\cdot e_2$, $v_1^p\approx 0$, $v_2^p\approx 0$, $v_3^p\approx e_1+\varepsilon e_2$ in such a way that

(α) the tangential singularity holds for σ_p;

(β) no other singularity occurs at $\sigma_1,\dots,\sigma_n$ on this perturbed orbit segment (This can be achieved by the inductive hypotheses (I)–(III) of 4.11.);

(γ) the perturbation is small enough, compared to the already small quantities $\varepsilon > 0$ and $\eta_2(\varepsilon) > 0$.

Then the condition (γ), along with $\langle \Delta V_2; e_1 \rangle = \eta_2(\varepsilon) > 0$ and the possibility of freely choosing $m_1 \approx 1$ and $m_2 \approx 1$ together imply that our perturbation leads out of the tangential degeneracy at σ_n. This proves that the tangential degeneracies at σ_p and σ_n can not globally (and, therefore, not even locally) coincide.
Hence the auxiliary lemma follows. □

Our last outstanding task is the handling of

Case 4/b/3: $\sigma_p = \sigma_n$ $(= \{1,2\})$, and for every $k \in \{p, \dots, n\}$ it is true that either $\sigma_k = \sigma_n$ or $\sigma_k \cap \sigma_n = \emptyset$.

In this case *not* all adjustment vectors a_k with $\sigma_k = \sigma_n$ $(p \le k \le n)$ are the same.

Again argue by contradiction, and assume that Ψ_p and Ψ_n locally coincide. By applying the globalization/substitution argument from the proof of (II) with the substitutions $m_3 = \dots = m_N = 0$, we get that in the 2-ball system $\{1, 2\}$ the tangential singularities Ψ_p and Ψ_n are globally the same. (Actually, in this case it is a bit frivolous thing to speak about the substitutions $m_3 = \dots = m_N = 0$, since there is no interaction between the balls $\{1, 2\}$ and $\{3, \dots, N\}$ during the collisions $\sigma_p, \dots, \sigma_n$.) The discussion of Case 4/b/3 and, therefore, the entire proof of Main Lemma 4.11 will be finished as soon as we prove our

Sublemma. *Suppose that $N = 2$, the triple $(\Sigma, \mathcal{A}, \vec{\tau})$ enjoys Property (A), and not all adjustment vectors $a_1, \dots, a_n$ of $\mathcal{A}$ are the same. Then Ψ_1 and Ψ_n can not locally (or, equivalently, globally) coincide.*

Proof. (A sketch).

1. We will construct a one-parameter family

$$\omega(s) \in \tilde{\Omega}\,(\Sigma, \mathcal{A}, \vec{\tau}) = \tilde{\Omega}, \qquad (s \in I \subset \mathbb{R}),$$

(I is an open and bounded interval of $\mathbb{R}$) of $(\Sigma, \mathcal{A}, \vec{\tau})$-orbit segments in such a way that this bunch of trajectories will explicitly show that Ψ_1 and Ψ_n are not the same. In fact, we will see that Ψ_1 and Ψ_n intersect each other transversally at a large collection of points of intersection.

2. Throughout the construction of the family $\{\omega(s) |\, s \in I\}$ we shall follow these guiding principles:

(a) The dimension ν in this construction will be two: The transversality of Ψ_1 and Ψ_n in a two-dimensional section $\tilde{q}_i^t(s) \in \mathbb{C}^2 \subset \mathbb{C}^\nu$, $v_i^t(s) \in \mathbb{C}^2 \subset \mathbb{C}^\nu$ clearly implies the transversality in the general set-up $\nu \ge 2$;

(b) All data $\tilde{q}_i^t(s) \in \mathbb{R}^2$, $v_i^t(s) \in \mathbb{R}^2$, $m_1(s) = m_2(s) = 1$, $r = 1/2$, $L(s) = L >> 1$ will be real, the side length L of the 2-torus being a very large, fixed positive constant L;

(c) Due to the translation invariance of the system, we will only deal with the relative motion

$$q^t(s) := \tilde{q}_1^t(s) - \tilde{q}_2^t(s) \in \mathbb{R}^2,$$

$$v^t(s) := v_1^t(s) - v_2^t(s) \in \mathbb{R}^2, \quad \left\|v^t(s)\right\|^2 = 1,$$

$(s \in I)$;

(d) In virtue of the condition $\sum_{i=1}^n ||a_i - a_1||^2 > 0$, by cutting off a suitable starting segment of Σ (if necessary), we can assume that $0 = a_1 \neq a_2$;

(e) The construction of the family $\omega(s)$ will be such that for every $k = 2, 3, \dots, n$ the polynomial (3.8) has two distinct, nonnegative real roots, and we will always take the greater one as τ_k out of these two roots. The inductive assumption on the irreducibility of these polynomials (3.8) over the field $\mathbb{K}_{k-1}$ (They are, in fact, irreducible, unless they have to be reducible, because of the repetitive adjustment vectors. This is part of the inductive hypothesis of 4.11.) and Property (A) of $(\Sigma, \mathcal{A}, \vec{\tau})$ together ensure that the time evolution of $\omega(s)$ will follow the algebraic pattern encoded in $(\Sigma, \mathcal{A}, \vec{\tau})$;

(f)

$$q^0(s) := (\cos(\phi_0 + s),\, \sin(\phi_0 + s)) + (1 - s) \cdot (-\sin(\phi_0 + s),\, \cos(\phi_0 + s)),$$

$$v^0(s) := (-\sin(\phi_0 + s),\, \cos(\phi_0 + s)),$$

where the ray $\left\{q^0(0) + tv^0(0) | \, t \geq 0\right\}$ passes through the point $L \cdot a_2$.

