Random version of Sperner's theorem

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A sequence $\mathcal{E}_n$ of events holds with high probablity (w.h.p) if $\mathbb{P}(\mathcal{E}_n) \to 1$.

Markov inequality: if X is a non-negative random variable and $a > 0$ is a real number, then $\mathbb{P}(X > a) \leq \frac{\mathbb{E}(X)}{a}$. In particular, if $X_{n\,n=1}^{\infty}$ are non-negative integer-valued random variables with $\mathbb{E}(X_n) \to 0$, then $X_n = 0$ w.h.p.

Chebyshev's inequality: if X is a finite random variable, then

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq k\sigma(X)) \leq \frac{1}{k^2}.$$

In particular, if $\mathbb{E}(X_n) \to \infty$ and $\sigma^2(X_n) = o(\mathbb{E}(X_n))$ (or equivalently $\mathbb{E}(X_n^2) = (1 + o(1))\mathbb{E}^2(X_n)$), then $X_n > 0$ w.h.p.

Chernoff bound: if $X_1, X_2, \ldots, X_n$ are independent random variables with $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$, then for $X = \sum_{i=1}^n X_i$, $\mu = \sum_{i=1}^n p_i$ and $\delta > 0$ we have

- $\mathbb{P}(X > (1+\delta)\mu) \leq \exp(-\frac{\delta^2}{2+\delta}\mu)$,
- $\mathbb{P}(X < (1-\delta)\mu) \leq \exp(-\frac{\delta^2}{2}\mu)$ provided $0 < \delta < 1$.

The inequality $\mathbb{P}(\cup_{i=1}^{\infty}\mathcal{E}_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(\mathcal{E}_i)$ is often referred to as the **union bound**.

The comparability graph Q_n of $2^{[n]}$ has vertex set $2^{[n]}$ and F, G are joined by an edge if and only if $F \subset G$ or $G \subset F$. $Q_n(p)$ is the probability space of all induced subgraphs of Q_n with $\mathbb{P}(Q_n(p) = Q_n[\mathcal{F}]) = p^{|\mathcal{F}|}(1-p)^{2^n - |\mathcal{F}|}$, i.e. every subset F of $[n]$ is a member of the vertex set of $Q_n(p)$ with probability p independently of all other subsets.

Independent sets of Q_n and $Q_n(p)$ correspond to antichains in $2^{[n]}$.

Theorem 1 (Balogh et al. [1], Colares Neto et al. [2])**.** *For every $\varepsilon > 0$, there exists a constant C such that if $p \geq C/n$, then the size of the largest antichain in $Q_n(p)$ is at most $(1+\varepsilon)p\binom{n}{\lfloor n/2 \rfloor}$ w.h.p.*

The threshold $p \geq C/n$ is sharp as if $p = o(n^{-1})$, then the expected number $p^2\frac{n}{2}\binom{n}{\lfloor n/2 \rfloor}$ of pairs in the middle two levels that remain in $Q_n(p)$ is negligable compared to the number of sets $2p\binom{n}{\lfloor n/2 \rfloor}$ remaining in $Q_n(p)$ in the middle two levels. So removing one set from each such pair leaves us with an antichain of size $(2 - o(1))p\binom{n}{\lfloor n/2 \rfloor}$.

Proof. During the proof we will write $m := \binom{n}{\lfloor n/2 \rfloor}$. We will need the following result that can be easily deduced from a result of Kleitman.

Lemma 2. *If $U \subseteq V(Q_n)$ with $|U| \geq (t+\varepsilon)m$ for some $0 < \varepsilon < 1/2$ and positive integer t, then $e(U) > \varepsilon n^t \frac{|U|}{(2t)^{t+1}}$.*

Lemma 3. *Let t be a positive integer, $0 < \varepsilon < 1/(2t)^{t+1}$, and n be sufficiently large. Then there exist functions $f : \binom{V(G_n)}{\leq n^{-(t+0.9)}2^n} \to \binom{V(G_n)}{(t+1+\varepsilon)m}$ and $g : \binom{V(G_n)}{\leq (t+2)m/(\varepsilon^2 n^t)} \to \binom{V(G_n)}{(t+\varepsilon)m}$ with the following property. For any independent set I in G_n, there exist disjoint subsets $S_1, S_2 \subseteq I$ with $|S_1| \leq n^{-(t+0.9)}2^n$ and $|S_1 \cup S_2| \leq (t+2)m/(\varepsilon^2 n^t)$, such that $S_1 \cup S_2$ is disjoint from $g(S_1 \cup S_2)$, $S_2 \subseteq f(S_1)$, and $I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$.*

