Resolution graphs of some surface singularities, I. (cyclic coverings)

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Abstract. The article starts with some introductory material about resolution graphs of normal surface singularities (definitions, topological/homological properties, etc). Then we discuss the problem of N-cyclic coverings $(X_{f,N},0)\to(X,0)$ of $(X,0)$, branched along $(\{f=0\},0)$, where $f:(X,0)\to(\mathbf{C},0)$ is the germ of an analytic function. We present non–trivial example in order to show that from the embedded resolution graph $\Gamma(X,f)$ of f it is not possible to recover the resolution graph of $(X_{f,N},0)$. The main result is the construction of a "universal covering graph" of $\Gamma(X,f)$ from the topology of the germ f, and the completely combinatorial construction of the resolution graph of $(X_{f,N},0)$ from this universal graph of f and the integer N. For this we also prove some classification theorems of "graph coverings", results which are purely graph-theoretical. In the last part, we connect the properties of the universal covering graph with the topological invariants of f, e.g. with the nilpotent part of its algebraic monodromy.

Introduction.

The present article has several goals. First of all, it is an article with some expository character, presenting several aspects of the resolution graphs of normal surface singularities. Moreover, it creates the good language and right point of view in the graph–codifications of important geometric constructions, such as cyclic coverings of surface singularities, series of singularities, degeneration of curves, etc. This is realized by the introduction of a new combinatorial object: the "covering of graphs" (section 1). We will present two main applications: the case of cyclic coverings of normal surface singularities (the present article, in the sequel: Part I), and the case of some series of singularities (a joint work with Ágnes Szilárd [30], in the sequel: Part II).

Consider a normal surface singularity $(X,0)$ and a germ of an analytic function $f:(X,0)\to(\mathbf{C},0)$. Let $X_{f,N}$ be the (normalized) cyclic N–covering of $(X,0)$ branched along $(\{f=0\},0)$. The final goal of Part I is to recover the resolution graph $\Gamma(X_{f,N})$ of $X_{f,N}$ from some invariants of the pair (X,f) and the integer N. Particular cases suggest that $\Gamma(X_{f,N})$ might be computable from the embedded resolution graph $\Gamma(X,f)$ of f and the integer N, but this is not true in general. In fact, $\Gamma(X,f)$ and N codify all the local data of the covering, but they miss some global information. This global data is in a close relationship with the monodromy representation arg_* of the Milnor fibration associated with f, which plays an important role in the global geometry of all the cyclic coverings.

In the second section we present the basic definitions and some of the (elementary) properties of embedded resolution graphs. This section also serves as an introduction to Part II. Then we discuss "local/global" properties, showing e.g. that the embedded resolution graph does not determine the representation arg_*. The additional global information of arg_* will be codified in another graph $G(X,f)$, called the "universal covering graph" of $\Gamma(X,f)$.

This already motivates the material of section 1, where a detailed presentation of the "cyclic covering of graphs" can be found. The section culminates in the classification theorems, which are indispensable in the applications.

In the third section we present an algorithm which provides the resolution graph $\Gamma(X_{f,N})$, in a purely combinatorial way, from the universal covering graph $G(X,f)$ of $\Gamma(X,f)$ and the integer N. In fact, each graph $\Gamma(X_{f,N})$ appears as a graph–covering $\Gamma(X_{f,N}) \to \Gamma(X,f)$ modified by Hirzebruch–Jung strings. The family of graphs $\{\Gamma(X_{f,N})\}_N$ behaves like a complicated series of graphs, which is coordinated by a unique graph, the universal covering of $\Gamma(X,f)$. The global nature of the construction is emphasized by non–trivial examples. On the other hand, we always stress the particular cases when the global information can be recovered from the local (i.e. when $G(X,f)$ is determined from $\Gamma(X,f)$), e.g. when the link of $(X,0)$ is a rational homology sphere.

Each section and subsection has its own introduction, where the reader can find some more guiding information (see also the introduction of Part II).

1§. Z–coverings of graphs

1.1. – In this section we present some graph–theoretical constructions. The construction of "cyclic covering of graphs" is motivated by the structure of the resolution graphs of cyclic coverings of the normal surface singularities $(X,0)$ with branch locus $(f^{-1}(0),0) \subset (X,0)$ where $f:(X,0)\to(\mathbf{C},0)$ is a germ of an analytic function (cf. 2.7.). All these resolution graphs are controlled by a single "universal" covering graph which is a $\mathbf{Z}$–covering of the embedded resolution graph of the pair $(f^{-1}(0),0)\subset(X,0)$.

1.2. Notations. For any graph Γ, we denote the set of vertices by $\mathcal{V}(\Gamma)$ and the set of edges by $\mathcal{E}(\Gamma)\subset\mathcal{V}(\Gamma)\times\mathcal{V}(\Gamma)$. If there is no danger of confusion, we denote them simply by $\mathcal{V}$ and $\mathcal{E}$.

For simplicity, we will assume that our graphs have no loops. The interested reader can easily extend all our results for graphs with loops. If $v\in\mathcal{V}$ is a fixed vertex, let $\mathcal{E}_v$ be the set of edges which have v as one of their end–points. If $|\Gamma|$ is the topological realization of the graph Γ, then we denote the rank of $H_1(|\Gamma|,\mathbf{Z})$ by c_Γ, i.e. c_Γ is the number of independent cycles of the graph Γ.

1.3. Definitions. *A morphism of graphs $p:\Gamma_1\to\Gamma_2$ consists of two maps $p_{\mathcal{V}}:\mathcal{V}(\Gamma_1)\to\mathcal{V}(\Gamma_2)$ and $p_{\mathcal{E}}:\mathcal{E}(\Gamma_1)\to\mathcal{E}(\Gamma_2)$, such that if $e\in\mathcal{E}(\Gamma_1)$ has end–points v_1 and v_2, then $p_{\mathcal{E}}(e)$ is an edge in Γ_2 with end–points $p_{\mathcal{V}}(v_1)$ and $p_{\mathcal{V}}(v_2)$. If $p_{\mathcal{V}}$ and $p_{\mathcal{E}}$ are isomorphisms of sets, then we say that p is an isomorphism of graphs.*

If Γ is a graph, we say that $\mathbf{Z}$ acts on Γ, if there are group–actions $a_{\mathcal{V}}:\mathbf{Z}\times\mathcal{V}\to\mathcal{V}$ and $a_{\mathcal{E}}:\mathbf{Z}\times\mathcal{E}\to\mathcal{E}$ of $\mathbf{Z}$ with the following compatibility property: if $e\in\mathcal{E}$ has end–points v_1 and v_2, then $a_{\mathcal{E}}(h,e)$ has end–points $a_{\mathcal{V}}(h,v_1)$ and $a_{\mathcal{V}}(h,v_2)$. The action is trivial if $a_{\mathcal{V}}$ and $a_{\mathcal{E}}$ are trivial actions.

If $\mathbf{Z}$ acts on both Γ_1 and Γ_2, then a morphism $p:\Gamma_1\to\Gamma_2$ is equivariant if the maps $p_{\mathcal{V}}$ and $p_{\mathcal{E}}$ are equivariant with respect to the actions of $\mathbf{Z}$. If additionally p is an isomorphism then it is called an equivariant isomorphism of graphs.

1.4. – The "segment graph". The simplest graph is the "segment graph" S which has two vertices $\mathcal{V} = \{v_1, v_2\}$ and one edge $\mathcal{E} = \{e = (v_1, v_2)\}$. In the sequel S will always carry the trivial action of the structure group $\mathbf{Z}$.

1.5. – The "standard blocks" of Z–coverings. A "standard block" B, which covers the "segment graph" S, can be constructed as follows.

Fix three strictly positive integers n_1, n_2 and d, and set $[n, m] = l.c.m.\{n, m\}$. Then the graph $B(n_1, n_2, d)$ consists of $n_1 + n_2$ vertices $\mathcal{V} = \{P_1^1, ..., P_{n_1}^1, P_1^2, ..., P_{n_2}^2\}$ and $d \cdot [n_1, n_2]$ edges $\mathcal{E} = \{e_1, ..., e_{d[n_1,n_2]}\}$

The end–points of the edge e_k are $P^1_{i(k)}$ and $P^2_{j(k)}$, where $i(k) \equiv k \ (mod\ n_1)$ and $j(k) \equiv k \ (mod\ n_2)$. Notice that P^1_i and P^2_j are connected by exactly d edges. The graph $B(n_1, n_2, d)$ has a natural $\mathbf{Z}$–action given by $a_{\mathcal{E}}(n, e_k) = e_l$, where $l \equiv k+n \ (mod\ d[n_1, n_2])$ and $a_{\mathcal{A}}(n, P^i_k) = P^i_l$, where $l \equiv k + n \ (mod\ n_i)$ $(i = 1, 2)$.

Consider the "segment graph" S and the trivial $\mathbf{Z}$–action on it. Then $p : B \to S$ defined by $p(P^i_j) = v_i$ $(i = 1, 2)$ and $p(e_k) = (v_1, v_2)$ is an equivariant morphism.

Actually, almost all the finite coverings of S are "standard blocks". Indeed, consider an arbitrary equivariant morphism $p : B \to S$ of graphs, where B is a finite connected graph with $\mathcal{E} \neq \emptyset$ and with a $\mathbf{Z}$–action, p is an equivariant morphism such that the restriction of the action of $\mathbf{Z}$ on $p^{-1}(v_i)$ $(i = 1, 2)$, respectively on $p^{-1}((v_1, v_2))$, is transitive. Then, it is not difficult to show that $p : B \to S$ is equivalent with the standard block $B(n_1, n_2, d)$, where $n_i\mathbf{Z}$ (respectively $d[n_1, n_2]\mathbf{Z}$) is the maximal subgroup of $\mathbf{Z}$ which acts trivially on $p^{-1}(v_i)$ $(i = 1, 2)$ (respectively on $p^{-1}((v_1, v_2))$).

1.6. – Cyclic (or Z–) coverings of graphs. Now, consider an arbitrary graph Γ. We assume that $\mathbf{Z}$ acts on Γ in a trivial way. Any edge e with end–points v_1 and v_2 determines a natural "segment subgraph" $S_e = (\{v_1, v_2\}, \{e\})$ of Γ.

1.7. Definition. *A $\mathbf{Z}$–covering, or cyclic covering, of the finite graph Γ consists of a finite graph G, that carries a $\mathbf{Z}$–action, together with an equivariant morphism $p : G \to \Gamma$ such that the restriction of the $\mathbf{Z}$–action on any set of type $p^{-1}(v)$ $(v \in \mathcal{V}(\Gamma))$, respectively $p^{-1}(e)$ $(e \in \mathcal{E}(\Gamma))$, is transitive.*

This definition can be reformulated in terms of "standard blocks" as follows. For any $v \in \mathcal{V}(\Gamma)$ (respectively edge $e \in \mathcal{E}(\Gamma)$ with end–points $\{v_1, v_2\}$), let $n_v\mathbf{Z}$ (respectively $d_e[n_{v_1}, n_{v_2}]\mathbf{Z}$) be the maximal subgroup of $\mathbf{Z}$ which acts trivially on $p^{-1}(v_i)$ $(i = 1, 2)$ (respectively on $p^{-1}(e)$). This defines a system of strictly positive integers $(\mathbf{n}, \mathbf{d}) = \Big\{\{n_v\}_{v\in\mathcal{V}(\Gamma)}; \{d_e\}_{e\in\mathcal{E}(\Gamma)}\Big\}$. This system will be called *covering data.*

1.8. Definition. *Fix a graph Γ with a trivial $\mathbf{Z}$–action, and a system of integers (covering data) $(\mathbf{n}, \mathbf{d}) = \{\{n_v\}_{v\in\mathcal{V}}; \{d_e\}_{e\in\mathcal{E}}\}$. A $\mathbf{Z}$–covering (or cyclic covering) of type $(\mathbf{n}, \mathbf{d})$ of the graph Γ consists of a graph G, that carries a $\mathbf{Z}$–action, together with an equivariant morphism $p : G \to \Gamma$ such that for any $v \in \mathcal{V}(\Gamma)$ (resp. $e \in \mathcal{E}(\Gamma)$) the set $p^{-1}(v)$ (resp. $p^{-1}(e)$) consists of n_v vertices (resp. $n_e = d_e \cdot [n_{v_1}, n_{v_2}]$ edges). Moreover, we assume that for any edge e, the subgraph $p^{-1}(S_e)$ is a "standard block" $B = B(n_{v_1}, n_{v_2}, d_e)$*

such that the restriction of the $\mathbf{Z}$*–action of* $\mathbf{Z}$ *to* $p^{-1}(S_e)$ *coincides with the natural* $\mathbf{Z}$*–action of* B*.*

The above definition shows that any cyclic covering $G \to \Gamma$ is constructed from standard blocks which are glued equivariantly along the vertices $\{p^{-1}(v)\}$, where v runs over the vertices of Γ with degree ≥ 2.

1.9. Definition. *Two cyclic coverings* $p_i : G_i \to \Gamma$ $(i = 1, 2)$ *are equivalent* $(G_1 \sim G_2)$ *if there is an equivariant isomorphism* $q : G_1 \to G_2$ *such that* $p_2 \circ q = p_1$*.*

The set of equivalence classes of cyclic coverings of Γ*, associated with a system of integers* $(\mathbf{n}, \mathbf{d})$*, is denoted by* $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$*.*

1.10. Examples.

a.) If $\Gamma = S$, then $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ has exactly one element for any $(\mathbf{n}, \mathbf{d})$.

b.) Let Γ be the cyclic graph with two vertices and two edges: $\mathcal{V}(\Gamma) = \{v_1, v_2\}$, $\mathcal{E}(\Gamma) = \{e_1, e_2\}$ with both e_i $(i = 1, 2)$ having v_1 and v_2 as end–points.

Set $n_1 = n_2 = n$. Then $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ has exactly n elements. Notice that we can have graphs that are not equivalent as cyclic coverings over Γ, but they are isomorphic as graphs. (Take e.g. the case $n = 3$.)

The fact that in this case $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ is independent of the choice of $\mathbf{d}$, is not a particularity of this example: it is true in the most general situation, cf. (1.15.).

1.11. – The trivial covering of $(\Gamma, (\mathbf{n}, \mathbf{d}))$**.** There is a special element in $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ which can be constructed as follows. Fix a distinguished edge in each standard block $B(n_{v_1}, n_{v_2}, d_e)$. Then construct G in such a way that whenever the vertex $v \in \mathcal{V}(\Gamma)$ of Γ is adjacent to the edges $\{e\}_{e \in \mathcal{E}_v}$ of Γ, then the distinguished edges of all the blocks $\{p^{-1}(e)\}_{e \in \mathcal{E}_v}$ have a common end–point (which is one of the vertices of G in $p^{-1}(v)$). Notice that this condition together with the existence of the $\mathbf{Z}$–action determines all the other adjacent relations.