The above constructed family $\left\{q^0(s),\, v^0(s) | \, |s| < \epsilon_0,\ s \in \mathbb{R}\right\}$ is a so called convex family of rays, or convex wave front, see, for example, [W(1986)]. This means that

$$\left\langle \frac{d}{ds} q^0(s);\, v^0(s) \right\rangle = 0, \quad \left\langle \frac{d}{ds} q^0(s);\, \frac{d}{ds} v^0(s) \right\rangle > 0.$$

The time evolution of such diverging families of trajectories is pretty well discussed in the literature, see [W(1986)]. It is shown in that paper (see Section 3, especially Theorem 3 there) that our two-dimensional billiard with *inner reflections* at the circles

$$C_a := \left\{q \in \mathbb{R}^2 | \, d(q,\, L \cdot a) = 1\right\}, \quad (a \in \mathbb{Z}^2)$$

has the so called convex scattering property, i. e. every such convex family of rays (with not too big curvature, see [W(1986)]) focuses after any inner reflection at C_a not later than when it reaches the midpoint of the chord of C_a along which it is traveling after the considered reflection.

4. Plainly, the trajectory $\omega(s)$ intersects the circle C_0 tangentially in negative time $\tau_1(s) = s - 1$, that is, the convex family of rays $\left\{q^0(s),\, v^0(s) | \, |s| < \epsilon_0\right\}$ is the "synchronized" family of trajectories belonging to Ψ_1. We recall that $r = 1/2$ and $a_1 = 0 \in \mathbb{Z}^2$.

5. The *inner reflections* at the circles $C_{a_2}, \dots, C_{a_n}$ (yet to be constructed) will be in good harmony with the principle 2/e above.

6. If L is large enough, then there is a small, open subinterval $I_2 \subset (-\epsilon_0, \epsilon_0)$ (even the closure of I_2 will be a subset of $(-\epsilon_0, \epsilon_0)$) such that the rays $\{q^0(s) + tv^0(s) | t \geq 0\}$ intersect the circle C_{a_2} in two distinct points for $s \in I_2$, while these rays are tangent to C_{a_2} for $s \in \partial I_2$. For $s \in I_2$ we elongate the above ray up to the second point of intersection with the circle C_{a_2}, and this way obtain the orbit segment up to the first collision with that circle.

7. Let $a_2 = a_3 = \cdots = a_{k(2)}$ $(2 \leq k(2) \leq n)$, where either $a_{k(2)+1} \neq a_{k(2)}$ or $k(2) = n$. Suppose that $k(2)$ is still less than n. By inspecting the geometry of the sequence of inner reflections $\sigma_2, \dots, \sigma_{k(2)}$ at the circle C_{a_2} it is easy to see that the outgoing velocities $v^{k(2)}(s)$ $(s \in I_2)$ sweep out the entire space $\mathbb{S}^1$ of velocity directions. Therefore, there is an even smaller interval I_3, $\mathrm{Cl}(I_3) \subset I_2$, such that the rays $\{q^{k(2)}(s) + tv^{k(2)}(s) | t \geq 0\}$ intersect the circle $C_{a_{k(2)+1}}$ in two distinct points for $s \in I_3$, while these rays are tangent to $C_{a_{k(2)+1}}$ for $s \in \partial I_3$. (Here $q^{k(2)}(s)$ and $v^{k(2)}(s)$ denote the position and the outgoing velocity at the reflection $\sigma_{k(2)}$ taking place at time $\sum_{i=1}^{k(2)} \tau_i$.)

For $s \in I_3$ we again elongate the above ray up to the second point of intersection with the circle $C_{a_{k(2)+1}}$, and this way obtain the orbit segment up to the first collision with that circle.

8. Now we keep going with the iterative construction of the orbit segments $\omega(s) \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ described in 7 above, by also constructing the nested sequence of open intervals $I_2 \supset I_3 \supset \cdots \supset I_l := I$, where $k(l) = n$. The two orbit segments $\{\omega(s) | \, s \in \partial I\}$ are now obviously in the transversal intersection of Ψ_1 and Ψ_n. $\square$

This finishes the proof of Main Lemma 4.11, as well. $\square$

Corollary 4.11/d. For every triple $(\Sigma, \mathcal{A}, \vec{\tau})$ with Property (A) and $N \geq 3$, there exists at least one derived scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$. $\square$

The next lemma establishes the link between the orbit segments of $(\Sigma, \mathcal{A}, \vec{\tau})$ and $(\Sigma', \mathcal{A}', \vec{\tau}')$. It is a direct consequence of Remark 4.10/a (actually, of the fact that the reducibility $\mathbb{K}_{l-1} = \mathbb{K}_l$ is inherited to the $(\Sigma', \mathcal{A}', \vec{\tau}')$-dynamics during the substitution $m_N = 0$) and part (2) of the definition of the derived schemes.

Lemma 4.12. *Fix a combinatorial scheme $(\Sigma, \mathcal{A}, \vec{\tau})$ satisfying Property (A) and an arbitrary scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$ derived from it. Then, for almost every choice of the vector of data $(\tilde{q}_N^0, v_N^0) \in \mathbb{C}^\nu \times \mathbb{C}^\nu$*

(1) *The subset $\{(x', \tilde{q}_N^0, v_N^0, m_N = 0) | \, x' \in \mathbb{C}^h\} \cap D(\Sigma, \mathcal{A}, \vec{\tau})$ is open and dense in the submanifold $\{(x', \tilde{q}_N^0, v_N^0, m_N = 0) | \, x' \in \mathbb{C}^h\}$ (here $h = (2\nu + 1)(N - 1) + 1$);*

(2) *For a nonempty, open set of orbit segments $\omega' \in \tilde{\Omega}(\Sigma', \mathcal{A}', \vec{\tau}')$ there exists a trajectory segment $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ for which $m_N(\omega) = 0$, $trunc(\omega) = \omega'$, and $(\tilde{q}_N^0(\omega), v_N^0(\omega)) = (\tilde{q}_N^0, v_N^0)$;*

(3) *For an open and dense set of orbit segments $\omega' \in \tilde{\Omega}(\Sigma', \mathcal{A}', \vec{\tau}')$ there exists a trajectory segment $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ for which $m_N(\omega) = 0$, $trunc(\omega) = \omega'$, and $(\tilde{q}_N^0(\omega), v_N^0(\omega)) = (\tilde{q}_N^0, v_N^0)$.*

(Here "trunc" denotes the cancellation of all kinetic data of the N-th ball with zero mass.)