Proof of Lemma. We fix an arbitrary order of $V(Q_n)$ and an independent set I. We set $U_0 = V(Q_n)$, and, during a process of two phases, we update U_i and add vertices first to S_1 and then to S_2, both of which are empty at the beginning of the process. At the beginning of each step we select the vertex $u \in U_{i-1}$ with $deg_{Q_n[U_{i-1}]}(u)$ maximal, and if there are several such vertices, we take the one with minimal index in the fixed order of $V(Q_n)$. The steps in the two phases are as follows:

Phase I

- If $u \notin I$, then we let $U_i = U_{i-1} \setminus \{u\}$, and proceed to Step $i+1$ in Phase I.
- If $u \in I$ with $deg_{Q_n[U_{i-1}]}(u) \geq n^{t+0.9}$, then we add u to S_1, let $U_i = U_{i-1} \setminus N_{Q_n}[u]$, and proceed to Step $i+1$ in Phase I.
- If $u \in I$ with $deg_{Q_n[U_{i-1}]}(u) < n^{t+0.9}$, then we add u to S_1, let $U_i = U_{i-1} \setminus N_{Q_n}[u]$, define $f(S_1) = U_i$, and proceed to Step $i+1$ in Phase II.

Phase II

- If $u \notin I$, then we let $U_i = U_{i-1} \setminus \{u\}$, and proceed to Step $i+1$ in Phase II.
- If $u \in I$ with $deg_{Q_n[U_{i-1}]}(u) \geq \varepsilon^2 n^t$, then we add u to S_2, let $U_i = U_{i-1} \setminus N_{Q_n}[u]$, and proceed to Step $i+1$ in Phase II.
- If $u \in I$ with $deg_{Q_n[U_{i-1}]}(u) < \varepsilon^2 n^t$, then we add u to S_2, let $U_i = U_{i-1} \setminus N_{Q_n}[u]$, define $g(S_1 \cup S_2) = U_i$, and stop the process.

Note that by definition, the sets S_1 and S_2 are disjoint and $S_1 \cup S_2 \subseteq I \subseteq S_1 \cup S_2 \cup g(S_1 \cup S_2)$ holds.

For all vertices u added to S_1 in Step i of Phase I (except for the last one), we have $deg_{Q_n[U_{i-1}]}(u) \geq n^{t+0.9}$; therefore $|S_1| \leq 1 + 2^n/(1 + n^{t+0.9}) \leq 2^n/n^{t+0.9}$. If Phase I ends after Step i_1, then, by definition, we know that $deg_{Q_n[U_{i_1}]}(v) \leq n^{t+0.9}$ for all $v \in U_{i_1}$; thus $e(U_{i_1}) \leq \frac{1}{2}|U_{i_1}|n^{t+0.9}$. Lemma 2 implies that $U_{i_1} = f(S_1)$ has size at most $(t + 1 + \varepsilon)m$.

For all vertices u added to S_2 in Step i of Phase II (except for the last one), we have $deg_{Q_n[U_{i-1}]}(u) \geq \varepsilon^2 n^t$, and their neighbors are never added to S_1 (they are not even in I); therefore $|S_2| \leq 1 + |f(S_1)|/(1 + \varepsilon^2 n^t)$ holds. Hence $|S_1 \cup S_2| \leq 2^n/n^{t+0.9} + (t + 1 + \varepsilon)m/(\varepsilon^2 n^t) \leq (t + 2)m/(\varepsilon^2 n^t)$. If Phase II ends after Step i_2, then, by definition we know that $deg_{Q_n[U_{i_2}]}(v) \leq \varepsilon^2 n^t$ for all $v \in U_{i_2}$; thus $e(U_{i_2}) \leq \frac{\varepsilon^2}{2}|U_{i_2}|n^t$. Lemma 2 implies that $U_{i_2} = g(S_1 \cup S_2)$ has size at most $(t + \varepsilon)m$.