The equivalence class of G constructed in this way does not depend on the choice of the distinguished edges in the standard blocks. Indeed, if we have two different choices of the distinguished edges in the standard blocks which provide two graphs G_1 and G_2 by the above construction (both covering Γ), then for any edge e of Γ denote the distinguished edges in the blocks $p_i^{-1}(S_e)$ by e_i $(i = 1, 2)$. Then the map $q(e_1) = e_2$ (for any e) can be extended to an equivariant isomorphism $q : G_1 \to G_2$ with $p_2 \circ q = p_1$.

The covering just constructed (or its class) is denoted by $p : T \to \Gamma$ and it is called the *trivial cyclic covering of* $(\Gamma, (\mathbf{n}, \mathbf{d}))$.

The trivial covering is characterized by the existence of a (non-equivariant !) morphism of graphs $s : \Gamma \to T$ with $p \circ s = id_\Gamma$ (i.e. s is a section with $s(e)$ =distinguished edge above e).

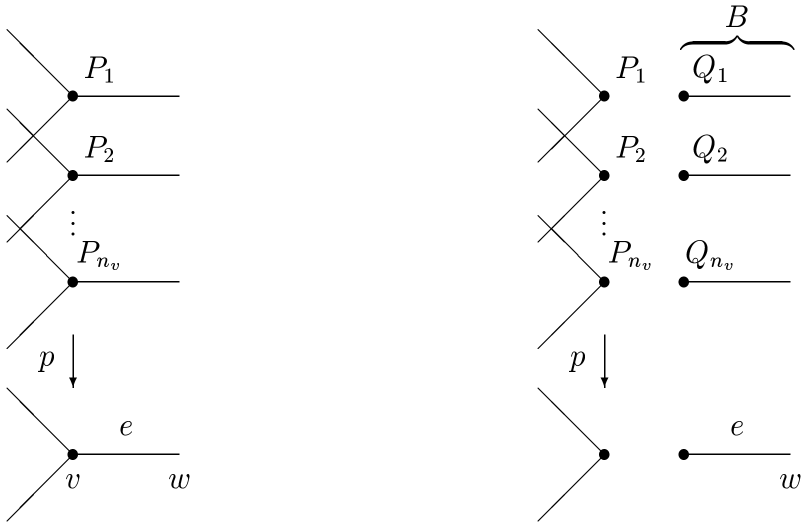
1.12. – $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ **as a homogeneous space. Classification.**

If $p : G \to \Gamma$ is a cyclic covering of Γ, then G is obtained by an equivariant gluing of $\#\mathcal{E}(\Gamma)$ standard blocks. Above any vertex $v \in \mathcal{V}(\Gamma)$ we have to glue together $\#\mathcal{E}_v$ standard blocks.

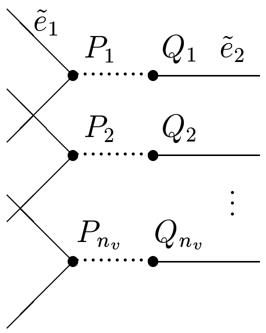
Regard Γ as a union of segments, with some end–points glued together. Then it is useful to introduce an index set of all the end–points of the segments. This is: $I = \coprod_{v} \mathcal{E}_v = \coprod_{v} \{e_v\}_{e_v \in \mathcal{E}_v}$.

Now consider the following group indexed exactly over this set: $B(\Gamma, \mathbf{n}) = \prod_{v} \prod_{e_v \in \mathcal{E}_v} \mathbf{Z}_{n_v} = \prod_{e_v \in I} \mathbf{Z}_{n_v}$. Then $B(\Gamma, \mathbf{n})$ describes the equivariant gluings in G. A typical element in $B(\Gamma, \mathbf{n})$ is $\{b_{e_v}\}_{e_v \in I}$, where $b_{e_v} \in \mathbf{Z}_{n_v}$. A generator set $\{g_{e_v}\}_{e_v \in I}$ has the form $g_{e_v} = (0, ..., \hat{1}, ...0)$, where all the entries are zero except the place e_v, where we put the generator $\hat{1}$ of $\mathbf{Z}_{n_v}$.

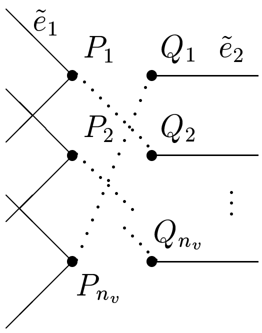
$B(\Gamma, \mathbf{n})$ acts in a natural way on the set $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$. We give the action of g_{e_v} for each $e_v \in I$. For this, fix $v \in \mathcal{V}(\Gamma)$ and an edge $e = e_v$ with end–points v and w. If $G \to \Gamma$ is a covering, then above the segment S_e we have the block $B(n_v, n_w, d_e)$. In particular, above the vertex v we have n_v vertices cyclically permuted by the $\mathbf{Z}$–action. Call them $P_1, ..., P_{n_v}$ so that the action is $P_1 \to P_2 \to ... \to P_{n_v} \to P_1$. We can detach the end–point v of the segment e from the graph Γ. In the graph G this means that we separate the end–points of the block $B(n_v, n_w, d_e)$, that stay above v, from G. In this way we obtain the graph $\tilde{G}_v$. It is represented in the right hand side of the next diagram.



Hence, we can re-obtain G from $\tilde{G}_v$ if we re-glue P_i and Q_i $(i = 1, ..., n_v)$. We emphasize that the action is $P_i \to P_{i+1}$ and similarly $Q_i \to Q_{i+1}$. Now, by definition, $g_{e_v} * G$ is obtained by the equivariant gluing $P_1 = Q_2,\ P_2 = Q_3, \ldots, P_{n_v} = Q_1$.



G



$g_{e_v} * G$

Notice that there is a canonical equivariant identification of the edges of G and $g_{e_v} * G$ (which, in general, cannot be extended to a morphism of graphs). Indeed, there are natural morphisms $q_1 : \tilde{G}_v \to G$ and $q_2 : \tilde{G}_v \to g_{e_v} * G$ which correspond to the gluings described above. They induce isomorphisms at the level of edges, hence an isomorphism of sets:

$$i_{g_{e_v}}(G) : \mathcal{E}(G) \to \mathcal{E}(g_{e_v} * G).$$

On the above diagram of graphs, we decorated by $\tilde{e}_1$ and $\tilde{e}_2$ two identified pairs of edges; this partial identification can be extended to a unique equivariant identification of edges.

It is not difficult to see, that this construction $(g_{e_v}, G) \mapsto g_{e_v} * G$ can be extended to a group action of $B(\Gamma, \mathbf{n})$. Moreover, for any $g \in B(\Gamma, \mathbf{n})$, one can define inductively an identification isomorphism:

$$i_g(G) : \mathcal{E}(G) \to \mathcal{E}(g * G).$$

The verification of the following easy facts are left to the reader.

1.13. Lemma. *a.) If $G_1 \sim G_2$ then $g * G_1 \sim g * G_2$ for any g. In particular, we obtain a group action $B(\Gamma, \mathbf{n}) \times \mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d})) \to \mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$.*

b.) The above action on $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$ is transitive.

For a fixed G let $Iso(G)$ be the isotropy subgroup $\{g \in B(\Gamma, \mathbf{n}) : g * G = G\}$ of G. Since $B(\Gamma, \mathbf{n})$ is abelian and the action is transitive, all the isotropy subgroups coincide. We denote this subgroup by $Iso \subset B(\Gamma, \mathbf{n})$. An immediate consequence of the previous lemma is that $\mathcal{G}$ can be represented as a homogeneous set $B(\Gamma, \mathbf{n})/Iso$.

In the next lemma, we will give a (possible) generator set of Iso. Fix a covering $p : G \to \Gamma$. If $e = (v, w)$ is an edge of Γ, then we can detach both end–points of e from Γ. We make the same twist above the separated end–points as before (but now above both end–points). This operation does not change the equivalence class of G, since we just rotate the block $p^{-1}(e)$, which is equivariant. In other words, for any edge $e = (v, w)$, the equivariance of block $p^{-1}(e)$ gives that $\hat{g}_e * G \sim G$, where $\hat{g}_e = g_{e_v} \cdot g_{e_w}$.

Similarly, if v is a vertex, then we take the product $\hat{g}_v = \prod_{e_v \in \mathcal{E}_v} g_{e_v}$ and equivariance of the fiber above v gives $\hat{g}_v * G \sim G$.

1.14. Lemma. *Iso is generated by the elements* $\{\hat{g}_e\}_{e\in\mathcal{E}(\Gamma)}$ *and* $\{\hat{g}_v\}_{v\in\mathcal{V}(\Gamma)}$.

Proof. We will prove that $Iso(T)$ is generated by the elements $\{\hat{g}_e\}_{e\in\mathcal{E}(\Gamma)}$ and $\{\hat{g}_v\}_{v\in\mathcal{V}(\Gamma)}$.

Fix a section $s:\Gamma\to T$ of T, which determines a set of distinguished edges $\{s(e)\}_{e\in\mathcal{E}(\Gamma)}$ of T (cf. 1.11.). Recall that for any $g\in B(\Gamma,\mathbf{n})$, as we already mentioned, there is a canonical equivariant identification $i_g=i_g(T):\mathcal{E}(T)\to\mathcal{E}(g*T)$.

Fix an element $g\in Iso(T)$, and set $s_g(e):=i_g(s(e))$. The correspondence $e\mapsto s_g(e)$, in general, cannot be extended to a section of $g*T\to\Gamma$, (because, for a fixed $v\in\mathcal{V}(\Gamma)$, the set of end–points above v of the edges $\{s_g(e_v)\}_{e_v\in\mathcal{E}_v}$ might contain more than one vertex).

On the other hand, since $g*T$ is trivial, there is a section $s':\Gamma\to g*T$, which again determines a set of distinguished edges $\{s'(e)\}_{e\in\mathcal{E}(\Gamma)}$.

For any block of $g*T$ above the segment S_e of Γ, we can find an integer $k_e\in\mathbf{Z}$ such that $e_{\mathcal{E}}(k_e,s_g(e))=s'(e)$. Set $h=\prod_e\hat{g}_e^{-k_e}$; then the end–points of the edges $\{i_{h\circ g}(s(e))\}$ above any vertex v consists of exactly one vertex, i.e. they form a section of $(h\circ g)*T$. But, the elements g' of $B(\Gamma,\mathbf{n})$, which have the property that for a section s of T, the set of edges $\{i_{g'}(s(e))\}_e$ can be extended to a section, have the form $g'=\prod_v\hat{g}_v^{k_v}$ (for some integers k_v). Hence $g=\prod\hat{g}_e^{k_e}\cdot\prod\hat{g}_v^{k_v}$. $\diamondsuit$

Now, for any edge e with end–points v_1 and v_2 define $n_e=d_e[n_{v_1},n_{v_2}]$. Let $A(\Gamma,\mathbf{n})$ be the group $\prod\limits_{e\in\mathcal{E}(\Gamma)}\mathbf{Z}_{n_e}\times\prod\limits_{v\in\mathcal{V}(\Gamma)}\mathbf{Z}_{n_v}$. The generators of $A(\Gamma,\mathbf{n})$ are $\tilde{g}_e=(0,...,\hat{1},...,0)$ (resp. $\tilde{g}_v$), whose all entries are zero except at the place e (resp. v), where we put the generator $\hat{1}$ of $\mathbf{Z}_{n_e}$ (resp. of $\mathbf{Z}_{n_v}$) . Consider the map $\theta:A(\Gamma,\mathbf{n})\to B(\Gamma,\mathbf{n})$ given by $\theta(\tilde{g}_e)=\hat{g}_e=g_{e_{v_1}}\cdot g_{e_{v_2}}$, and $\theta(\tilde{g}_v)=\hat{g}_v=\prod\limits_{e_v\in\mathcal{E}_v}g_{e_v}$. Then by the above discussion $\mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d}))$ can be identified with the *factor group* $\operatorname{coker}\theta$. Indeed, just notice that we have a distinguished element of $\mathcal{G}$, namely the trivial covering T. The action $g\in B(\Gamma,\mathbf{n})\to g*T\in\mathcal{G}$ identifies $\mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d}))$ with the factor group $\operatorname{coker}\theta=B(\Gamma,\mathbf{n})/Iso$.

The above discussions are summarized in the following theorem:

1.15. Theorem. *a.) One has the following exact sequence of abelian groups:*

$$A(\Gamma,\mathbf{n})\xrightarrow{\theta}B(\Gamma,\mathbf{n})\xrightarrow{*T}\mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d}))\to 0$$

In particular, $\mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d}))$ *is independent of* $\mathbf{d}$ *(therefore, sometimes we will use the more simple notation* $\mathcal{G}(\Gamma,\mathbf{n})$ *for it).*

b.) If $\Gamma=\coprod_{i=1}^k\Gamma_i$ *has* k *connected components* $\{\Gamma_i\}_{i=1}^k$*, then*

$$\mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d}))=\oplus_{i=1}^k\mathcal{G}(\Gamma_i,(\mathbf{n},\mathbf{d})).$$

1.16. **– $\mathcal{G}(\Gamma',(\mathbf{n},\mathbf{d}))$ for a subgraph $\Gamma'\subset\Gamma$.** Fix a graph Γ and integers $(\mathbf{n},\mathbf{d})=\{\{n_v\}_{v\in\mathcal{V}};\{d_e\}_{e\in\mathcal{E}}\}$ as above. Then any subgraph Γ' (i.e. graph Γ' with $\mathcal{V}(\Gamma')\subset\mathcal{V}(\Gamma)$ and $\mathcal{E}(\Gamma')\subset\mathcal{E}(\Gamma)$) inherits a system of integers $\{\{n_v\}_{v\in\mathcal{V}(\Gamma')};\{d_e\}_{e\in\mathcal{E}(\Gamma')}\}$ still denoted by

$(\mathbf{n},\mathbf{d})$. The natural projections $pr_A : A(\Gamma,\mathbf{n}) \to A(\Gamma',\mathbf{n})$ and $pr_B : B(\Gamma,\mathbf{n}) \to B(\Gamma',\mathbf{n})$ can be inserted in the following diagram:

1.17.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & \\
\ker(pr_A) \arrow[r, "\theta"] \arrow[d] & \ker(pr_B) \arrow[r, two heads] \arrow[d] & \text{coker }\theta \arrow[r] \arrow[d, "j"] & 0 \\
A(\Gamma,\mathbf{n}) \arrow[r, "\theta(\Gamma)"] \arrow[d, two heads, "pr_A"] & B(\Gamma,\mathbf{n}) \arrow[r, two heads, "*T"] \arrow[d, two heads, "pr_B"] & \mathcal{G}(\Gamma,\mathbf{n}) \arrow[r] \arrow[d, two heads, "pr"] & 0 \\
A(\Gamma',\mathbf{n}) \arrow[r, "\theta(\Gamma')"] \arrow[d] & B(\Gamma',\mathbf{n}) \arrow[r, two heads, "*T"] \arrow[d] & \mathcal{G}(\Gamma',\mathbf{n}) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Then the horizontal lines and vertical columns are exact. This diagram will be used several times in the next paragraphs. As a first immediate fact, one has:

1.18. Corollary. *For any subgroup $\Gamma' \subset \Gamma$, $pr : \mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d})) \to \mathcal{G}(\Gamma',(\mathbf{n},\mathbf{d}))$ is onto.*

The construction of the coverings $G \to \Gamma$ suggests that the group $\mathcal{G}(\Gamma)$ describes the possible twists above Γ. Intuitively it is clear that a non–trivial twist can be realized only above cycles of Γ. The next theorems state this fact rigorously.