Proof. Statement (1) is an obvious consequence of 4.11/(I) and the Fubini theorem. The second assertion follows immediately from the definition of the derived schemes, see also Remark 4.10/a. For demonstrating (3), we select and fix a typical pair $(\tilde{q}_N^0, v_N^0)$ according to (1) of 4.12. Intersect the open and dense subset of $\mathbb{C}^h$ – featuring in (1) – with the open and dense domain $D(\Sigma', \mathcal{A}', \vec{\tau}')$ and obtain the open and dense intersection $D' \subset \mathbb{C}^h$. Suppose now that $\omega' \in \tilde{\Omega}(\Sigma', \mathcal{A}', \vec{\tau}')$, and the vector $x' = x(\omega')$ of the initial variables of ω' belongs to D'. (This is an open and dense condition on ω'.) Launch the $(\Sigma, \mathcal{A}, \vec{\tau})$-dynamics with the initial data $(x', \tilde{q}_N^0, v_N^0, m_N = 0) \in D(\Sigma, \mathcal{A}, \vec{\tau})$ and try to continue the $(\Sigma, \mathcal{A}, \vec{\tau})$-dynamics by also preserving the time evolution of the given $\omega' \in \tilde{\Omega}(\Sigma', \mathcal{A}', \vec{\tau}')$. Remark 4.10/a (more precisely, the inheritance of the reducibility proved there) and the definition of the derived schemes (especially (2) thereof) ensure that the continuation of the $(\Sigma, \mathcal{A}, \vec{\tau})$-dynamics is, indeed, possible by also preserving the given time evolution of ω'.
Hence Lemma 4.12 follows. $\square$

The next lemma will use

Definition 4.13/a. *Suppose that two indices $1 \le p < q \le n$ and two labels of balls $i, j \in \{1, \dots, N\}$ are given with the additional requirement that if $i = j$, then $i \in \bigcup_{l=p+1}^{q-1} \sigma_l$. Following the proof of Lemma 4.2, denote by $Q_1(\vec{x}), Q_2(\vec{x}), \dots, Q_\nu(\vec{x})$ ($\vec{x} \in \mathbb{C}^{(2\nu+1)N+1}$) the polynomials with the property that for every vector of initial data $\vec{x} \in D(\Sigma, \mathcal{A}, \vec{\tau})$ and for every k, $k = 1, \dots, \nu$, the following equivalence holds true:*

$$\left(\exists\, \omega \in \tilde{\Omega} \text{ such that } \vec{x}(\omega) = \vec{x} \;\&\; \left(v_i^p(\omega)\right)_k = \left(v_j^{q-1}(\omega)\right)_k\right) \iff Q_k(\vec{x}) = 0.$$

Lemma 4.13. *Assume that the combinatorial-algebraic scheme $(\Sigma, \mathcal{A}, \vec{\tau})$ has Property (A), see Definition 3.31, and use the assumptions and notations of the above definition.*

We claim that at least one of the polynomials $Q_1, \dots, Q_\nu$ is nonzero, i. e. $v_i^p(\omega) \ne v_j^{q-1}(\omega)$ for almost every $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$, see also the "dichotomy corollary" before Lemma 4.7.

Proof. Induction on the number $N \ge 2$.

1. Base of the induction, $N = 2$: First of all, by performing the substitution $L = 0$, we can annihilate all adjustment vectors, see the proofs of parts 4/a–4/b in Main Lemma 4.11, in accordance with statement (VI) of that lemma. Then an elementary inspection shows that for any selection of *positive* masses (m_1, m_2), indeed, $v_i^p(\omega) \ne v_j^{q-1}(\omega)$ almost surely in the section $\tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau}, \vec{m})$ of $\tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ corresponding to the selected masses, since any trajectory segment of a two-particle system with positive masses and $\mathcal{A} = 0$ has a very nice, totally real (and essentially two-dimensional) representation in the relative coordinates of the particles: the

consecutive, elastic bounces of a point particle moving uniformly inside a ball of radius $2r$. Therefore, the statement of the lemma is true for $N = 2$.

We note here that there are merely three points in the entire proof of Key Lemma 4.1 where we need to explicitly use Property (A): The case $N = 2$, $i = j$ in the proof of this lemma, the Auxiliary Lemma, and the Sublemma in the proof of Main Lemma 4.11. Whenever a collision of type $\{1, 2\}$ is repeated an even number of times with the same adjustment vector and zero time slots, the resulting velocity change is identically zero, because an even power of the same velocity reflection is applied. (See Example I at the end of this section.)

2. Assume that $N \geq 3$ and the lemma has been proved for all smaller numbers of balls. By re-labeling the particles, if necessary, we can achieve that

(i) $N \neq i$, $N \neq j$ and

(ii) if $i = j$, then the ball i has at least one collision between σ_p and σ_q with a particle different from N.

For the fixed combinatorial scheme $(\Sigma, \mathcal{A}, \vec{\tau})$, select a scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$ derived from it. By applying the inductive hypothesis to $(\Sigma', \mathcal{A}', \vec{\tau}')$, we get that for almost every $\omega' \in \tilde{\Omega}\,(\Sigma', \mathcal{A}', \vec{\tau}')$ the inequality $v_i^p(\omega') \neq v_j^{q-1}(\omega')$ holds. Moreover, by virtue of Lemma 4.12/(1), for almost every $(\tilde{q}_N^0,\, v_N^0) \in \mathbb{C}^\nu \times \mathbb{C}^\nu$ one has $(x',\, \tilde{q}_N^0,\, v_N^0,\, m_N = 0) \in D\,(\Sigma, \mathcal{A}, \vec{\tau})$ for a dense, open subset of $x' \in \mathbb{C}^h$. Let us fix one such pair $(\tilde{q}_N^0,\, v_N^0)$. By Lemma 4.12/(2), for a non-empty, open set of $\omega' \in \tilde{\Omega}\,(\Sigma', \mathcal{A}', \vec{\tau}')$ it is true that $\big(x(\omega'),\, \tilde{q}_N^0,\, v_N^0,\, m_N = 0\big) \in D\,(\Sigma, \mathcal{A}, \vec{\tau})$, $v_i^p(\omega') \neq v_j^{q-1}(\omega')$, and there exists an $\omega \in \tilde{\Omega}\,(\Sigma, \mathcal{A}, \vec{\tau})$ with $m_N(\omega) = 0$ and $\mathrm{trunc}(\omega) = \omega'$.