We can define $f(S)$ and $g(S)$ arbitrarily for sets S that are not obtained as S_1 or $S_1 \cup S_2$ for any independent set I in the above process. We are left to prove that if we obtain S_1 or $S_1 \cup S_2$ for two different independent sets I and I', then $f(S_1)$ and $g(S_1 \cup S_2)$ do not differ in the two cases. But this follows from the fact that if for I and I' we have $S_1(I) = S_1(I')$, then Phase I is the same for I and I'. Similarly, if $S_1(I) \cup S_2(I) = S_1(I') \cup S_2(I')$, then the whole process is the same for I and I' (in particular, $S_1(I) = S_1(I'), S_2(I) = S_2(I')$). These statements can be seen by induction on i. If the two processes are identical up to Step $i - 1$ (if $i = 1$, this trivially holds), then U_{i-1} is the same for both processes, therefore the same vertex u is chosen in Step i, and since $S_1(I) = S_1(I')$ (resp. $S_1(I) \cup S_2(I) = S_1(I') \cup S_2(I')$), u is either added to S_1 (resp. S_2) in both cases or dropped in both cases. □

We prove the following more general statement of which Theorem 1 is the special case $t = 1$: for every $\varepsilon > 0$ and positive integer t, there exists a constant C such that if $p \geq \frac{C}{n^t}$, then $Q_n(p)$ does not contain an antichain larger than $(1 + \varepsilon)pt\binom{n}{\lfloor n/2 \rfloor}$ w.h.p.

It is enough to prove this statement for $\varepsilon < \frac{1}{(2t)^{t+1}}$. Let us define $C = 10^{10}\varepsilon^{-5}$ and $\varepsilon_1 = \varepsilon/4$. We will show that w.h.p. for every I that is an independent set in Q_n of size at least $(1+\varepsilon)ptm$, not all sets corresponding to vertices in I remain in $Q_n(p)$. We will use the union bound, and to bound the number of possibilities we apply Lemma 3 with ε_1 in the role of ε. For every independent set I it gives us S_1, S_2 such that

- $S_1 \in \binom{V(Q_n)}{\leq n^{-(t+0.9)}2^n}$; therefore the number of possible S_1's is at most

$$\sum_{a \leq n^{-(t+0.9)}2^n} \binom{2^n}{a}.$$

  Clearly, we have $\mathbb{P}(S_1 \subseteq Q_n(p)) = p^{|S_1|}$.

- $S_2 \in \binom{V(Q_n)}{\leq (t+2)m/(\varepsilon_1^2 n^t)}$ and $S_2 \subseteq f(S_1) \in \binom{V(G_n)}{(t+1+\varepsilon_1)m}$, so for fixed S_1 the number of possible S_2's is at most

$$\left|\binom{f(S_1)}{\leq (t+2)m/(\varepsilon_1^2 n^t)}\right| \leq \sum_{b\leq (t+2)m/(\varepsilon_1^2 n^t)} \binom{(t+2)m}{b}.$$

Also, $\mathbb{P}(S_2 \subseteq Q_n(p)) = p^{|S_2|}$.

- for fixed S_1 and S_2 the corresponding I's are all subsets of $S_1 \cup S_2 \cup g(S_1 \cup S_2)$ and contain $S_1 \cup S_2$. Let $\mathcal{E}_{S_1,S_2}$ be the event that there exists *any* I with $S_1(I) = S_1, S_2(I) = S_2$ and $|I| \geq (1+\varepsilon)ptm$, $I \subseteq Q_n(p)$ and $\mathcal{E}_{g(S_1\cup S_2)}$ be the event that $|Q_n(p) \cap g(S_1 \cup S_2)| \geq (1+\varepsilon)ptm - |S_1 \cup S_2|$ holds.