1.19. Theorem. *Assume that Γ is a tree. Then $\mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d})) = 0$ for any $(\mathbf{n},\mathbf{d})$.*

Proof. We can assume that Γ is connected (cf. 1.15.). We prove the vanishing of $\mathcal{G}$ by induction on the number of edges of Γ. If $\#\mathcal{E} = 1$, then $\mathcal{G} = 0$ by (1.10. a) or by (1.15.). If $\#\mathcal{E} \geq 2$, fix a connected subgroup $\Gamma' \subset \Gamma$ such that $\mathcal{V}(\Gamma') = \mathcal{V}(\Gamma) \setminus \{v\}$, and $\mathcal{E}(\Gamma') = \mathcal{E}(\Gamma) \setminus \{(v,w)\}$ (i.e. Γ' is obtained from Γ by deleting a vertex v with $\#\mathcal{E}_v = 1$). Then, in (1.17.), $\ker(pr_A) = \mathbf{Z}_{n_v} \times \mathbf{Z}_{d_{(v,w)}[n_v,n_w]}$, $\ker(pr_B) = \mathbf{Z}_{n_v} \times \mathbf{Z}_{n_w}$, and $\theta : \ker(pr_A) \to \ker(pr_B)$ is given by $(\hat{a},\hat{b}) \mapsto (\widehat{a+b},\hat{b})$, hence θ is onto. Therefore $pr : \mathcal{G}(\Gamma) \to \mathcal{G}(\Gamma')$ is an isomorphism. Hence the theorem follows. $\diamondsuit$

In the second part (part II, 5.53.), we will need the following generalization of (1.19.).

1.20. Theorem. *Fix a graph Γ and $(\mathbf{n},\mathbf{d})$ as above. Denote by $\mathcal{V}^1$ the set of vertices $\{v \in \mathcal{V}(\Gamma) : n_v = 1\}$. Let $\Gamma_{\neq 1}$ be the subgraph of Γ obtained from Γ by deleting the vertices $\mathcal{V}^1$ and all those edges which have at least one endpoint in $\mathcal{V}^1$. If each connected component of $\Gamma_{\neq 1}$ is a tree, then $\mathcal{G}(\Gamma,(\mathbf{n},\mathbf{d})) = 0$.*

The proof is similar to the proof of 1.19. (using the diagram 1.17.), and it is left to the reader.

Now, let Γ be a cyclic graph, i.e. $\mathcal{V}(\Gamma) = \{v_1, v_2, \ldots, v_k\}$, $\mathcal{E}(\Gamma) = \{(v_1,v_2),(v_2,v_3), \ldots, (v_k,v_1)\}$.

1.21. Theorem. *If Γ is a cyclic graph as above, and $d = g.c.d.\{n_v : v \in \mathcal{V}(\Gamma)\}$, then $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d})) = \mathbf{Z}_d$.*

Proof. Let Γ' be the subgraph of Γ with $\mathcal{V}(\Gamma') = \mathcal{V}(\Gamma)$ and $\mathcal{E}(\Gamma') = \mathcal{E}(\Gamma) \setminus \{(v_k, v_{k+1})\}$. Then in (1.17.) $\ker(pr_A) = \mathbf{Z}_{d_k[n_{v_k}, n_{v_{k+1}}]}$, $\ker(pr_B) = \mathbf{Z}_{n_{v_k}} \times \mathbf{Z}_{n_{v_{k+1}}}$, and $\mathcal{G}(\Gamma', \mathbf{n}) = 0$ (because Γ' is a tree). Hence $\operatorname{coker}\theta = \mathbf{Z}_{(n_{v_k}, n_{v_{k+1}})} \to \mathcal{G}(\Gamma, \mathbf{n})$ is onto for any k. Therefore, $\mathcal{G}(\Gamma, \mathbf{n})$ is cyclic of order d' with $d'|d$. Now, consider the map $B(\Gamma, \mathbf{n}) = \prod_k \mathbf{Z}^2_{n_{v_k}} \to \mathbf{Z}_d$ given by $((a_1, b_1), (a_2, b_2), \ldots) \mapsto \sum_i a_i - \sum_i b_i$. This is obviously onto. The subgroup Iso is generated by $(\ldots, (0,0), (1,1), (0,0), \ldots)$ and $(\ldots, (0,1), (1,0), \ldots)$. Therefore, there is a well–defined map $\mathcal{G}(\Gamma, \mathbf{n}) \to \mathbf{Z}_d$, which is onto. Hence $d = d'$. $\diamondsuit$

The proofs of (1.19.) and (1.21.) combined give:

1.22. Corollary. *Let Γ be a graph with $c_\Gamma = 1$ and fix the system of integers $\mathbf{n} = \{n_v\}_v$. Let Γ' be the (unique) minimal cyclic subgraph of Γ, and set $d := g.c.d.\{n_v : v \in \mathcal{V}(\Gamma')\}$. Then $\mathcal{G}(\Gamma, \mathbf{n}) = \mathbf{Z}_d$.*

We end this sphere of thought with the following fact:

1.23. Lemma. *If $p : G \to \Gamma$ is a cyclic covering then $c_G \geq c_\Gamma$.*
Proof. If $|G|$ and $|\Gamma|$ denote the topological realizations of the corresponding graphs, then $H_1(|G|, \mathbf{Q})^{\mathbf{Z}} = H_1(|\Gamma|, \mathbf{Q})$, hence the result follows. Cf. also with (3.11.), and also with (2.14.), (3.21.) and (3.25.). $\diamondsuit$

1.24. – The mod_N–construction. Fix an arbitrary graph Γ and system of integers $\mathbf{n} = \{n_v\}_{v \in \mathcal{V}}$ and $\mathbf{d} = \{d_e\}_{e \in \mathcal{E}}$ as above. For any strict positive integer N, we introduce a new set of integers $(N, \mathbf{n}) := \{(N, n_v)\}_{v \in \mathcal{V}}$ and $\mathbf{d}'$, where $d'_e := (d_e, N/(N, [n_{v_1}, n_{v_2}]))$. (In other words: if $n_e = d_e \cdot [n_{v_1}, n_{v_2}]$, then $(n_e, N) = d'_e \cdot [(n_{v_1}, N), (n_{v_2}, N)]$.) We define a natural map:

$$mod_N : \mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d})) \to \mathcal{G}(\Gamma, ((N, \mathbf{n}), \mathbf{d}'))$$

as follows. Let G be a representative of an element of $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$. Then $mod_N(G)$ is the "orbit graph" of the induced action of $N\mathbf{Z}$. More precisely, we introduce the equivalence relation $\overset{N}{\sim}$ on $\mathcal{V}(G)$, respectively on $\mathcal{E}(G)$: $v_1 \overset{N}{\sim} v_2$ (resp. $e_1 \overset{N}{\sim} e_2$) if there is an integer k such that $a_{\mathcal{V}}(kN, v_1) = v_2$ (resp. $a_{\mathcal{E}}(kN, e_1) = e_2$). Then $mod_N(G) = G/\overset{N}{\sim}$.

This construction will be crucial in section 3.

1.25. – Variations on the theme of coverings. We can extend our set of coverings if we change the definition of the "standard block". Actually, what is really important in the definition of a block B is summarized in the following two principles:

(a) B must be equivariant (i.e. it has a $\mathbf{Z}$–action);

(b) B must be rigid in the following sense: if $a_{\mathcal{E}}(h, e) = e$ for some $h \in \mathbf{Z}$ and $e \in \mathcal{E}(B)$, then all the edges and vertices of B are left invariant by the action of h.

Otherwise, the block can be as complicated as we want.

(1) First variation. Assume that the vertices of all our graphs have two types: arrowhead vertices $\mathcal{A}$ and non–arrowhead vertices $\mathcal{W}$, i.e. $\mathcal{V} = \mathcal{A} \coprod \mathcal{W}$. Then in the definition of the coverings $p : G \to \Gamma$ we add the following axiom: $\mathcal{A}(G) = p^{-1}(\mathcal{A}(\Gamma))$.

(2) Second variation. Assume that our graphs have some decorations. Then for coverings $p : G \to \Gamma$ we require additionally, that the decoration of G must be equivariant. (At this moment, we require no connection between the decorations of G and Γ.)

(3) Third variation. In chapter 3 the following construction will be intensively used: we will change (in an equivariant way) any edge of G into a string. More precisely, we start with the following data:

(a) a graph Γ and a system $(\mathbf{n}, \mathbf{d})$ as above;

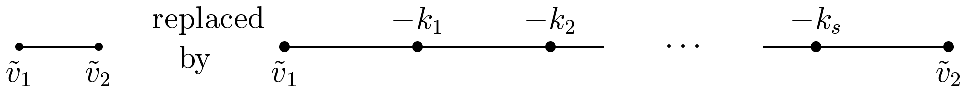
(b) a covering $p : G \to \Gamma$, as an element of $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$;

(c) for any edge e of Γ, we fix a string $Str(e)$ (which can have maybe some decorations):

$Str(e)$:

We denote the collection $\{Str(e)\}_{e\in\mathcal{E}(\Gamma)}$ of strings by **Str**.

Then the new graph $G(\mathbf{Str})$ is constructed as follows: we replace any edge $\tilde{e} \in p_{\mathcal{E}}^{-1}(e)$ (with end–points $\tilde{v}_1$ and $\tilde{v}_2$) of G with the decorated string $\overline{Str}(e)$, as it is shown below:



2§. Graphs associated with analytic germs

Let (X, x) be a normal surface singularity and fix the germ $f : (X, x) \to (\mathbf{C}, 0)$ of an analytic function. In the first subsection we review the definition of the embedded resolution graph $\Gamma(X, f)$ of f, and we introduce our basic notations. For more details the reader is invited to consult the books of H. Laufer [19] and Eisenbud–Neumann [10], and also the survey article of J. Lipman [22]. In the next subsection, we discuss Jung's method of desingularization of normal surface singularities. Here we also recall the basic arithmetical properties of the resolution graph of Hirzebruch–Jung singularities. The third subsection deals with the topology of the link L_X of (X, x) and of the pair $(X, f^{-1}(0))$. We discuss in details the representation $arg_*(f)$ provided by the Milnor fibration associated with f. This representation will be crucial in the next subsection, and in section 3. In the last subsection we introduce the "universal cyclic $\mathbf{Z}$–covering" $G(X, f)$ of the embedded resolution $\Gamma(X, f)$. This covering graph coordinates all the resolution graphs of the cyclic coverings of (X, x) branched along $f^{-1}(0)$ (see next section).

In the body of this section we present many examples in order to clarify the relationship between the objects $\Gamma(X, f)$, $arg_*(f)$, L_X and $G(X, f)$. The verification of these examples sometimes is not absolutely trivial, but basically all of them are based on the main result (3.7.). Even if that theorem is presented only in the next section, we prefer to give all these examples already here. The reader can skip the verification of them at the first reading.

I. The embedded resolution graph $\Gamma(X, f)$

2.1. – The embedded resolution. Let (X, x) be a normal surface singularity and let $f : (X, x) \to (\mathbf{C}, 0)$ be the germ of an analytic function.

An embedded resolution $\phi : (\mathcal{Y}, D) \to (U, f^{-1}(0))$ of $(f^{-1}(0), x) \subset (X, x)$ is characterized by the following properties. There is a sufficiently small neighborhood U of x in X, smooth analytic manifold $\mathcal{Y}$, and analytic proper map $\phi : \mathcal{Y} \to U$ such that:

1) if $E = \phi^{-1}(x)$, then the restriction $\phi|_{\mathcal{Y}\setminus E} : \mathcal{Y} \setminus E \to U \setminus \{x\}$ is biholomorphic, and $\mathcal{Y} \setminus E$ is dense in $\mathcal{Y}$;

2) $D = \phi^{-1}(f^{-1}(0))$ is a divisor with only normal crossing singularities, i.e. at any point P of E, there are local coordinates (u, v) in some small neighbourhood of P, such that in these coordinates $f \circ \phi = u^a v^b$ for some non–negative integers a and b.

If such an embedded resolution ϕ is fixed, then $E = \phi^{-1}(x)$ is called the exceptional divisor associated with ϕ. Let $E = \cup_{w\in\mathcal{W}} E_w$ be its decomposition in irreducible divisors. The closure S of $\phi^{-1}(f^{-1}(0) \setminus \{0\})$ is called the strict transform of $f^{-1}(0)$. Let $\cup_{a\in\mathcal{A}} S_a$ be its irreducible decomposition. Obviously, $D = E \cup S$.

In this section, for simplicity, we will assume that $\mathcal{W} \neq \emptyset$, any two irreducible components of E has at most one intersection point, and no irreducible exceptional divisor has a self–intersection. This can always be realized by some additional blow-ups.

2.2. – The embedded resolution graph. We construct the embedded resolution graph $\Gamma(X, f)$ of the pair (X, f), associated with a fixed resolution ϕ, as follows. Its vertices $\mathcal{V} = \mathcal{W} \coprod \mathcal{A}$ consist of the nonarrowhead vertices $\mathcal{W}$ corresponding to the irreducible exceptional divisors, and arrowhead vertices $\mathcal{A}$ correponding to the irreducible divisors of the strict transform S. If two irreducible divisors corresponding to $v_1, v_2 \in \mathcal{V}$ have an intersection point then (v_1, v_2) $(= (v_2, v_1))$ is an edge of $\Gamma(X, f)$. The set of edges is denoted by $\mathcal{E}$.

For any $w \in \mathcal{W}$, we denote by $\mathcal{V}_w$ the set of vertices $v \in \mathcal{V}$ adjacent to w. Its cardinality $\#\mathcal{V}_w$ is called the degree δ_w of w. In section 1 we introduced for any graph the set $\mathcal{E}_w$, the set of edges adjacent to w. Since, by our assumption, any two vertices are connected by at most one edge, one has the identification $\mathcal{E}_w = \mathcal{V}_w$.

The graph $\Gamma(X, f)$ is decorated as follows. Any vertex $w \in \mathcal{W}$ is decorated with three numbers. The first is the self–intersection $e_w := E_w \cdot E_w$. Equivalently, e_w is the Euler–number of the normal bundle of E_w in $\mathcal{Y}$. The second is the genus g_w of E_w. The third decoration is given by the multiplicities of f. More precisely, for any $v \in \mathcal{V}$, let m_v be the vanishing order of $f \circ \phi$ along the irreducible divisor corresponding to v. For example, if f defines an isolated singularity, then for any $a \in \mathcal{A}$ one has $m_a = 1$.

In all our graph–diagrams, we put the multiplicities in parentheses (e.g.: (3)) and the genera in brackets (e.g.: [3]). In order to simplify the graphs, if $g_w = 0$ for some w, then we omit [0] from the graph.

2.3. – First properties of the graph.

(1) Notice that the combinatorics of the graph and the self-intersection numbers codify completely the intersection matrix $(E_w \cdot E_v)_{(w,v)\in\mathcal{W}\times\mathcal{W}}$ of the irreducible components of E. Moreover, this matrix is negative definite, see e.g. [25] page 230; [19] page 49, or [11].