The assertion of the lemma now follows for the N-ball systems since the bare existence of a single orbit segment with the above non-equality implies that not all polynomials $Q_l(\vec{x})$ are zero. $\square$

Lemma 4.14. *Assume that the combinatorial scheme $(\Sigma, \mathcal{A}, \vec{\tau})$ satisfying Property (A) is given for the N ($N \geq 3$) ball system $\{1, 2, \dots, N\}$ and, furthermore, there are two integers $1 \leq p < q \leq n$ such that $N \in \sigma_p \cap \sigma_q$, $N \notin \sigma_j$ for $p < j < q$, and if $\sigma_p = \sigma_q$ then we require additionally that there exists an index j, $p < j < q$, such that $\sigma_p \cap \sigma_j \neq \emptyset$. Denote by $(\Sigma', \mathcal{A}', \vec{\tau}')$* **an arbitrary derived scheme** *of $(\Sigma, \mathcal{A}, \vec{\tau})$.*

Denote by $P_1(\vec{x}), \dots, P_s(\vec{x})$ the canonically determined complex polynomials, the simultaneous vanishing of which is the consequence of non-sufficiency of the $(\Sigma, \mathcal{A}, \vec{\tau})$ orbit segments in the sense of Lemma 4.2. We claim that if all these polynomials are zero, then all the analogous polynomials associated with the derived scheme $(\Sigma', \mathcal{A}', \vec{\tau}')$ are also zero.

In other words, if all $(\Sigma, \mathcal{A}, \vec{\tau})$-orbit segments are non-sufficient, then the same is true for all $(\Sigma', \mathcal{A}', \vec{\tau}')$-orbit segments, see also the corollary at the end of the proof of Lemma 4.2.

Proof. Assume that the assertion is false for the $(\Sigma', \mathcal{A}', \vec{\tau}')$-orbit segments, that is, there exists at least one sufficient $(\Sigma', \mathcal{A}', \vec{\tau}')$-orbit segment or, equivalently, by Lemma 4.7, almost all $(\Sigma', \mathcal{A}', \vec{\tau}')$-orbit segments are sufficient.

By putting together Lemmas 4.12/(1-2), 4.13 and 4.8, we obtain a sufficient $(\Sigma, \mathcal{A}, \vec{\tau})$-orbit. Indeed, for typical elements $x' \in D\,(\Sigma', \mathcal{A}', \vec{\tau}')$ we have that

$$(\dagger) \qquad \begin{array}{c} \text{for all } \ \omega' \in \tilde{\Omega}(\Sigma', \mathcal{A}', \vec{\tau}') \ \text{ with } \ x(\omega') = x' \\ \omega' \text{ is sufficient and } \ v^p_{i_p}(\omega') \neq v^{q-1}_{i_q}(\omega'). \end{array}$$

By Lemma 4.12/(1) for almost every $(\tilde{q}^0_N, v^0_N) \in \mathbb{C}^\nu \times \mathbb{C}^\nu$ it is true that for a dense open subset of $x' \in \mathbb{C}^h$

$$(\dagger\dagger) \qquad (x',\, \tilde{q}^0_N,\, v^0_N,\, m_N = 0) \in D\,(\Sigma, \mathcal{A}, \vec{\tau})\,.$$

Thus, for almost every $(\tilde{q}^0_N, v^0_N) \in \mathbb{C}^\nu \times \mathbb{C}^\nu$ we have the following: For a dense, open subset of $x' \in \mathbb{C}^h$ ($\dagger$) and ($\dagger\dagger$) hold. Further the velocity $v^{q-1}_N(\omega) = v^p_N(\omega)$ can be chosen in such a *typical* manner that it be not contained by the complex line

$$v^p_{i_p}(\omega') + \lambda \cdot \left(v^{q-1}_{i_q}(\omega') - v^p_{i_p}(\omega') \right) \qquad (\lambda \in \mathbb{C}),$$

so we can achieve that condition (2) of Lemma 4.8 be satisfied. (The possibility of the desired choice for $v^p_N(\omega) = v^{q-1}_N(\omega)$ is easily seen if we represent our variables at the pth collison, cf. the proof of Lemma 3.19.)

By Lemma 4.12/(1-2), all this implies that for a typical $(\tilde{q}^0_N, v^0_N) \in \mathbb{C}^\nu \times \mathbb{C}^\nu$ there exists a nonempty, open set G of orbit segments $\omega' \in \tilde{\Omega}(\Sigma', \mathcal{A}', \vec{\tau}')$, with $(x(\omega'),\, \tilde{q}^0_N,\, v^0_N,\, m_N = 0) \in D\,(\Sigma, \mathcal{A}, \vec{\tau})$, such that for every $\omega' \in G$ there exists an $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ satisfying $m_N(\omega) = 0$, $\mathrm{trunc}(\omega) = \omega'$, $(\tilde{q}^0_N(\omega),\, v^0_N(\omega)) = (\tilde{q}^0_N,\, v^0_N)$, and, according to Lemma 4.8, ω is sufficient. Hence Lemma 4.14 follows. $\square$

The next, purely combinatorial lemma is the last ingredient of the inductive proof of the key lemma.

Lemma 4.15. *Define the sequence of positive numbers $C(N)$ recursively by taking $C(2) = 1$ and $C(N) = \frac{N}{2} \cdot max\{C(N-1); 3\}$ for $N \geq 3$. Let $N \geq 3$, and suppose that the symbolic collision sequence $\Sigma = (\sigma_1, \ldots, \sigma_n)$ for N particles is $C(N)$-rich. Then we can find a particle, say the one with label N, and two indices $1 \leq p < q \leq n$ such that*

(i) $N \in \sigma_p \cap \sigma_q$,

(ii) $N \notin \bigcup_{j=p+1}^{q-1} \sigma_j$,

(iii) $\sigma_p = \sigma_q \Longrightarrow \exists j\ p < j < q \,\&\, \sigma_p \cap \sigma_j \neq \emptyset$,

and

(iv) Σ' is $C(N-1)$-rich on the vertex set $\{1, \ldots, N-1\}$.