  We bound the probability of the event $\mathcal{E}_{S_1,S_2}$ by the probability of the event $\mathcal{E}_{g(S_1\cup S_2)}$. Note that
  $$(1+\varepsilon)ptm - |S_1 \cup S_2| \geq (1+\varepsilon/2)ptm$$
  and
  $$|g(S_1 \cup S_2)| \leq (t+\varepsilon_1)m \leq (1+\varepsilon/4)tm.$$
  Therefore, $|Q_n(p) \cap g(S_1 \cup S_2)|$ is binomially distributed with $\mathbb{E}(|Q_n(p) \cap g(S_1 \cup S_2)|) \leq (1+\varepsilon/4)pmt$, so by Chernoff's inequality we have
  $$\mathbb{P}(\mathcal{E}_{S_1,S_2}) \leq \mathbb{P}[|Q_n(p) \cap g(S_1 \cup S_2)| \geq (1+\varepsilon/2)ptm] \leq e^{-\varepsilon^2 pmt/100}.$$

Note that S_1, S_2, and $g(S_1 \cup S_2)$ are disjoint, so the above three events are independent; hence the probability, that for fixed S_1 and S_2 there is a corresponding large antichain, is at most $p^{|S_1|+|S_2|}e^{-\varepsilon^2 pmt/100}$. Summing up for all possible S_1 and S_2, we obtain that the probability Π, that there is an antichain in $Q_n(p)$ of size $(1+\varepsilon)ptm$, is at most

$$\sum_{0\leq a\leq n^{-(t+0.9)}2^n} \sum_{0\leq b\leq (t+2)m/(\varepsilon_1^2 n^t)} \binom{2^n}{a} p^a \binom{(t+2)m}{b} p^b e^{-\varepsilon^2 pmt/100}.$$

The assumptions $p \geq Cn^{-t}$ and $a \leq n^{-(t+0.9)}2^n$ imply $\frac{\binom{2^n}{a+1}p^{a+1}}{\binom{2^n}{a}p^a} > 1$. Similarly, $p \geq Cn^{-t}$ and $b \leq (t+2)m/(\varepsilon_1^2 n^t)$ imply $\frac{\binom{(t+2)m}{b+1}p^{b+1}}{\binom{(t+2)m}{b}p^b} > 1$, so the largest summand in the above sum belongs to the largest possible values of a and b. Therefore the above expression is bounded from above by

$$(n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t) + 1)\binom{2^n}{n^{-(t+0.9)}2^n}\binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)}$$
$$\cdot\, e^{-\varepsilon^2 pmt/100} p^{n^{-(t+0.9)}2^n} p^{(t+2)m/(\varepsilon^2 n^t)}.$$

Note that $pm = \Omega(n^{-(t+1/2)}2^n)$. Therefore

$$(n^{-(t+0.9)}2^n + 1)((t+2)m/(\varepsilon_1^2 n^t) + 1) \le e^{O(n)} \le e^{\varepsilon^2 pmt/(400)}.$$

Also, using $\binom{n}{k} \le (\frac{en}{k})^k$, we have

$$\begin{aligned}\binom{2^n}{n^{-(t+0.9)}2^n} p^{n^{-(t+0.9)}2^n} &\le (en^{t+0.9}p)^{n^{-(t+0.9)}2^n} \\ &= e^{n^{-(t+0.9)}2^n \ln n} \le e^{\varepsilon^2 pmt/(400)}.\end{aligned}$$

Finally, by $C = 10^{10}\varepsilon^{-5}$ and the monotone decreasing property of $\frac{\ln x}{x}$ when x is large enough, we have

$$\begin{aligned}\binom{(t+2)m}{(t+2)m/(\varepsilon_1^2 n^t)} p^{(t+2)m/(\varepsilon_1^2 n^t)} &\le (e\varepsilon_1^2 n^t p)^{(t+2)m/(\varepsilon_1^2 n^t)} \\ &\le e^{(t+2)mp\frac{\ln(n^t p)}{(\varepsilon^2 n^t p)}} \le e^{\varepsilon^2 pmt/(400)}\end{aligned}$$

Therefore, the probability Π is at most $e^{-\varepsilon^2 pmt/400} = o(1)$, as required. ■

Corollary 4. *For every $\varepsilon > 0$ and positive integer k, there exists a constant C such that if $p \ge C/n$, then the size of the largest k-Sperner family in $Q_n(p)$ is at most $(k+\varepsilon)p\binom{n}{\lfloor n/2 \rfloor}$ w.h.p.*

Proof. This follows from the fact that every k-Sperner family is the union of k antichains, and Theorem 1. ■

References

[1] Balogh, J., Mycroft, R., and Treglown, A. A random version of Sperner's theorem. *Journal of Combinatorial Theory, Series A 128* (2014), 104–110.

[2] Neto, M. C., and Morris, R. Maximum-size antichains in random set-systems. *Random Struct. Algorithms* (2016).