(2) The following compatibility property holds between self–intersections and multiplicities. For any $w \in \mathcal{W}$ one has:

$$e_w m_w + \sum_{v \in \mathcal{V}_w} m_v = 0.$$

Obviously, $m_v > 0$ for any $v \in \mathcal{V}$, hence the set of multiplicities determine the self–intersection numbers completely. The advantage of this fact is the following: a multiplicity can always be determined by a local computation, on the other hand the Euler–number e_w is a global characteristic class.

Similarly, since the intersection matrix $(E_w \cdot E_v)_{(w,v) \in \mathcal{W} \times \mathcal{W}}$ is negative definite, these relations determine the multiplicities $\{m_w\}_{w \in \mathcal{W}}$ in terms of the self–intersection numbers $\{e_w\}_w$ and the multiplicities $\{m_a\}_{a \in \mathcal{A}}$.

2.4. – The resolution of (X, x). We say that $\phi : \mathcal{Y} \to U$ is a resolution of (X, x) if $\mathcal{Y}$ is a smooth analytic manifold, U a neighbourhood of x in X, ϕ is a proper analytic map, such that $\mathcal{Y} \setminus E$ (where $E = \phi^{-1}(x)$) is dense in $\mathcal{Y}$ and the restriction $\phi|_{\mathcal{Y} \setminus E} : \mathcal{Y} \setminus E \to U \setminus \{x\}$ is biholomorphic.

The topology of the resolution and the combinatorics of the the irreducible exceptional divisors $\cup_w E_w$ can by codified in the graph $\Gamma(X)$, which is called the dual resolution graph of (X, x) associated with ϕ. If the divisor E is not a normal crossing divisor, then this codification can be slightly complicated, so in the sequel we will assume that the irreducible components of E are smooth and intersect each other transversally, the irreducible components have no self–intersections, and there is no intersection point which is contained in more than two components. In this case, similarly as in the situation of the embedded resolution, the vertices of the dual graph correspond to the irreducible components of E, the edges to the intersections of these components, and each vertex w carry two decorations: the genus g_w of E_w, and the self–intersection E_w^2.

Actually, one can obtain a possible graph $\Gamma(X)$ from any $\Gamma(X, f)$ by deleting all the arrows and multiplicities of the graph $\Gamma(X, f)$.

The graphs $\Gamma(X, f)$ and $\Gamma(X)$ are connected (see [19], or Zariski's Main Theorem, e.g. in [12]).

2.5. Example. – Plane curve singularities. (see e.g. [7]) Assume that (X, x) is smooth. Then the singular germ $(f^{-1}(0), 0) \subset (X, x)$ can be resolved only by quadratic modifications. In this case, the graph $\Gamma(X, f)$ is a tree, and $g_w = 0$ for any $w \in \mathcal{W}$.

2.6. Example. – The normalization of an arbitrary surface singularity. If (X, x) is a surface singularity, but it is not normal, then there is a canonical way to construct its normalization $\tilde{X}$ (see e.g. [19]). If (X, x) has k local irreducible components, then its normalization will split in k normal singular space–germs. The corresponding resolution graph of $(\tilde{X}, \{x_1, \ldots, x_k\})$ will have k connected components, they correspond exactly to resolution graphs of the irreducible components $(\tilde{X}, x_i)$.

2.7. Example. – Cyclic coverings. Start with a normal surface singularity (X,x) and a germ $f:(X,x)\to(\mathbf{C},0)$. Consider the covering $b:(\mathbf{C},0)\to(\mathbf{C},0)$ given by $z\mapsto z^N$, and construct the fiber product:

$$(X,x)\prod_{f,b}(\mathbf{C},0)=\{(x',z)\in(X\times\mathbf{C},x\times 0)\ :\ f(x')=z^N\}.$$

By definition, $X_{f,N}$ is the normalization of $(X,x)\prod_{f,b}(\mathbf{C},0)$. There is a natural ramified covering $X_{f,N}\to X$ branched along $f^{-1}(0)$.

2.8. Example. – Hirzebruch–Jung singularities. (See [14, 19, 5]). For a normal surface singularity, the following conditions are equivalent. If (X,x) satisfy (one of) them, then it is called Hirzebruch–Jung singularity.

a) The resolution graph $\Gamma(X)$ is a string, and $g_w=0$ for any $w\in\mathcal{W}$. (If the graph is minimal then $e_w\leq -2$ for any w.)

b) There is a finite proper map $\pi:(X,x)\to(\mathbf{C}^2,0)$ such that reduced discriminant locus of π, in some local coordinates (u,v) of $(\mathbf{C}^2,0)$, is $\{uv=0\}$.

c) (X,x) is isomorphic with exactly one of the "model spaces" $\{A_{n,q}\}_{n,q}$, where $A_{n,q}$ is the normalization of $(\{xy^{n-q}+z^n=0\},0)$, where $0<q<n,\ (n,q)=1$.

If there is a map π as in (b) with smooth reduced discriminant locus, then (X,x) is automatically smooth.

II. Jung's method. Hirzebruch–Jung singularities.

2.9. – Jung's method [17] gives not only a qualitative proof of the existence of the resolution $\phi:(\mathcal{Y},E)\to(X,x)$ of a normal surface singularity (X,x) (see, e.g. [19] Theorem 2.1; or [5] Theorem 6.1), but also a rather clear recipe how this resolution can be constructed in concrete cases (see, e.g. [19] chapter III, or [22]). The Jungian strategy (presented for the case of a normal surface singularity (X,x)) can be summarized in the following diagram:

$$\begin{tikzcd}
X^{res} \arrow[r] \arrow[drr, "\rho"'] & \tilde{X}' \arrow[r] \arrow[dr, "\tilde{\pi}'"'] & X' \arrow[r, "\phi'_\Delta"] \arrow[d, "\pi'"] & (X,x) \arrow[d, "\pi"] \\
 & & (\mathcal{Y},D) \arrow[r, "\phi_\Delta"] & (\mathbf{C}^2,0)\supset(\Delta,0)
\end{tikzcd}$$

where:

a) $\pi:(X,x)\to(\mathbf{C}^2,0)$ is a proper finite map with (reduced) discriminant locus $(\Delta,0)\subset(\mathbf{C}^2,0)$.

b) ϕ_Δ is an embedded resolution of $(\Delta,0)\subset(\mathbf{C}^2,0)$. In particular, $D=\phi_\Delta^{-1}(\Delta)$ is a divisor with normal crossing singularities.

c) $\pi':X'\to\mathcal{Y}$ is the pullback of π via ϕ_Δ, and $\tilde{X}'$ is the normalization of X'. Then $\tilde{X}'$ has only normal singularities, and the discriminant of the projection $\tilde{\pi}'$ has normal crossing singularities only. This property characterizes exactly the Hirzebruch–Jung singularities (cf. 2.8.).

d) $X^{res} \to \tilde{X}'$ is the resolution of the (Hirzebruch–Jung) singularities of $\tilde{X}'$.

2.10. Example. In general, the computation of the discriminant locus Δ of some projection π, and the whole process, can be rather complicated. But in some cases, the above recipe is really nice. For example, if $(X,x) = (\{f(x,y)+z^N=0\},0) \subset (\mathbf{C}^3,0)$, and π is induced by $(x,y,z)\mapsto(x,y)$, then $(\Delta,0)=\{f=0\}\subset(\mathbf{C}^2,0)$. It turns out, that the dual resolution graph of (X,x) can be recovered from the embedded resolution graph of f and the integer N (see 3.12.).

The above strategy shows that, in order to resolve (X,x), we have to know two things: the embedded resolution of plane curve singularities, and the resolution of Hirzebruch–Jung singularities.

2.11. – Hirzebruch–Jung singularities. [14, 19, 5] (cf. also 2.8.).

From our point of view, (and also from the point of view of the strategy 2.9.), it is more convenient to consider a biger class of "models" instead of $\{A_{n,q}\}_{n,q}$.

For any three strict positive integers a,b and N, with $g.c.d.(a,b,N)=1$, we define $(X,x)=(X(a,b,N),x)$ as the unique singularity lying over the origin in the normalization of $(\{\alpha^a\beta^b+\gamma^N=0\},0)$. Let the germ $\gamma:(X(a,b,N),x)\to(\mathbf{C},0)$ be induced by $(\alpha,\beta,\gamma)\mapsto\gamma$. In the sequel, we give the embedded resolution graph $\Gamma(X,\gamma)$ of the germ γ. Obviously, if we delete the arrows and multiplicities of $\Gamma(X,\gamma)$ we obtain the resolution graph $\Gamma(X)$ of $(X(a,,b,N),x)$.

First, consider the unique $0\le\lambda<N/(a,N)$ and $m_1\in\mathbf{N}$ with:

$$b+\lambda\cdot\frac{a}{(a,N)} = m_1\cdot\frac{N}{(a,N)}.$$

If $\lambda\neq 0$, then consider the continuous fraction:

$$\frac{N/(a,N)}{\lambda} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_s}}}, \quad k_1,\ldots,k_s\ge 2.$$

Then the following "string", denoted by $Str(a,b;N)$, is the embedded resolution graph $\Gamma(X,\gamma)$ of γ:

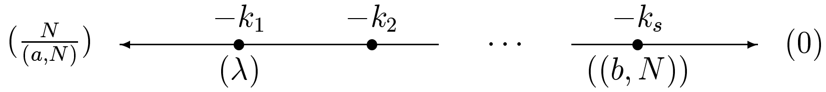
$$Str(a,b;N):\quad \left(\tfrac{a}{(a,N)}\right) \longleftarrow \underset{(m_1)}{\overset{-k_1}{\bullet}} \text{---} \underset{(m_2)}{\overset{-k_2}{\bullet}} \text{---} \cdots \text{---} \underset{(m_s)}{\overset{-k_s}{\bullet}} \longrightarrow \left(\tfrac{b}{(b,N)}\right)$$

The arrow at the left (resp. right) hand side codifies the strict transform of $\{\alpha=0\}$ (resp. of $\{\beta=0\}$). All vertices have genus $g_w=0$, i.e. they represent rational irreducible exceptional divisors. The first vertex has multiplicity m_1 given by the above congruence.

Hence $m_2, \ldots, m_s$ can be easily computed using (2.3.), namely:

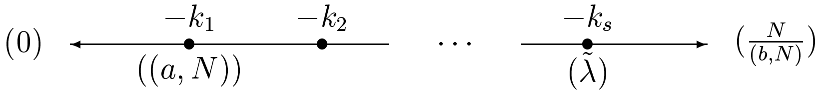
$$-k_1 m_1 + \frac{a}{(a,N)} + m_2 = 0, \text{ and } -k_i m_i + m_{i-1} + m_{i+1} = 0 \text{ for } i \geq 2.$$

This resolution resolves also the germ α (induced by the projection $(\alpha, \beta, \gamma) \mapsto \alpha$). The multiplicities of α along the (same!) divisors (exceptional divisors and strict transforms of $\{\alpha = 0\}$ and $\{\beta = 0\}$) are given in the next graph. (This is, in fact, the graph $\Gamma(X, \alpha)$, if we delete the arrow with zero multiplicity):



The other multiplicities can be again computed by (2.3.). We emphasize again: the arrows codifies the *same strict transforms* as the arrows of $\Gamma(X, \gamma)$.

The multiplicity of β along the corresponding irreducible divisors can be determined symmetrically:



where $0 \leq \tilde{\lambda} < N/(N,b)$ and

$$a + \tilde{\lambda} \cdot \frac{b}{(b,N)} = m_s \cdot \frac{N}{(b,N)}.$$

Obviously, the embedded resolution graph $\Gamma(X, \alpha^i \beta^j \gamma^k)$ of the function $\alpha^i \beta^j \gamma^k$ defined on X can be deduced easily from the above resolution graphs. It has the same shape, the same self–intersections and genera, and the multiplicity m_v (for any vertex v) satisfies:

$$m_v(\alpha^i \beta^j \gamma^k) = i \cdot m_v(\alpha) + j \cdot m_v(\beta) + k \cdot m_v(\gamma).$$

Notation. The decorated string $\Gamma(X, \alpha^i \beta^j \gamma^k)$ sometimes will be denoted by:

$$Str(a, b; N \,|\, i, j; k).$$

Form the point of view of the classification theorem (2.8. c), $X(a, b, N)$ is an $A_{n,q}$–singularity, where $n = N/(a,N)(b,N)$ and $q = \lambda/(b,N)$ (cf. e.g. [5], page 83-84).

If $\lambda = 0$, then the string has no vertices, in particular $(X(a,b,N), x)$ is smooth. Moreover, in this case, the zero set of γ (on X) has only a normal crossing singularity: in some local coordinates (u, v) of (X, x), it can be represented as $\gamma = u^{a/(a,N)} v^{b/(b,N)}$. (In this case the above string becomes a double arrow, without any non–arowhead vetices.)

III. The topology of the link of f.

2.12. – The link of (X, x)**.** Let (X, x) be a normal surface singularity, and fix an embedding $(X, x) \subset (\mathbf{C}^N, 0)$ for some N. Then, for sufficiently small $\epsilon_0 > 0$, all the spheres $S_\epsilon = \{z \in \mathbf{C}^N \ : \ ||z|| = \epsilon\}$ $(0 < \epsilon \leq \epsilon_0)$ intersects (X, x) transversally (see, e.g. [24, 23]), and the differentiable manifold $S_\epsilon \cap X$ does not depend on the choice of ϵ and of the embedding $(X, x) \subset \mathbf{C}^N, 0)$. It inherits a natural orientation and it is always connected. It is called *the link of* (X, x), and denoted by L_X.

From topological point of view, L_X characterizes completely (X, x). If B_ϵ denotes the ball $\{z \in \mathbf{C}^N \ : \ ||z|| \leq \epsilon\}$, then for ϵ sufficiently small $(B_\epsilon \cap X, x)$ is homeomorphic to $(\text{Cone}(L_X), \text{vertex of the cone})$. We will write $U = B_\epsilon \cap X$ for a small ϵ. For such a U, consider an embedded resolution $\phi : \mathcal{Y} \to U$. Then the inclusion $\phi^{-1}(0) = E \hookrightarrow \mathcal{Y}$ admits a strong deformation retract $r : \mathcal{Y} \to E$, and $\mathcal{Y}$ is a manifold with smooth boundary. Moreover, the restriction of ϕ to $\partial\mathcal{Y}$ identifies $\partial\mathcal{Y}$ with $L_X = \partial U$. This shows that L_X is the *plumbed manifold* $M(L_X)$ associated with the graph $\Gamma(X)$ (for details, see [31]), i.e. $\Gamma(X)$ determines completely the 3–manifold L_X. The converse is also true: W. Neumann in [31] proved that the topology of the (minimal) resolution of the singularity (X, x) is determined by the oriented homeomorphism type of the link L_X.