Here, just as in the case of derived schemes, we denote by Σ' the symbolic sequence that can be obtained from Σ by discarding all edges containing N.

Proof. The hypothesis on Σ implies that there exist subsequences $\Sigma_1, \ldots, \Sigma_r$ of Σ with the following properties:

(1) For $1 \leq i < j \leq r$ every collision of Σ_i precedes every collision of Σ_j,

(2) the graph of Σ_i $(1 \leq i \leq r)$ is a tree (a connected graph without loop) on the vertex set $\{1, \ldots, N\}$,

and

(3) $r \geq C(N)$.

Since every tree contains at least two vertices with degree one and $C(N) = \frac{N}{2} \cdot \max\{C(N-1); 3\} = \frac{N}{2} \cdot b$ (b is a shorthand for $\max\{C(N-1); 3\}$) we have that there is a vertex, say the one labelled by N, such that N is a degree-one vertex of $\Sigma_{i(1)}, \dots, \Sigma_{i(t)}$, where $1 \leq i(1) < \cdots < i(t) \leq r$ and $t \geq b$.

Now $t \geq C(N-1)$ and, therefore, (iv) obviously holds.

Let $\sigma_{p'}$ the edge of $\Sigma_{i(1)}$ that contains N and, similarly, let $\sigma_{q'}$ be the edge of $\Sigma_{i(t)}$ containing the vertex N. Then the fact $t \geq 3$ ensures that the following properties hold:

(i)' $N \in \sigma_{p'} \cap \sigma_{q'}$,

(iii)' $\sigma_{p'} = \sigma_{q'} \Longrightarrow \exists j\ p' < j < q'\ \&\ \sigma_{p'} \cap \sigma_j \neq \emptyset,\ \sigma_j \neq \sigma_{p'}$.

Let σ_p, σ_q ($1 \leq p < q \leq n$) be a pair of edges $\sigma_{p'}$, $\sigma_{q'}$ ($1 \leq p' < q' \leq n$) fulfilling (i)' and (iii)' and having the minimum possible value of $q' - p'$. Elementary inspection shows that then (ii) must also hold for σ_p, σ_q. Lemma 4.15 is now proved. □

We are now able to prove the key lemma by an induction on the number $N \geq 2$. Indeed, for $N = 2$ there are no non-sufficient trajectories with initial data in $D(\Sigma, \mathcal{A}, \vec{\tau})$ and, therefore, the assertion of the key lemma is obviously true.

The inductive step $(N-1) \to N$ can be obtained by simply putting together lemmas 4.2, 4.7, 4.14 and 4.15.

The proof of the key lemma is now complete in the complex case. □

Finally, since a nonzero polynomial $P_i(\vec{x})$ ($1 \leq i \leq s$) takes nonzero values almost everywhere on the real space $\mathbb{R}^{(2\nu+1)N+1}$, we immediately obtain the validity of the real version of Key Lemma 4.1. □

We conclude this paragraph by presenting to the reader two interesting examples. The first of them shows the absolute necessity of imposing Property (A) on the discrete structure $(\Sigma, \mathcal{A}, \vec{\tau})$, see Definition 3.31, while the second one sheds light on the limitations of our algebraic approach developed in sections 3–4.

Example I. Suppose that the collision graph of Σ is a tree (a connected graph without loop) with the possiblity of repeated edges. (This allows an arbirarily large number of consecutive, connected collision subgraphs.) Assume further that the combinatorial-algebraic scheme $(\Sigma, \mathcal{A}, \vec{\tau})$ is given in such a way that it has the following property:

There exists a sequence $1 = k_1 < k_2 < \cdots < k_r < k_{r+1} = n+1$ such that

(1) $k_{j+1} - k_j$ is an even number for $j = 1, \dots, r$,

and

(2) $\sigma_p = \sigma_{k_j}$, $a_p = a_{k_j}$, and $\tau_p = 0$ for $k_j < p < k_{j+1}$, $j = 1, \dots, r$.

Plainly, the orthogonal velocity reflections corresponding to the collisions

$$\sigma_{k_j}, \sigma_{k_j+1}, \dots, \sigma_{k_{j+1}-1}$$

are the same ($j = 1, \dots, r$) and, therefore, their product is the identity operator. Thus $v_i^0(\omega) = v_i^{k_j-1}(\omega)$ for all $i = 1, \dots, N$, $j = 1, \dots, r+1$, $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$. It is an easy exercise then to show that these orbit segments ω are very far from being sufficient. Actually, $\dim_{\mathbb{C}}\{\alpha_1, \dots, \alpha_n\} = N-1$ and $\dim_{\mathbb{C}}\mathcal{N}(\omega) = \nu + N - 1$, see also (4.4).

Example II. Suppose now that the $C(N)$-rich symbolic sequence Σ only contains the collisions of type $\{i, i+1\}$ for $i = 1, \dots, N-1$, i. e. the collision graph is the simple path of length N with the allowance of repeated edges. Consider the set X of orbit segments $\omega \in \tilde{\Omega}\,(\Sigma, \mathcal{A}, \vec{\tau})$ with the side conditions $m_1 = m_3 = 0$. (Of course, $m_2 \neq 0$.) It is easy to see that the orbit segments from X are *never* sufficient: the vectors

$$(\delta\tilde{q}_1^0, \dots, \delta\tilde{q}_N^0) = \left(\lambda \cdot (v_1^0 - v_2^0), 0, \dots, 0\right)$$

($0 \neq \lambda \in \mathbb{C}$) are nontrivial neutral vectors. The interesting feature of this example is that the set X has two codimensions in $\tilde{\Omega}\,(\Sigma, \mathcal{A}, \vec{\tau})$, and the codimension of X decreases to one at the crucial step of the induction when we perform the substitution $m_1 = 0$. (In earlier notations $m_N = 0$.)