2.13. – The homology of L_X**.** Consider the intersection matrix A given by $A_{v,w} = E_v \cdot E_w$. This defines a bilinear form $(\mathbf{Z}^{\#\mathcal{W}})^{\otimes 2} \to \mathbf{Z}$, or equivalently a $\mathbf{Z}$–linear map $A : \mathbf{Z}^{\#\mathcal{W}} \to (\mathbf{Z}^{\#\mathcal{W}})^*$ (where for a $\mathbf{Z}$–module M, M^* denotes its dual $Hom_{\mathbf{Z}}(M, \mathbf{Z})$). Since A is non–degenerate, $\operatorname{coker} A$ is a torsion group with $|\operatorname{coker} A| = |\det A|$. Then the following holds:

2.14. Proposition. [35, 15, 25] $H_1(L_X, \mathbf{Z}) \approx \operatorname{coker} A \oplus \mathbf{Z}^{2g+c_\Gamma}$.

In particular, L_X is an integer (resp. rational) homology sphere if and only if $g = c_\Gamma = 0$ and $\det A = \pm 1$ (resp. $g = c_\Gamma = 0$; i.e. Γ is a tree with $g_w = 0$ for all w).

2.15. – The links of germs $f : (X, x) \to (\mathbf{C}, 0)$**.** By similar arguments (and notations) as above, for sufficiently small ϵ the intersection $S_\epsilon \cap f^{-1}(0) \subset S_\epsilon \cap X = L_X$ defines a 1–dimensional compact (in general non–connected) orientable submanifold L_f of L_X, called the link of f. The link–components (=connected components of L_f) are indexed by $\mathcal{A}$ in $\Gamma(X, f)$. Similarly as above, one can recover the link $L_f \subset L_X$ from the graph $\Gamma(X, f)$ by a plumbing construction.

2.16. – The homology group $H_1(L_X \setminus L_f, \mathbf{Z})$**.** Let $\mathbf{Z}^{\mathcal{V}}$ be the free abelian group generated by $\{[v]\}_{v \in \mathcal{V}}$ (recall $\mathcal{V} = \mathcal{W} \coprod \mathcal{A}$). Define the group H_Γ as the quotient of $\mathbf{Z}^{\mathcal{V}}$ factorized by the subgroup generated by:

$$e_w[w] + \sum_{v \in \mathcal{V}_w} [v] \quad (\text{for all } w \in \mathcal{W}).$$

Let i be the composed map $\mathbf{Z}^{\mathcal{A}} \hookrightarrow \mathbf{Z}^{\mathcal{V}} \to H_\Gamma$. Then one has the following exact sequence:

2.17. $$0 \to \mathbf{Z}^{\mathcal{A}} \xrightarrow{i} H_\Gamma \to \operatorname{coker} A \to 0.$$

2.18. Proposition. ([35, 15, 31]) *There is a natural exact sequence:*

$$0 \to H_\Gamma \xrightarrow{j} H_1(L_X \setminus L_f, \mathbf{Z}) \xrightarrow{q} H_1(E, \mathbf{Z}) \to 0.$$

Geometrically, the map j can be described as follows. Identify L_X with $\partial\mathcal{Y}$ and L_f with the intersection $\cup_a S_a \cap L_X$ of L_X with the strict transform S. For each $v \in \mathcal{V}$, define $M_v \subset L_X \setminus L_f$ as a naturally oriented circle in the transversal slice of the corresponding irreducible divisor of $D = (\cup_w E_w) \cup (\cup_a S_a)$. Equivalently, for $w \in \mathcal{W}$ we can take M_w as a generic fiber of $T_w \to B_w$, and for $a \in \mathcal{A}$, M_a is a topological standard meridian (see e.g. [10]) of $S_a \cap L_X \subset L_X$. Then $j(\widehat{[v]}) = [M_v]$ for any $v \in \mathcal{V}$.

If S is the strict transform, then the strong deformation retract $r : \mathcal{Y} \to E$ can be chosen such that it preserves S, hence induces a map $r' : \mathcal{Y} \setminus D \to E \setminus S$. Then q is the composed map:

$$H_1(L_X \setminus L_f, \mathbf{Z}) \overset{\phi_*^{-1}}{\approx} H_1(\mathcal{Y} \setminus D, \mathbf{Z}) \xrightarrow{r'_*} H_1(E \setminus S, \mathbf{Z}) \to H_1(E, \mathbf{Z}).$$

2.19. – The Milnor fibration. Fix a map $f : (X, x) \to (\mathbf{C}, 0)$. Then $arg = f/|f| : L_X \setminus L_f \to S^1$ is a $\mathcal{C}^\infty$ fibration ([24, 23]). This induces $\pi_1(arg) : \pi_1(L_X \setminus L_f) \to \mathbf{Z}$ at the fundamental group level, and $arg_* : H_1(L_X \setminus L_f, \mathbf{Z}) \to \mathbf{Z}$ at homology level. Obviously, if $ab : \pi_1 \to \pi_1/[\pi_1, \pi_1] = H_1$ is the abelianization map, then $\pi_1(arg) = ab \circ arg_*$.

The connection with the exact sequence (2.18.) is the following. Let $\mathbf{m} : \mathbf{Z}^{\mathcal{V}} \to \mathbf{Z}$ be defined by $\mathbf{m}([v]) = m_v$ (where m_v is the multiplicity of f along E_w as in 2.2.). Then by (2.3.), this induces a well–defined map $\mathbf{m} : H_\Gamma \to \mathbf{Z}$. Then $arg_* \circ j = \mathbf{m}$ (cf. [10]):

$$\begin{tikzcd} 0 \arrow[r] & H_\Gamma \arrow[r, "j"] \arrow[dr, "\mathbf{m}"'] & H_1(L_X \setminus L_f, \mathbf{Z}) \arrow[r, "q"] \arrow[d, "arg_*"] & H_1(E, \mathbf{Z}) \arrow[r] & 0. \\ & & \mathbf{Z} & & \end{tikzcd}$$

In particular, arg_* contains all the information about the multiplicities, but as we will see later, from the multiplicities we cannot recover the representation arg_* (cf. Examples 2.22.–2.26.). On the other hand, in the study of cyclic coverings of (X, x) branched along $f^{-1}(0)$, the representation arg_* plays a crucial role. In subsection IV, the additional information about arg_* which is not contained in $\Gamma(X, f)$ will be codified in a $\mathbf{Z}$–covering graph of $\Gamma(X, f)$. Moreover, one can verify the following:

2.20. Proposition. *The fibration $arg : L_X \setminus L_f \to S^1$ is completely determined (up to isotopy) by the induced representation $arg_* : H_1(L_X \setminus L_f, \mathbf{Z}) \to \mathbf{Z}$. Moreover, if $\mathbf{Z}_d := coker(p_* : H_1(L_X \setminus L_f, \mathbf{Z}) \to \mathbf{Z})$, then the (Milnor) fiber of arg has d connected components which are cyclically permuted by the monodromy.*

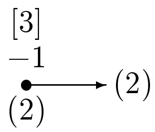
In general one has the following divisibility conditions: $d \,|\, g.c.d.\{m_v : v \in \mathcal{V}\} \,|\, g.c.d.\{m_a : a \in \mathcal{A}\}$, but in is possible that $d \neq g.c.d.\{m_v : v \in \mathcal{V}\}$. This means that d cannot be determined from $\mathbf{m}$.

Since $\mathbf{m}$, in general, does not determine the representation arg_* (cf. 2.19.), the graph $\Gamma(X, f)$ alone does not determine the Milnor fibration associated with f.

2.21. – The particular case when L_X is rational homology sphere. Set $f : (X, x) \to (\mathbf{C}, 0)$ as above, and assume that L_X is a rational homology sphere. This is equivalent to the vanishing of $H_1(E, \mathbf{Z})$. Then by (2.18. – 2.19.), $\pi_1(arg) : \pi_1(L_X \setminus L_f) \to \mathbf{Z}$ is completely determines by $\mathbf{m} : H_\Gamma \to \mathbf{Z}$, and $\mathbf{m}$ is determined by $\{m_a\}_{a \in \mathcal{A}}$ via (2.3.). Hence, the set $\{m_a\}_{a\in\mathcal{A}}$ determines completely the Milnor fibration up to an isotopy. Moreover, if $d = g.c.d.\{m_v : v \in \mathcal{V}\}$, then the fiber has d connected components, and they are cyclically permuted by the monodromy.

The next examples show that these properties are not true if L_X is not a rational homology sphere, i.e. even with the same multilink, different representations $H_1(L_X \setminus L_f, \mathbf{Z}) \to \mathbf{Z}$ do can occur.

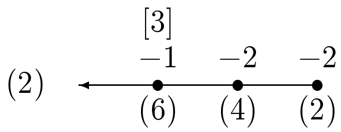
2.22. Example. Set $(X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbf{C}^3, 0)$ and take $f_1(x, y, z) = z^2$ and $f_2(x, y, z) = z^2 - y$. Then $\Gamma(X, f_1) = \Gamma(X, f_2)$ is :



Since coker $(\mathbf{m}) = \mathbf{Z}_2$, in both cases coker (arg_*) is a factor group of $\mathbf{Z}_2$ (cf. 2.19.). We will show that in the first case coker $(arg_*) = \mathbf{Z}_2$ and in the second case arg_* is onto. Indeed, the Milnor fibration of z^2 is the pullback by $z \mapsto z^2$ of the Milnor fibration of z, hence coker $(arg_*) = \mathbf{Z}_2$. In order to prove the second statement, it is enough to verify that the double covering $\{x^2 + y^7 - z^{14} = w^2 + y - z^2 = 0\} \subset \mathbf{C}^4$ is irreducible (notice that our equations are quasi–homogeneous, so we can replace a small ball centered at the origin with the whole affine space). But this is true if its intersection with $y = 1$, i.e. $C := \{x^2 = z^{14} - 1; w^2 = z^2 - 1\} \subset \mathbf{C}^3$, is irreducible. The covering $C \to \mathbf{C}$ ($(x, w, z) \mapsto z$) is a $\mathbf{Z}_2 \times \mathbf{Z}_2$ covering. The monodromy around ± 1 is $(-1, -1)$, and around any α with $\alpha^{14} = 1$ and $\alpha^2 \neq 1$ is $(-1, +1)$ (here $\mathbf{Z}_2 = \{+1, -1\}$). Hence the global monodromy group is the whole group $\mathbf{Z}_2 \times \mathbf{Z}_2$, in particular C is irreducible.

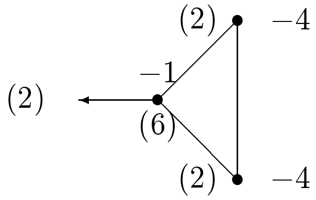
Notice also that the multiplicities of f_2 are all even numbers, but there is no germ $g : (X, x) \to (\mathbf{C}, 0)$ with $f_2 = g^2$. (Actually, there is no homotopy between f_2 and g^2 for any g.)

2.23. Example. Set $(X, x) = (\{z^2 + y(x^{12} - y^{18}) = 0\}, 0)$ and $f_1 = x^2$ and $f_2 = x^2 - y^3$. Then $\Gamma(X, f_1) = \Gamma(X, f_2)$ is the graph:



By similar argument as in (2.22.) one has that $arg_*(f_1)$ has cokernel $\mathbf{Z}_2$, and $arg_*(f_2)$ is onto.

2.24. Example. Set $(X,x)=(\{z^2+(x^2-y^3)(x^3-y^2)=0\},0)$ and $f_1=x^2$ and $f_2=x^2-y^3$. Then $\Gamma(X,f_1)=\Gamma(X,f_2)$ is the following graph:

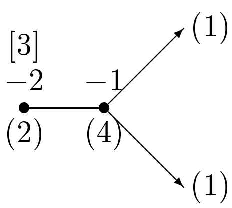


Then again: $arg_*(f_1)$ has cokernel $\mathbf{Z}_2$, and $arg_*(f_2)$ is onto.

Notice that in the above examples, $(X,x)=(\{z^2+h(x,y)=0\},0)$, and f_2 divides h but it is not equal to h. For all such cases the monodromy argument given in (2.22.) is valid. (So the interested reader can construct many–many similar examples, with even more additional properties.) But all these examples define non–isolated singularities. In order to construct examples of germs which define isolated singularities, we will use the well–known construction of series of singularities. Namely, assume that f_1 and f_2 have the same graph but have different representations arg_*, and their zero sets have non–isolated singularities (e.g. they are constructed by the above method). Next, we find a germ g such that the zero set of f_i and g have no common components (for $i=1,2$). Then, for k sufficiently big, the germs f_1+g^k and f_2+g^k, will define isolated singularities with the same embedded resolution graphs, but different representations arg_*.

2.25. Example. Set $(X,x)=(\{x^2+y^7-z^{14}=0\},0)\subset(\mathbf{C}^3,0)$ and take $f_1(x,y,z)=z^2$ and $f_2(x,y,z)=z^2-y$ as in (2.22.). Let P be the intersection point of the strict transform S_a of $\{f_i=0\}$ with the exceptional divisor E. Then, in some local coordinate system (u,v) of P, $\{u=0\}$ represents E (in a neighborhood of P), $\{v=0\}$ represents S_a, and $f_i=u^2v^2$. Consider $g=y$. Since y in the neighborhood of P can be represented as $y=u^2$ (modulo a local invertible germ), f_i+g^k near P has the form $u^2v^2+u^{2k}$. For example, if $k=2$, then one needs one more blowing up in order to resolve f_i+g^k.

Therefore, $\Gamma(X,z^2+y^k)=\Gamma(X,z^2-y+y^k)$ for any $k\geq 2$; and for $k=2$, the graphs have the following form:



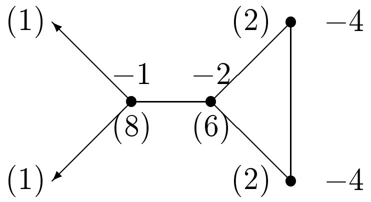
Notice that now $\mathbf{m}$ is onto, hence for both $i=1,2$, $arg_*(f_i)$ is onto. Nevertheless, $arg_*(f_1)\neq arg_*(f_2)$ because their restrictions to a subgroup of $H_1(L_X\setminus L_f)$ are different.

Indeed, let $\mathcal{Y}'$ be a tubular neighborhood of the irreducible exceptional divisor E of genus 3. This curve E can be contracted by Grauert theorem [11]. Then E contracted in $\mathcal{Y}'$ give birth to a singularity (X',x) with the same resolution graph as the surface singularity in (2.22.). Moreover, the germs f_i $(i=1,2)$ induces germs $f'_i:(X',x)\to(\mathbf{C},0)$, such that they have the same embedded resolution graphs as the germs in (2.22.). In particular, coker $(arg_*(f_1')) = \mathbf{Z}_2$ and $arg_*(f_2')$ is onto. But $arg_*(f_i')$ is the composed map

$$H_1(\mathcal{Y}' \setminus D, \mathbf{Z}) \to H_1(\mathcal{Y} \setminus D, \mathbf{Z}) \stackrel{arg_*(f_i)}{\longrightarrow} \mathbf{Z},$$

hence $arg_*(f_1) \neq arg_*(f_2)$.

2.26. Example. Set $(X, x) = (\{z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0)$ and $f_1 = x^2 + y^k$ and $f_2 = x^2 - y^3 + y^k$, where $k \geq 4$. Then by similar argument as before, $\Gamma(X, f_1) = \Gamma(X, f_2)$. This graph for $k = 4$ is:



Let E_i $(i = 1, 2, 3)$ be the irreducible exceptional divisors with self intersection numbers $-2, -4, -4$ respectively, and $\mathcal{Y}'$ be the union of small tubular neighborhoods of them. Then collapsing the curve $\cup_i E_i$ in $\mathcal{Y}'$ creates a singularity (X', x) with the same graph as in (2.24.). Repeating the arguments of (2.25.) (but using (2.24.) instead of (2.22.)) one has $arg_*(f_1) \neq arg_*(f_2)$.

IV. The universal cyclic covering of $\Gamma(X, f)$.

2.27. – The covering $p : G(X, x) \to \Gamma(X, x)$. As we already noticed, the embedded resolution graph $\Gamma(X, 0)$ does not codifies all the information about arg_*. On the other hand, this information is needed in the study of cyclic coverings of (X, x). In this section, we define a cyclic covering of $\Gamma(X, x)$ (cf. section 1) which will control the behavior of the resolution graphs of all the cyclic coverings $\{X_{f,N}\}_N$ (cf. 2.7.). The graph $\Gamma(X, x)$ was already considered in the literature by Ph. Du Bois and F. Michel from a completely different point of view, see [8].

Let (X, x) be a normal surface singularity and $f : (X, x) \to (\mathbf{C}, 0)$ the germ of an analytic function. Fix an embedded resolution $\phi : (\mathcal{Y}, D) \to (X, f^{-1}(0))$ of $(f^{-1}(0), x) \subset (X, x)$ (as in 2.1.) with embedded resolution graph $\Gamma(X, f)$.

Let $T(E_w)$ $(w \in \mathcal{W})$ be a small tubular neighborhood of the irreducible divisor E_w. By our assumption that any two irreducible exceptional divisor has at most one intersection point (see the first subsection of this chapter), for any $e = (v, w) \in \mathcal{E}$, the intersection $T(E_w) \cap T(E_v)$ $((v, w) \in \mathcal{W} \times \mathcal{W})$ is homeomorphic to a multidisc $D \times D$. This will be denoted by T_e. If $T(S_a)$ $(a \in \mathcal{A})$ is a small tubular neighborhood of the irreducible component S_a of the strict transform S (cf. 2.1.), and a is adjacent to $w_a \in \mathcal{W}$, then corresponding to the edge $e = (a, w_a)$ we introduce the multidisc $T_e = T(S_a) \cap T(E_{w_a})$. Set $T = (\cup_w T(E_w)) \cup (\cup_a T(S_a))$.

Now, we consider the smooth nearby fiber $f^{-1}(\delta) \subset X$ lifted via ϕ. For sufficiently small $\delta > 0$, the fiber $F := (f \circ \phi)^{-1}(\delta) \subset \mathcal{Y}$ is in T. Set $F_w = F \cap T(E_w)$ for any $w \in \mathcal{W}$, $F_a = F \cap T(S_a)$ for any $a \in \mathcal{A}$, and $F_e = F \cap T_e$ for any $e \in \mathcal{E}$.

It is not very difficult to construct a geometric monodromy $h_g : F \to F$ of the fibration $f^{-1}(S^1_\delta) \to S^1_\delta$ (where $S^1_\delta = \{z \in \mathbf{C} : |z| = \delta\}$ and δ is sufficiently small) which preserves the subspaces $\{F_v\}_{v \in \mathcal{V}}$ and $\{F_e\}_{e \in \mathcal{E}}$. Then the connected components of F_v (resp. of F_e) are cyclically permuted by the geometric monodromy. Let n_v (resp. n_e) be the number of connected components of F_v (resp. F_e). Then, for any $e = (v_1, v_2)$, $n_e = d_e \cdot [n_{v_1}, n_{v_2}]$ for some $d_e \geq 1$.

Now, we are able to construct the covering $p : G(X, f) \to \Gamma(X, f)$ associated with the resolution ϕ. Above a vertex $v \in \mathcal{V}(\Gamma(X, f))$ there are exactly n_v vertices of $G(X, f)$, they correspond to the connected components of F_v. The $\mathbf{Z}$–action is induced by the monodromy (by the identification $1_{\mathbf{Z}} = (h_g)_*$). If v is an arrowhead in Γ then by our agreement, all the vertices in G above v are arrowheads (cf. 1.25.1). Above an edge e of Γ, there are n_e edges of G. They corresponds to the connected components of F_e. The $\mathbf{Z}$–action again is generated by the monodromy.

Fix an edge $\tilde{e}$ of G (above the edge e of Γ) which corresponds to the connected component $F_{\tilde{e}}$ of F_e. Similarly, take a vertex $\tilde{v}$ of G (above the vertex v of Γ) which corresponds to the connected component $F_{\tilde{v}}$ of F_v. Then $\tilde{e}$ has as an endpoint the vertex $\tilde{v}$ (in G) if and only if $F_{\tilde{e}} \subset F_{\tilde{v}}$. In particular, with the same notations as above, $\tilde{v}_1$ and $\tilde{v}_2$ are connected in G by (at least) one edge if and only if $F_{\tilde{v}_1} \cap F_{\tilde{v}_2} \neq \emptyset$. If $F_{\tilde{v}_1} \cap F_{\tilde{v}_2} \neq \emptyset$, then $F_{\tilde{v}_1} \cap F_{\tilde{v}_2}$ has exactly $d_e = [n_{v_1}, n_{v_2}]/n_e$ ($e = (v_1, v_2)$) connected components, hence $\tilde{v}_1$ and $\tilde{v}_2$ are connected exactly by d_e edges. Therefore, the "standard blocks" of G above the segments of Γ are exactly the blocks described in (1.5.).

In the sequel, we will use the notation $(\mathbf{n}, \mathbf{d}) = \{\{n_v\}_{v \in \mathcal{V}}; \{d_e\}_{e \in \mathcal{E}}\}$, where $n_e = d_e[n_{v_1}, n_{v_2}]$ for any edge $e = (v_1, v_2)$.

The next lemma establishes the number of connected components of $G(X, f)$.

2.28. Lemma. *The number of connected components of the graph $G(X, f)$ is equal to the number of connected components of the Milnor fiber F of the germ f.*
Proof. Let $|G|$ be the topological realization of the graph G considered as a 1–dimensional simplicial complex. Then it is not difficult to construct a continuous map $\alpha : F \to |G|$ which maps $F_{\tilde{v}}$ to the zero-cell (vertex) $\tilde{v}$, maps $F_{\tilde{e}}$ to the one–cell (edge) $\tilde{e}$, and for any $P \in |G|$, the space $\alpha^{-1}(P)$ is connected. (Notice that $F_{\tilde{e}} \approx S^1 \times \tilde{e}$, then α restricted to $F_{\tilde{e}}$ can be identified with the second projection $S^1 \times \tilde{e} \to \tilde{e}$.) Hence $\pi_0(\alpha) : \pi_0(F) \to \pi_0(|G|)$ is an isomorphism. $\diamondsuit$

Now, it is well–known, that the fibrations $f : f^{-1}(S^1_\delta) \to S^1_\delta$ and $arg = f/|f| : L_X \setminus L_f \to S^1$ are equivalent. Hence by the long homotopy exact sequence:

$$\pi_1(L_X \setminus L_f) \overset{\pi_1(arg)}{\to} \pi_1(S^1) = \mathbf{Z} \to \pi_0(F) \to 0$$

we obtain that $|\pi_0(F)| = |\mathrm{coker}\,(arg_*)|$ (where $\pi_1(arg) = ab \circ arg_*$, cf. 2.19.). Therefore:

2.29. Corollary. *$G(X, f)$ has $|coker\,(arg_*(f))|$ connected components. In particular, if f defines an isolated singularity, then $G(X, f)$ is a connected graph. (But we can have connectedness of $G(X, f)$ even for germs f with g.c.d.$\{m_v : v \in \mathcal{V}\} \neq 1$.)*

Before we state the second part of this corollary, we make the following discussion.

Fix a connected subgraph Γ' of $\Gamma(X,f)$ with non–arrowhead vertices $\mathcal{W}'$. Since the intersection form associated with the exceptional divisors $E' := \cup_{w\in\mathcal{W}'}$ is negative definite, by Grauert theorem [11] $E' \subset \mathcal{Y}$ can be contracted to a singular point. Let (X',x') be this singular point. Moreover, since $f\circ\phi$ is zero along E', it gives rise to a germ f' defined on (X',x'). It is obvious, that Γ' is the embedded resolution graph of a singularity $((f')^{-1}(0),0) \subset (X',x')$. Let the corresponding representation be denoted by arg'_*.

2.30. Corollary. *Consider the universal cyclic covering $p: G(X,f) \to \Gamma(X,f)$ associated with f and the resolution ϕ. Fix a connected subgraph Γ' of $\Gamma(X,f)$. Then $p^{-1}(\Gamma')$ has $|coker(arg_*(f'))|$ connected components.*

2.31. – The case when L_X is a rational homology sphere. If L_X is a rational homology sphere, then the Milnor fibration is completely determined by $\mathbf{m}$ (cf. 2.21.). Hence, $G(X,f)$ contains the same amount of information as $\Gamma(X,f)$, and it can be always recovered from $\Gamma(X,f)$.

2.32. Lemma. *Assume that L_X is a rational homology sphere. Then $n_v := g.c.d.\{m_w : w \in \mathcal{V}_v \cup \{v\}\}$ for any $v \in \mathcal{V}(\Gamma)$; and $n_e := g.c.d.(m_{v_1}, m_{v_2})$ for any $e = (v_1,v_2) \in \mathcal{E}(\Gamma)$. (In particular, for $a \in \mathcal{A}(\Gamma)$ one has $n_a = n_e$, where $e = (a, w_a) \in \mathcal{E}(\Gamma)$.) Moreover, the number of connected components of $G(X,f)$ is exactly $g.c.d.\{m_v : v \in \mathcal{V}(\Gamma)\}$.*

Proof. If T_v is a tubular neighborhood of E_v as in (2.27.), then coker $(\pi_1(T_v \setminus \phi^{-1}(f^{-1}(0))) \to \mathbf{Z})$ is $\mathbf{Z}/\{m_w : w \in \mathcal{V}_v \cup \{v\}\}\mathbf{Z}$, hence the statement about n_v follows (by 2.30.). Similarly, for $e=(v,w)$, the cokernel of $\pi_1(T_v \cap T_w \setminus \phi^{-1}(f^{-1}(0))) \to \mathbf{Z})$ is $\mathbf{Z}/(m_v\mathbf{Z} + m_w\mathbf{Z})$. The number of connected components of G is $|\mathrm{coker}\,(arg_*)| = |\mathrm{coker}\,(\mathbf{m} : H_\Gamma \to \mathbf{Z})|$. $\diamondsuit$

Notice also that in this case $\mathcal{G}(\Gamma(X,f),(\mathbf{n},\mathbf{d}))$ contains only one element (because Γ is a tree, cf. 1.19.). This class is represented by $G(X,f)$.

For example, if (X,x) is smooth, then $L_X = S^3$, hence for any plane curve singularity, the universal cyclic covering $G(X,f) \to \Gamma(X,f)$ can be completely determined from the embedded resolution graph $\Gamma(X,f)$ of f.

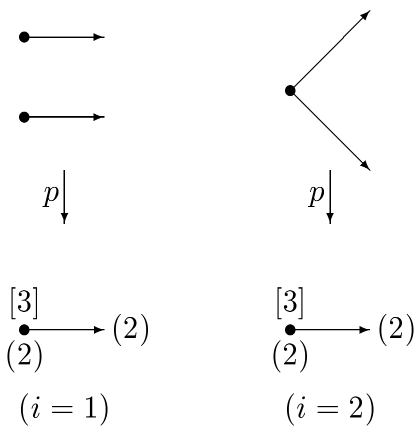
2.33. Corollary. *Consider an arbitrary germ $f : (X,x) \to (\mathbf{C},0)$ (i.e. without any restriction about L_X), and consider the universal cyclic covering $p : G(X,f) \to \Gamma(X,f)$. Then for any $w \in \mathcal{W}(\Gamma)$ with $g_w = 0$, $p^{-1}(w)$ consists of exactly $n_w = g.c.d.\{m_v : v \in \mathcal{V}_w \cup \{w\}\}$ vertices of G. Similarly, for any $a \in \mathcal{A}(\Gamma)$, $\#p^{-1}(a) = n_a = g.c.d.(m_a, m_{w_a})$, where $(a,w_a) \in \mathcal{E}(\Gamma)$. Moreover, the number of edges n_e above $e=(v_1,v_2)$ is $g.c.d.(m_{v_1}, m_{v_2})$.*

For $w \in \mathcal{W}(\Gamma)$ with $g_w > 0$ the following divisibility holds: $n_w \,|\, g.c.d.\{m_v : v \in \mathcal{V}_w \cup \{w\}\}$.

Proof. Notice that if an irreducible rational exceptional divisor is contracted to a singular point then the link of this singular point is a rational homology sphere. Then use (2.29. –2.32.). $\diamondsuit$

2.34. – The case when $\Gamma(X, f)$ is a tree. Recall that if $\Gamma(X, f)$ is a tree, then for any system of integers $(\mathbf{n}, \mathbf{d})$, by (1.19.) $\mathcal{G}(\Gamma(Xf), (\mathbf{n}, \mathbf{d})) = 0$, hence all the coverings are equivalent. One the other hand, if L_X is not rational homology sphere, even if $\Gamma(X, f)$ is a tree, the covering data $\{n_x\}_{x \in \mathcal{V} \cup \mathcal{E}}$ of the universal covering $G \to \Gamma$ (more precisely, the integers n_w with $g_w > 0$) are not determined by $\Gamma(X, f)$ (see next examples). Hence already in this case, the covering $G(X, f) \to \Gamma(X, f)$ contains some additional information about the representation arg_*.

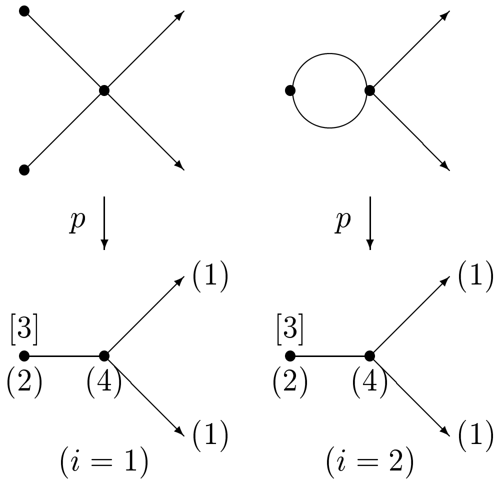
2.35. Example. Set $(X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbf{C}^3, 0)$ and take $f_1(x, y, z) = z^2$ and $f_2(x, y, z) = z^2 - y$ (cf. 2.22.). Then the coverings $p : G(X, f_i) \to \Gamma(X, f_i)$ (for $i = 1, 2$) are:



In order to count the number of vertices above the irreducible divisor (with $g = 3$), we have to consider the representations $arg_*(f_i)$ (cf. 2.29.). In the first case, the cokernel of this representation is $\mathbf{Z}_2$, in the second case it is trivial. Hence, above E, in the first case one has two vertices, and in the second case only one.