5. Proof of the Main Theorem

Richness will throughout be understood in the sense of Defintion 2.5 by choosing $C := C(N)$ as prescribed by Key Lemma 4.1 (see the Remark 2 after the lemma).

According to the results of [V(1979)], [G(1981)] and [B-F-K(1995)], in a semi-dispersing billiard system there are no orbits with a finite accumulation point of collision moments. Thus, it is sufficient to concentrate our attention to the set M^0 of orbits containing no singular collision, since it is well-known and easy to see that phase points with a singular orbit form a countable union of proper submanifolds.

Theorem 5.1. *Consider a system of N (≥ 3) particles on the L-torus $\mathbb{T}_L^\nu$ $(\nu \geq 2)$ satisfying $r \in R_0$. Let $P = \{P_1, P_2\}$ be a given, two–class partition of the N particles, where, for simplicity, $P_1 = \{1, \dots, n\}$ and $P_2 = \{n+1, \dots, N\}$ $(n < N-1)$. Then the closed set*

$$F_+ = \left\{x \in \mathbf{M} : S^{[0,\infty)}x \textit{ is partitioned by } P\right\}$$

has measure zero.

This statement would, in fact, allow to fix $C(N)$ (the number of consecutive, connected collision graphs required for sufficiency) arbitrarily large.

Proof. Our reasoning is reminiscent to the proofs of Theorem 5.1 of [Sim(1992)-I] and of Proposition 2.8 of [S-Sz(1995)] irrespective of the fact that the notion of center of mass has lost its sense now. The two cases $\min\{n, N-n\} \geq 2$ and $\min\{n, N-n\} = 1$ can be treated similarly, and thus we only consider the first one.

Every point $x \in \mathbf{M}$ can be characterized by the following coordinates in an essentially unique way:

$$(1)\ \ \pi_{P_1}(x) = x_1 \in \mathbf{M}_1, \qquad (2)\ \ \pi_{P_2}(x) = x_2 \in \mathbf{M}_2,$$

$$(3)\ \ C_1(x) = \tilde{q}_{n+1} - \tilde{q}_1 \in \mathbb{R}^\nu, \qquad (4)\ \ \frac{I_1(x)}{\|I_1(x)\|} \in \mathbb{S}^{\nu-1},$$

$$(5)\ \ \|I_1(x)\| \in \mathbb{R}_+, \qquad (6)\ \ E_1(x) = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2(x) \in \mathbb{R}_+.$$

where $I_1(x) = \dfrac{\sum_{i=1}^n m_i v_i(x)}{\sum_{i=1}^n m_i}$. (Non-uniqueness only arises in choosing $C_1(x)$ as an arbitrary representant of $q_{n+1} - q_1 \in \mathbb{T}_L^\nu$.) In what follows the six coordinates corresponding to the characterization given before will, in general, be denoted by $b_1, \dots, b_6$, thus it will always be assumed that $b_1 \in \mathbf{M}_1, b_2 \in \mathbf{M}_2, b_3 \in \mathbb{R}^\nu, b_4 \in \mathbb{S}^{\nu-1}, b_5, b_6 \in \mathbb{R}_+$. The relation $\mu(F_+) = 0$ will certainly follow if we show that for almost every such choices of the b_i's

$$\mu_{b_1,b_2,b_3,b_5,b_6}(F_+(b_1,b_2,b_3,b_5,b_6)) = 0, \tag{5.2}$$

where

$$F_+(b_1,b_2,b_3,b_5,b_6) = \{y_4 \in \mathbb{S}^{\nu-1}\colon\ (b_1,b_2,b_3,y_4,b_5,b_6) \in F_+\},$$

and $\mu_{b_1,b_2,b_3,b_5,b_6}$ denotes the conditional measure of μ under the conditions corresponding to fixing the values of b_1, b_2, b_3, b_5, b_6. (This conditional measure is equivalent to the Lebesgue measure on its support.)

The relation $x \in F_+$ is equivalent to saying that for every pair $i \in P_1, j \in P_2$ and every $t \geq 0$

$$\varrho\left(q_i^t(x) - q_j^t(x), L \cdot \mathbb{Z}^\nu\right) \geq 2r, \tag{5.3}$$

where $\varrho(.,.)$ denotes the euclidean distance. For simplicity, fix $i = 1$ and $j = n+1$.

Now we will be considering the subdynamics corresponding to our two-class partition (cf. the "Subsystems, decompositions" part of Section 2 in the paper [Sim(1992)-I]), and will denote them, for simplicity, by S_1 and S_2, respectively (their phase spaces are $\mathbf{M}_1$ and $\mathbf{M}_2$, of course).

It is worthwile to note here that — according to the construction, given in the introduction, of the configuration space of our hard ball system with an arbitrary mass vector — we can not talk about the absolute configuration vector of a particle but rather about the relative position $q_i^t(x) - q_j^t(x)$ for any pair of particles, only. Also, it makes sense to talk about the time-displacement $q_i^t(x) - q_i^0(x)$ for any $1 \leq i \leq N$, $t \in \mathbb{R}$ and $x \in \mathbf{M}$ since

$$q_i^t(x) - q_i^0(x) = \int_0^t v_i^s(x)ds \qquad (\mathrm{mod}\ L \cdot \mathbb{Z}^\nu).$$

Our remarks naturally apply to the subdynamics S_1 and S_2 as well.