Notice, that the number of connected components (and even the Euler–characteristic) of the graphs $G(X, f_i)$ are different.

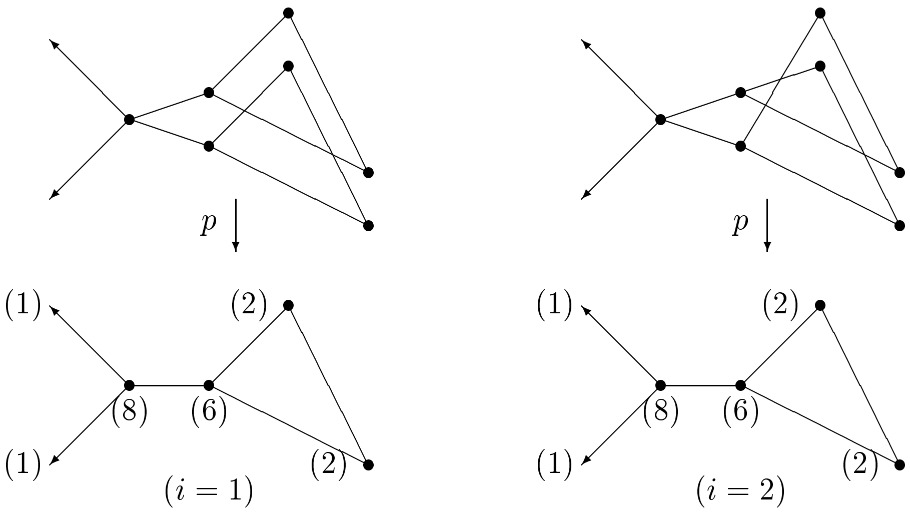
2.36. Example. Set $(X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbf{C}^3, 0)$ and take $f_1(x, y, z) = z^2 + y^2$ and $f_2(x, y, z) = z^2 - y + y^2$ (cf. 2.25.). By a similar argument as above, the coverings $p : G(X, f_i) \to \Gamma(X, f_i)$ (for $i = 1, 2$) are:



In this case the number of independent cycles of the graphs $G(X, f_i)$ is different.

2.37. – If all the irreducible exceptional divisors of ϕ are rational (i.e. $g_w = 0$ for all w), then the type $(\mathbf{n}, \mathbf{d})$ of the covering is completely determined by $\Gamma(X, f)$ (cf. 2.33.). But if $\Gamma(X, f)$ is not a tree, then $\mathcal{G}(\Gamma(X, f), \mathbf{n})$ can be non–trivial. Hence again, $G(X, f)$ carries some additional information about arg_*.

2.38. Example. Set $(X, x) = (\{z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0)$ and $f_1 = x^2 + y^4$ and $f_2 = x^2 - y^3 + y^4$ (cf. 2.26.). Then the coverings $p : G(X, f_i) \to \Gamma(X, f_i)$ (for $i = 1, 2$) are:



In this case, $\mathcal{G}(\Gamma(X, f), \mathbf{n}) = \mathbf{Z}_2$ (cf. 1.22.). The above examples provide the two non–equivalent representatives of $\mathcal{G} = \mathbf{Z}_2$.

The number of independent cycles c_G for both graphs $G(X, f_i)$ are the same (= 2). This is a general fact: for a connected graph G one has: $1 - c_G = \#\mathcal{V}(G) - \#\mathcal{E}(G)$, but $\#\mathcal{V}(G(X, f_i))$ and $\#\mathcal{E}(G(X, f_i))$ is given by Γ (provided that $g_w = 0$ for any w).

3§. The resolution graph of cyclic coverings

I. The monodromy representation of cyclic coverings.

3.1. Let (X,x) be a normal surface singularity and $f:(X,x)\to(\mathbf{C},0)$ the germ of an analytic function. For any integer $N\geq 1$, take $b:(\mathbf{C},0)\to(\mathbf{C},0)$ given by $z\longmapsto z^N$, and let $X_{f,N}$ be the normalization of the fiber product $\{(x',z)\in(X\times\mathbf{C},x\times 0):f(x')=z^N\}$ (cf. 2.7.). The second projection $(x',z)\in X\times\mathbf{C}\longmapsto z\in\mathbf{C}$ induces an analytic map $X_{f,N}\to\mathbf{C}$, still denoted by z. The first projection $(x',z)\longmapsto x'$ gives rise to a ramified cyclic N-covering $pr:X_{f,N}\to X$, branched along $f^{-1}(0)$. If $f^{-1}(0)$ has an isolated singular point at x, then there is only one (singular) point of $X_{f,N}$ lying above $x\in X$. But in general $pr^{-1}(x)$ contains more than one point (cf. 2.6.).

We regard $\mathbf{Z}_N$ as the group of N^{th}-roots of unity $\{\xi_k=e^{2\pi ik/N};\ 0\leq k\leq N-1\}$, then $(x',z)\longmapsto(x',\xi_k z)$ induces a $\mathbf{Z}_N$–Galois action of $X_{f,N}$ over X.

If $P\in X\setminus f^{-1}(0)$ is a point in the complement of the branch locus, then $pr^{-1}(P)$ consists of N points and they are cyclically permuted by the Galois action of $\mathbf{Z}_N$. The monodromy representation of the regular covering $pr|_{X_{f,N}\setminus\{z=0\}}:X_{f,N}\setminus\{z=0\}\to X\setminus\{f=0\}$ is denoted by $\varphi_N:\pi_1(X\setminus\{f=0\})\to\mathbf{Z}_N$. Notice that $X\setminus\{f=0\}$ is connected, and $\mathbf{Z}_N$ is abelian, hence the monodromy representation does not depend on the choice of the basepoint. So, we will omit the basepoint of the fundamental group.

By the local cone structure of $(X,\{f=0\})$, we can replace the group $\pi_1(X\setminus\{f=0\})$ by $\pi_1(L_X\setminus L_f)$. The following property of the cyclic coverings is well-known:

3.2. Lemma. *Let $\pi_1(arg):\pi_1(L_X\setminus L_f)\to\pi_1(S^1)=\mathbf{Z}$ be the morphism induced by the Milnor fibration of f (at the fundamental group level) (cf. 2.19.); and let $mod_N:\mathbf{Z}\to\mathbf{Z}_N$ be the natural projection $1\longmapsto\hat{1}$. Then $\varphi_N=mod_N\circ\pi_1(arg)$.*

Since $\mathbf{Z}_N$ is abelian, $\varphi_N=r_N\circ ab$ for some $r_N:H_1(L_X\setminus L_f,\mathbf{Z})\to\mathbf{Z}_N$ (where $ab:\pi_1\to\pi_1/[\pi_1,\pi_1]=H_1$ is the abelianization map). Hence, $r_N=mod_N\circ arg_*$, where $arg_*:H_1(L_X\setminus L_f,\mathbf{Z})\to\mathbf{Z}$ is induced by the Milnor fibration $arg=f/|f|$. This implies the following:

3.3. Lemma. *If the Milnor fiber F of f has k connected components, then $X_{f,N}$ has (k,N) connected components.*

Proof. By the long homotopy exact sequence of the Milnor fibration $k=|\mathrm{coker}\,arg_*|$, hence the range of arg_* is $k\mathbf{Z}\subset\mathbf{Z}$. But $mod_N(k\mathbf{Z})=k\mathbf{Z}_N$ has index (k,N) in $\mathbf{Z}_N$, hence $\mathrm{coker}\,r_N=\mathrm{coker}\,\varphi_N\approx\mathbf{Z}_{(k,N)}$. Again, by the long homotopy exact sequence of the regular covering pr over $X\setminus\{f=0\}$, $|\mathrm{coker}\,\varphi_N|$ is the number of connected components of $X_{f,N}\setminus\{z=0\}$. But this is exactly the number of connected (or irreducible) components of $X_{f,N}$. $\diamondsuit$

The number (N,k) is exactly the number of points of $X_{f,N}$ lying above $x\in X$ (i.e. $\#pr^{-1}(x)$). In the sequel, we will use the germ–notation $(X_{f,N},\{x_1,...,x_{(N,k)}\})$, which means that $X_{f,N}$ consists of (N,k) disjoint space germs $(X_{f,N},x_i)_{i=1}^{(N,k)}$. Obviously, they are all isomorphic with each other – an isomorphism is given by the Galois action (which permutes the points $\{x_i\}_{i=1}^{(N,k)}$).

3.4. Remark. The number of (singular) points of $X_{f,N}$ lying above $x \in X$ cannot be determined from the embedded resolution graph of f and the integer N.

Indeed, consider the situation described in (2.22.) or (2.35.). Then $\#pr^{-1}(x) = 2$ in the first case, and $\#pr^{-1}(x) = 1$ in the second case.

The above remark already suggests (and we will see a lot of other examples later) that the resolution graph of $X_{f,N}$ cannot be reconstructed from the graph $\Gamma(X, f)$ and N. In fact, this was the very reason why we constructed the universal cyclic covering $G(X, f) \to \Gamma(X, f)$. For example, related to the above discussion: the number of connected components of $G(X, f)$ is exactly k (cf. 2.28.), hence $(k, N) = \#pr^{-1}(x)$ is determined by $G(X, f)$ and the integer N.

3.5. Definitions.

a.) The resolution graph $\Gamma(X_{f,N})$*, by definition, is the union of the resolution graphs of* $(X_{f,N}, x_i)_{i=1}^{(k,N)}$.

b.) The embedded resolution graph $\Gamma(X_{f,N}, z)$ *of* $z : (X_{f,N}, \{x_1, ..., x_{(k,N)}\}) \to (\mathbf{C}, 0)$ *is the union of the embedded resolution graphs of* $z : (X_{f,N}, x_i) \to (\mathbf{C}, 0)$ $(1 \le i \le (k, N))$.

In both cases Γ has (k, N) identical connected components.

II. The embedded resolution graph of $\Gamma(X_{f,N}, z)$.

3.6. – The main construction. Let $p : G(X, f) \to \Gamma(X, f)$ be the universal cyclic graph–covering of the embedded resolution graph $\Gamma = \Gamma(X, f)$ associated with the germ $f : (X, x) \to (\mathbf{C}, 0)$ (cf. 2.27.). This covering is an element of $\mathcal{G}(\Gamma, (\mathbf{n}, \mathbf{d}))$, where $n_v = \#p^{-1}(v) \subset \mathcal{V}(G)$ for any vertex $v \in \mathcal{V}(\Gamma)$, and $n_e = d_e[n_{v_1}, n_{v_2}] = \#p^{-1}(e) \subset \mathcal{E}(G)$ for any $e \in \mathcal{E}(\Gamma)$ (where we abridge $G(X, f)$ into G). Recall that the graph Γ also has some decorations: multiplicities $\{m_v\}_{v\in\mathcal{V}(\Gamma)}$, genera $[g_w]_{w\in\mathcal{W}(\Gamma)}$ (and self intersection numbers, which are less important in this construction). By our convention: $\mathcal{A}(G) = p^{-1}(\mathcal{A}(\Gamma))$.

Now, for any fixed integer $N \ge 1$, we start to construct a new graph in four steps.

Step 1. The graph G has a $\mathbf{Z}$–action. The “orbit graph” of the subgroup $N\mathbf{Z} \subset \mathbf{Z}$ is denoted by $mod_N(G)$ (for details, see 1.24.). The new covering $mod_N(p) : mod_N(G) \to \Gamma$ is an element of $\mathcal{G}(\Gamma, ((N, \mathbf{n}), \mathbf{d}'))$, where $(N, \mathbf{n}) = \{g.c.d.(N, n_v)\}_{v\in\mathcal{V}(\Gamma)}$, and $d'_e = (d_e, N/(N, [n_{v_1}, n_{v_2}]))$ for any edge $e = (v_1, v_2) \in \mathcal{E}(\Gamma)$ (i.e. $\#mod_N(p)^{-1}(e) = g.c.d.(N, \#p^{-1}(e))$ for any e).

Step 2. We put some decorations (multiplicities and genera) on $mod_N(G)$ as follows.

(a) The multiplicity $(m_{\tilde{v}})$ of any vertex $\tilde{v} \in \mathcal{V}(mod_N(G))$, which lies above $v \in \mathcal{V}(\Gamma)$ is $m_v/(m_v, N)$;

(b) The genus $[\tilde{g}_{\tilde{w}}]$ of any vertex $\tilde{w} \in \mathcal{W}(mod_N(G))$, which lies above $w \in \mathcal{W}(\Gamma)$ with genus g_w, is given by:

$$2 - 2\tilde{g}_{\tilde{w}} = \frac{(2 - 2g_w - \delta_w) \cdot (m_w, N) + \sum_{v \in \mathcal{V}_w(\Gamma)} g.c.d.(m_v, m_w, N)}{(N, n_w)},$$

where $\delta_w = \#\mathcal{V}_w(\Gamma)$.

Step 3. Any edge $\tilde{e}$ of $mod_N(G)$, with end–points $\tilde{v_1}$ and $\tilde{v_2}$, lying above e with end–points v_1 and $v_2 \in \mathcal{E}(\Gamma)$, will be replaced by a string $\overline{Str}(e)$ as follows (cf. 2.11. and 1.25.3).

If $t_e = g.c.d.(m_{v_1}, m_{v_2}, N)$, then set $N' = n/t_e$ and $m'_{v_i} = m_{v_i}/t_e$ for $i = 1, 2$. Consider the unique $0 \le \lambda < N'/(m'_{v_1}, N')$ and $m_1 \in \mathbf{N}$ with:

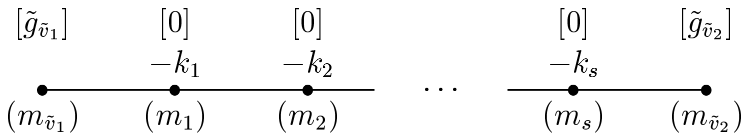
$$m'_{v_2} + \lambda \cdot \frac{m'_{v_1}}{(m'_{v_1}, N')} = m_1 \cdot \frac{N'}{(m'_{v_1}, N')}.$$

If $\lambda = 0$, then the edge $\tilde{e}$ remains unchanged.
If $\lambda \neq 0$, then take the continuous fraction:

$$\frac{N'/(m'_{v_1}, N')}{\lambda} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_s}}}, \quad k_1, \ldots, k_s \ge 2.$$

If both v_1 and v_2 are non–arrowhead vertices, then $\overline{Str}(e)$ denotes the following decorated string:

$\overline{Str}(e)$:

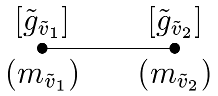


with genera $[\tilde{g}_{\tilde{v}_1}]$, $[0]$, $\ldots, [0]$, $[\tilde{g}_{\tilde{v}_2}]$, self intersection numbers $-k_1, \ldots, -k_s$, and multiplicities $m_{\tilde{v}_1}$, m_1, $\ldots, m_s$, $m_{\tilde{v}_2}$.