Now, assuming for simplicity $1 < n < N-1$, a standard calculation, which is given for instance in the proof of Lemma 5.8 of [Sim(1992)-I], yields that

$$(5.4)\quad q^t_{n+1}(x) - q^t_1(x) = \int_0^{\beta t} \tilde{v}^s_{n+1}(x_2)ds - \int_0^{\alpha t} \tilde{v}^s_1(x_1)ds + C_1(x) + \tilde{I}t \pmod{L\cdot\mathbb{Z}^\nu}$$

where $\tilde{v}^s_1(x_1)$ and $\tilde{v}^s_{n+1}(x_2)$ denote the time evolution of the corresponding velocity vectors under the subdynamics S_1 and S_2, respectively and $\tilde{I}$ is the relative velocity of the "baricenters" of the second and first subsystems (this term appears since in $\mathbf{M}_1$ and $\mathbf{M}_2$ the momenta of the subsytems are scaled to be equal to 0; actually,

$$(5.5)\qquad \tilde{I} = \frac{-\mathcal{M}_1 - \mathcal{M}_2}{\mathcal{M}_2} I_1 = \frac{-\mathcal{M}_1 - \mathcal{M}_2}{\mathcal{M}_2}\|I_1\|\frac{I_1}{\|I_1\|}$$

where

$$\mathcal{M}_1 = \sum_{i=1}^{n} m_i \qquad \text{and} \qquad \mathcal{M}_2 = \sum_{i=n+1}^{N} m_i$$

are the masses of the subsystems) and, finally,

$$\alpha = \sqrt{2E_1(x) - \mathcal{M}_1\|I_1(x)\|^2},$$

$$\beta = \sqrt{1 - 2E_1(x) - \frac{\mathcal{M}_1^2}{\mathcal{M}_2}\|I_1(x)\|^2}$$

are the corresponding time scalings.

Our task is to show that the event

$$(5.6)\qquad \forall t \geq 0 \quad \varrho\left(\lambda t \frac{I_1}{\|I_1\|} + f(t), L\cdot\mathbb{Z}^\nu\right) \geq 2r$$

has measure zero for every fixed b_1, b_2, b_3, b_5, b_6 where $\lambda = (\mathcal{M}_2)^{-1}(-\mathcal{M}_1 - \mathcal{M}_2)\|I_1\|$ and

$$f(t):\ \mathbb{R} \to \mathbb{R}^\nu$$

is an arbitrary fixed function such that $f(t) = f(t, x_1, x_2, C_1, \|I_1\|, E_1)$. Actually, by (5.4),

$$f(t) = \int_0^{\beta t} \tilde{v}^s_{n+1}(x_2)ds - \int_0^{\alpha t} \tilde{v}^s_1(x_1)ds + C_1(x).$$

In (5.6), the canonical meaning of zero measure is that in $\frac{I_1}{\|I_1\|}$.

Denote by $\mathcal{L}_{2r,L}$ the lattice of balls of radius $2r$ centered at points of $L\cdot\mathbb{Z}^\nu$. Our proof of Theorem 5.1 will be based on the following well-known elementary lemma.

Lemma 5.7. *Fix a vector $\vec{n} \in \mathbb{S}^{\nu-1}$ for which at least one ratio of coordinates is irrational. Consider arbitrary hyperplanes H perpendicular to $\vec{n}$, and denote by $B_R(z)$ the $(\nu-1)$-dimensional ball of radius R in H centered at $z \in H$. Then, for a suitable $\gamma(\vec{n}) > 0$*

$$\liminf_{R\to\infty} \inf_{H} \inf_{z\in H} \frac{meas\,(B_R(z) \cap \mathcal{L}_{2r,L})}{meas\,(B_R(z))} \geq \gamma(\vec{n}). \tag{5.8}$$

Proof. The lemma easily follows from the fact that the set $B_R(z)$ modulo $\mathbb{Z}^\nu$ is "uniformly dense" in the torus $\mathbb{T}^\nu_L$, i. e. it is an $\epsilon(R)$-net in $\mathbb{T}^\nu_L$, where $\epsilon(R) \to 0$ as $R \to \infty$. This statement is, however, an easy consequence of the well-known density of the hyperplane H (mod $\mathbb{Z}^\nu$) in the torus $\mathbb{T}^\nu_L = \mathbb{R}^\nu / L \cdot \mathbb{Z}^\nu$ and the compactness of $\mathbb{T}^\nu_L$. Indeed, since being an ϵ-net is a translation invariant property, the positioning of the foot point z is irrelevant. For a given positive ϵ, we choose first a finite $\epsilon/2$-net $F \subset \mathbb{T}^\nu$, and then for every point $y \in F$ we select a radius $R_y > 0$ with the property that $B_{R_y}(z)$ is closer to y than $\epsilon/2$. The maximum value $R(\epsilon)$ of the arising finitely many radii R_y ($y \in F$) provides now an ϵ-net $B_{R(\epsilon)}(z)$. Now select and fix $\epsilon = r$. Observe that at least a fixed , positive percentage of the ball $B_{R(r)}(z)$ belongs to $\mathcal{L}_{2r,L}$. However, for every large enough R, also a fixed, positive percentage of the ball $B_R(z)$ can be covered by disjoint balls $B_{R(r)}(z_i)$.
Hence Lemma 5.7 follows. □

Assume that the statement (5.6) is not true, i. e. the measure of the subset K of $\mathbb{S}^{\nu-1}$ described by (5.6) is positive. Select then and fix a Lebesgue density point $\vec{n}$ of K with the property that at least one ratio of the components of $\vec{n}$ is irrational. Denote by $G_\varepsilon \subset \mathbb{S}^{\nu-1}$ the ball of radius ε around $\vec{n}$. By (5.8) we can choose R_0 so large that for $R \geq R_0$

$$\inf_{H} \inf_{z\in H} \frac{\mathrm{meas}\,(B_R(z) \cap \mathcal{L}_{2r,L})}{\mathrm{meas}\,(B_R(z))} \geq \frac{\gamma(\vec{n})}{2}.$$

The set $\lambda t G_\varepsilon$ can be arbitrarily well approximated by a ball of radius $\lambda t \varepsilon = R$ (R is fixed, $R \geq R_0$) in the hyperplane orthogonal to $\vec{n}$ through the point $\lambda t \vec{n} + f(t)$ if only t is sufficiently large. Consequently, if $R \geq R_0$, then by choosing t sufficiently large and at the same time putting $\varepsilon = (\lambda t)^{-1} R$, we have

$$\frac{\mathrm{meas}\,((\lambda t G_\varepsilon + f(t)) \cap \mathcal{L}_{2r,L})}{\mathrm{meas}\,((\lambda t G_\varepsilon + f(t))} \geq \frac{\gamma(\vec{n})}{4}.$$

But this inequality contradicts to the fact that $\vec{n}$ was chosen as a Lebesgue density point of the subset $K \subset \mathbb{S}^{\nu-1}$. Hence Theorem 5.1 follows. □

Return now to the demonstration of the Main Theorem. Restrict our attention to the subset M^0. From Theorem 5.1 it follows that almost every point is $C(N)$-rich whatever the value of $C(N)$ may be. Let us fix $C(N)$ according to the requirement of Key Lemma 4.1. Denote by $\hat{R} \subset M^0$ the subset of $C(N)$-rich phase points. By the real version of Key Lemma 4.1, apart from a countable union of proper analytic submanifolds of the outer geometric parameters $(\vec{m}, L)$, for every $(\Sigma, \mathcal{A}, \vec{\tau})$

satisfying Property (A), almost every non-singular orbit segment with the combinatorial scheme $(\Sigma, \mathcal{A}, \vec{\tau})$ is sufficient. Since the number of combinatorial schemes is countable, and the combinatorial schemes of actual (real) hard ball trajectories necessarily satisfy Property (A), we can conclude that, indeed, almost every orbit in $\hat{R}$ is sufficient.

At this point we will use the following simple consequence of Poincaré recurrence and the ergodic theorem.

Lemma 5.9. *(see [S-Ch(1987)]) If $S^{[a,b]}x$ is a non-singular orbit segment and it is sufficient, then, in a suitable open neighbourhood of x, all relevant Lyapunov exponents are non-zero almost everywhere.*

Adding Lemma 5.9 to what has been said before, one also obtains that for almost every phase point in $\hat{R}$, the relevant Lyapunov exponents of the flow are non-zero. Thus our Main Theorem is proved. □

Proof of the Corollary. The positivity of the ergodic components is a consequence of the Katok-Strelcyn theory (cf. [K-S(1986)]), where the underlying assumption is that the relevant Lyapunov-exponents of the system do not vanish. (We note that, in fact, the Katok-Strelcyn theory is a generalization of Pesin's theory (see [P(1977)] for hyperbolic systems with singularities, also containing billiards isomorphic to hard ball systems). □

Finally, it is worthwile to note here that, in fact, on any positive ergodic component (i. e. on any ergodic component of positive measure) the Bernoulli property of the standard billiard flow $(\mathbf{M}, \{S^t\}, \mu)_{\vec{m},L}$ follows from its hyperbolicity and weak mixing, as it has been recently shown by Chernov–Haskell and Ornstein–Weiss, cf. [C-H(1996)] and [O-W(1996)]. For a brief summary of how the K-property follows from ergodicity and hyperbolicity, see – for instance – Section 7 of [Sim(1992)-I] , or [K-S(1986)].

6. Concluding remarks

1. Had we been able to strengthen the statement of Key Lemma 4.1 from *"for almost every $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$"*, i. e. from "for every $\omega \in \tilde{\Omega}(\Sigma, \mathcal{A}, \vec{\tau})$ apart from a closed algebraic subset of codimension one" to a similar statement with an *at least two-codimensional algebraic subset*, we would have obtained the global ergodicity of the system. As shown by our earlier manuscript [S-Sz(1996)], a crucial step in getting to codimension two in the Key Lemma would be the strengthening of the last claim of Lemma 4.13 to stating $v_i^p(\omega) \neq v_j^{q-1}(\omega)$ apart from a subset of codimension two. Both lemmas would be implied by the following analogous algebraic properties: all common divisors of the polynomials

(1) $P_1, \dots, P_s$ (in case of Key Lemma 4.1);
(2) $Q_1, \dots, Q_\nu$ (in case of Lemma 4.13)

are purely homogeneous polynomials of the masses (Example II at the end of section 4 illustrastes the essential difficulty arising in the inductive proof of this claim). In this way one can formulate a *hypothetic statement about the Boltzmann-Sinai* *ergodic hypothesis: if this stronger form of Key Lemma 4.1 is true, then, in the Main Theorem, we can also claim the K-property instead of just full hyperbolicity of the system.*

2. The system of $N = 2$ hard discs (i. e. $\nu = 2$) with arbitrary masses m_1, m_2 was studied by Simányi and Wojtkowski [S-W(1989)]. They were considering the same dynamics as here, but in the larger phase space $\tilde{\mathbf{Q}} \times S^1$. By using the fact that this system is an isometric $\mathbb{T}^2$-extension of the standard billiard ball flow $(\mathbf{M}, S^{\mathbb{R}}, \mu)_{(m_1,m_2)}$, which is a planar dispersing billiard, and was known to possess the Bernoulli property since [G-O(1974)], they showed that this extension is also a Bernoulli flow as long as the ratio m_1/m_2 is irrational. If one thoroughly studies the steps of that proof, he realizes that, once the difficulties outlined in the above remark have been resolved, the methods of the present paper and [S-W(1989)] together with the results of [C-H(1996)] and [O-W(1996)] prove even the Bernoulli property of the flow on the extended phase space $\tilde{\mathbf{Q}} \times \mathcal{E}$ (without the factorization Ψ, see the introduction), provided that *not all ratios m_i/m_j are rational.* (The latter condition means precisely that the "center of mass" is defined modulo a dense subgroup of $\mathbb{T}^\nu$.)

3. In [Sz(1994)], the second author obtained necessary and sufficient conditions for the K-property of the so-called "orthogonal cylindric billiards". The subclass of cylindric billiards, introduced in [Sz(1993)], within the family of semi-dispersing billiards is of particular interest since it is this class for which the formulation of general — and at the same time in some sense constructively verifiable — necessary and sufficient conditions of the K-property seems possible. (We have seen in the proof of Lemma 2.1 that the system of N balls with an arbitrary mass vector is also isomorphic to a cylindric billiard.) Indeed, such a condition is found and shown to be necessary for the systems of N hard balls of identical masses in a forthcoming paper of the present authors [S-Sz(1997)], which is based upon the ideas from [S-Sz(1995-I)]. One apparent difficulty in establishing the sufficiency of this condition is the fact that so far no characterization of the neutral subspace of a trajectory segment of a general cylindric billiard is known (in the case of hard ball systems such a characterization was obtained by using the CPF, whose derivation, on the other hand, used the conservation of momentum, a property present in hard ball systems but absent in general cylindric billiards).

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