The multiplicities $m_{\tilde{v}_1}$ and $m_{\tilde{v}_2}$ were already determined in step 2, namely $m_{\tilde{v}_i} = m_{v_i}/(m_{v_i}, N)$; and m_1 is the number given by the the above congruence. Moreover, the multiplicities $m_2, \ldots, m_s$ can be easily determined using (2.3.2), namely

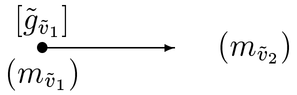
$$m_2 = k_1 m_1 - m_{\tilde{v}_1}; \text{ and } m_{i+1} = k_i m_i - m_{i-1} \text{ for } i \ge 2.$$

Then any edge $\tilde{e}$:



of $mod_N(G)$, lying above e, is replaced by the string $\overline{Str}(e)$.

Moreover, if $v_2 \in \mathcal{A}(\Gamma)$, then the edge $\tilde{e}$:



of $mod_N(G)$, lying above e, is replaced by the "modified" string $\overline{Str}(e)$ (which has the same decorations as the "original $\overline{Str}(e)$"):

This new graph is denoted by $mod_N(G)(\mathbf{Str})$.

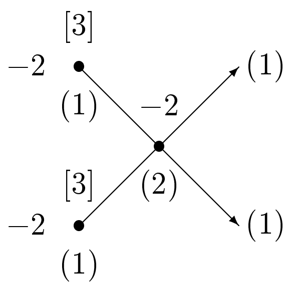
Step 4. The decoration of $mod_N(G)(\mathbf{Str})$ is not complete. All the vertices have multiplicities, all the non–arrowhead vertices have genera, but some of the self intersection numbers are missing (corresponding exactly to the vertices $\tilde{v}$ of $mod_N(G)$). Now, we add these numbers using the relation (2.3.2) applied for $mod_N(G)(\mathbf{Str})$ (namely, for any $w \in \mathcal{W}$, the relation $e_w m_w + \sum_{v \in \mathcal{V}_w} m_v = 0$ provides e_w).

3.7. Theorem. *Let $\phi : (\mathcal{Y}, D) \to (X, x)$ be any embedded resolution of $(f^{-1}(0), x) \subset (X, x)$, and $p : G(X, f) \to \Gamma(X, f)$ the universal graph–covering associated with ϕ. Then the graphs $\Gamma(X_{f,N}, z)$ and $\Gamma(X_{f,N})$ can be determined from $p : G(X, f) \to \Gamma(X, f)$ and from the integer N. Namely:*

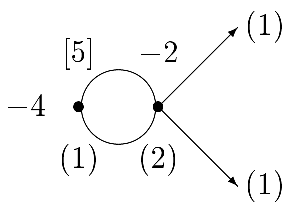
a) The decorated graph $mod_N(G)(\mathbf{Str})$ constructed in (3.6.) is (a possible) embedded resolution graph $\Gamma(X_{f,N}, z)$.

b) If we delete all the arrows and multiplicities of $mod_N(G)(\mathbf{Str})$, then we obtain a resolution graph $\Gamma(X_{f,N})$ of $X_{f,N}$.

3.8. Example. Set $(X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbf{C}^3, 0)$ and take $f_1(x, y, z) = z^2 + y^2$ and $f_2(x, y, z) = z^2 - y + y^2$ (cf. 2.25. and 2.36.). Then $\Gamma(X, f_1) = \Gamma(X, f_2)$, but in general, the graphs $\Gamma(X_{f_i,N}, z)$ $(i = 1, 2)$ are not the same. For example, these graphs for $N = 2$ are:

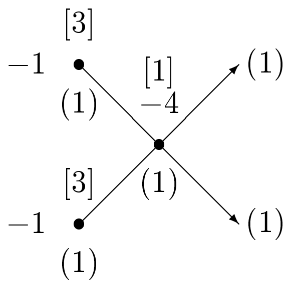


$(i = 1)$

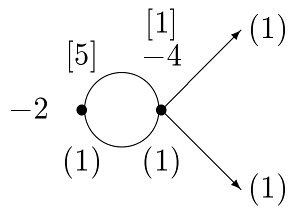


$(i = 2)$

For $N = 4$, the graphs $\Gamma(X_{f_i,N}, z)$ are:

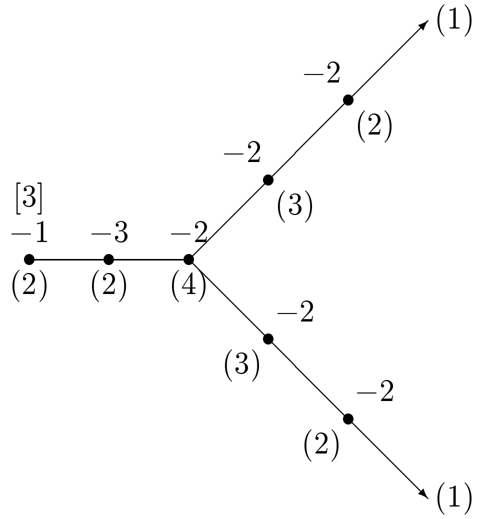


$(i = 1)$

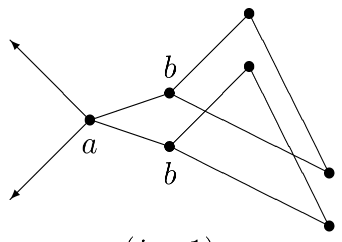


$(i = 2)$

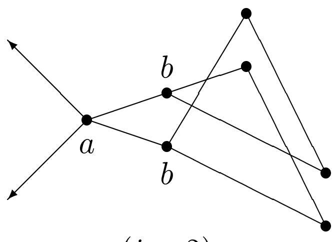
For any odd N the orbit graphs $mod_N(G)$, for $i = 1, 2$, are the same. Hence $\Gamma(X_{f_1,N}, z) = \Gamma(X_{f_2,N}, z)$. For $N = 3$ this graph is:



3.9. Example. Set $(X, x) = (\{z^2 + (x^2 - y^3)(x^3 - y^2) = 0\}, 0)$ and $f_1 = x^2 + y^4$ and $f_2 = x^2 - y^3 + y^4$ (cf. 2.26. and 2.38.). The graphs $\Gamma(X_{f_i,N}, z)$ for $N = 2$ are:



$(i = 1)$



$(i = 2)$

The decorations are the following. The arrows have multiplicity (1), $m_a = 4,\ m_b = 3$, and the multiplicities of the unmarked nodes are $m = 1$; $e_a = e_b = -2$, and the self intersection of the unmarked nodes are $e = -4$. All the genera are zero.

3.10. Remark. The construction (3.6.), in general, gives non–minimal resolution graphs. They can be simplified by blowing–down the (-1)–rational exceptional divisors E_w with $\delta_w \leq 2$.

3.11. Remark. Using the formula of (Step 2.), it is easy to prove that $\tilde{g}_{\tilde{w}} \geq g_w$. Moreover, $c_{mod_N(G)} \geq c_\Gamma$ (cf. also with (1.23.) and (3.21.)). Therefore:

$$rank\ H_1(L_{X_{f,N}}) \geq rank\ H_1(L_X).$$

3.12. – Modification of (3.6.) for the case when L_X is a rational homology sphere. If L_X is a rational homology sphere, then the above algorithm can be simplified. In this case the universal covering $G(X,f) \longrightarrow \Gamma(X,f)$ can be completely reconstructed from $\Gamma(X,f)$ (see 2.31. –2.32.). This means that the graph $\Gamma(X_{f,N}, z)$ is completely determined by the embedded resolution graph $\Gamma(X,f)$ of f and the integer N, and the reader can easily reconstruct this new algorithm.

In particular, if $(X,x) = (\mathbf{C}^2, 0))$, and $f : (\mathbf{C}^2,0) \longrightarrow (\mathbf{C},0)$ is an arbitrary isolated plane curve singularity, then $(X_{f,N}, 0) = (\{f(x,y) = z^N\}, 0) \subset (\mathbf{C}^3, 0)$, and $z : (X_{f,N}, 0) \longrightarrow (\mathbf{C}, 0)$ is induced by the projection $(x,y,z) \longmapsto z$. Therefore, our algorithm also gives the resolution graph of $(Y,0) = (\{g=0\},0)$, where $g = f(x,y) + z^N$. For the algorithm in this case, see also [28]. The idea of the construction in the case of $f + z^N$ is already in the book of Laufer [19]. For other particular cases, see [4, 33, 34].

The general algorithm can be compared with some results of Eriko Hironaka, who considers the global case of cyclic coverings.

3.13. Proof of the Theorem (3.7.). Consider the "Jungian diagram" (cf. 2.9.):

$$\begin{array}{ccccccc}
X^{res} & \longrightarrow & \tilde{X}' & \xrightarrow{n} & X' & \xrightarrow{\phi'} & (X_{f,N}, x) \\
 & \searrow^{\rho} & & \searrow^{\tilde{\pi}'} & \downarrow \pi' & & \downarrow \pi = pr \\
 & & & & (\mathcal{Y}, D) & \xrightarrow{\phi} & (X,x)
\end{array}$$

where:

a) the proper map $\pi : (X_{f,N}, x) \to (X,x)$ is induced by the projection $(x', z) \mapsto x'$, and it is an N–covering with branch locus $f^{-1}(0)$ (cf. 3.1.).

b) ϕ is a fixed embedded resolution of $(f^{-1}(0), x) \subset (X,x)$, and $D = \phi^{-1}(f^{-1}(0))$. The dual graph of D is exactly $\Gamma(X,f)$.

c) $\pi' : X' \to \mathcal{Y}$ is the pullback of π via ϕ, and $\tilde{X}'$ is the normalization of X'. Then $\tilde{X}'$ has only Hirzebruch–Jung singularities (cf. 2.8. and 2.9.).

d) $X^{res} \to \tilde{X}'$ is the resolution of the Hirzebruch–Jung singularities of $\tilde{X}'$.

If P is a generic point of the irreducible exceptional divisor $E_w \subset E = \phi^{-1}(x)$ ($w \in \mathcal{W}(\Gamma(X,f))$), then we fix local coordinates (u,v) in a neighbourhood U of P such that

$\{u=0\} = E_w \cap U$, and $f \circ \phi|_U = u^{m_w}$. Let P' be the unique point in X' above P and consider its neighbourhood $U' := (\pi')^{-1}(U)$. Then $U' = \{(u,v,z) \in (U\times \mathbf{C}, P\times 0) : u^{m_w} = z^N\}$. This shows that $n^{-1}(P')$ contains exactly (N, m_w) points, which corresponds to the irreducible components of U'. Indeed, $U' = \cup_i U'_i$, where $U'_i = \{(u,v,z) : u^{m_w/(N,m_w)} = \epsilon_i \cdot z^{N/(N,m_w)}$, where ϵ_i, for $i = 0, 1, \dots, (N, m_w) - 1$, are the (N, m_w)-roots of unity. Moreover, the normalization of any U'_i is smooth. For example, for $i = 0$, a possible normalization map of U'_0 is $(\mathbf{C}^2, 0) \to U'_0$ given by $(t, v) \mapsto (u(t), v, z(t))$, where

$$(*) \qquad u(t) = t^{N/(N,m_w)}, \quad z(t) = t^{m_w/(N,m_w)}.$$

Therefore, $(\tilde{\pi}')^{-1}(E_w) \to E_w$ is a regular covering of degree (N, m_w) above $E^0_w := E_w \setminus \overline{D \setminus E_w}$. The covering $U' \to U$ is a Galois covering with Galois group action of $\mathbf{Z}_N = \{\xi : \xi^N = 1\}$ given by $(u,v,z) \mapsto (u,v,\xi z)$. Notice that ξ^{m_w} preserves the components U'_i, therefore the covering $(\tilde{\pi}')^{-1}(E^0_w) \to E^0_w$ is a Galois covering with Galois group $\mathbf{Z}_N/m_w\mathbf{Z}_N = \mathbf{Z}_{(N,m_w)}$. This regular covering extends to a branched covering $(\tilde{\pi}')^{-1}(E_w) \to E_w$, with branch locus $E_w \setminus E^0_w$. The curve $(\tilde{\pi}')^{-1}(E_w)$ contains exactly the irreducible exceptional divisors of $\tilde{X}'$ lying above E_w. Then the number i_w of these irreducible exceptional components is exactly the number of irreducible components of $(\tilde{\pi}')^{-1}(E^0_w)$, and this number coincides with the number of connected components of $(\tilde{\pi}')^{-1}(E^0_w)$. The representation associated with the covering $(\tilde{\pi}')^{-1}(E^0_w) \to E^0_w$ is denoted by:

$$\rho_w : \pi_1(E^0_w) \to \mathbf{Z}_{(N,m_w)},$$

and the number i_w of connected components of $(\tilde{\pi}')^{-1}(E^0_w)$ is $|\mathrm{coker}\,(\rho_w)|$.

Now, let T_w be a tubular neighborhood of E_w in $\mathfrak{Y}$ (cf. 2.27.). Then $(\tilde{\pi}')^{-1}(T_w \setminus D) \to T_w \setminus D$ is a regular N–covering. The corresponding representation is denoted by:

$$r_w : \pi_1(T_w \setminus D) \to \mathbf{Z}_N.$$

This representation is induced by arg_*, as it is explained in (3.1. –3.2.). Hence, the following composed map:

$$\pi_1(T_w \setminus D) \xrightarrow{j} \pi_1(L_X \setminus L_f) \xrightarrow{arg_*} \mathbf{Z} \xrightarrow{pr_N} \mathbf{Z}_N$$

is exactly the map r_w (above, j is induced by the natural inclusion).

Now, notice that $T_w \setminus D = E^0_w \times S^1$, where S^1 can be represented by an oriented circle in a transversal slice of E^0_w in T_w. In particular, $\pi_1(T_w \setminus D) = \pi_1(E^0_w) \times \mathbf{Z}$, and $arg_*((0, 1_\mathbf{Z})) = m_w \in \mathbf{Z}$. This shows that $r_w((0, 1_\mathbf{Z}))$ is the class of m_w in $\mathbf{Z}_N$. The above discussion provides the following commutative diagram:

$$\begin{array}{rccc} \pi_1(T_w \setminus D) = & \pi_1(E^0_w) \times \mathbf{Z} & \xrightarrow{r_w} & \mathbf{Z}_N \\ & \uparrow & & \downarrow {\scriptstyle pr_{(N,m_w)}} \\ & \pi_1(E^0_w) & \xrightarrow{\rho_w} & \mathbf{Z}_{(N,m_w)} \end{array}$$

Since the class of m_w in $\mathbf{Z}_{(N,m_w)}$ is zero, one has: $|\mathrm{coker}\,(pr_{(N,m_w)} \circ r_w)| = |\mathrm{coker}\,(\rho_w)|$.

Therefore, $i_w = |\text{coker}\,(\rho_w)|$ is the cardinality of the cokernel of the following composed map:

$$p_w : \pi_1(T_w \setminus D) \xrightarrow{arg_* \circ j} \mathbf{Z} \xrightarrow{pr_N} \mathbf{Z}_N \xrightarrow{pr_{(N,m_w)}} \mathbf{Z}_{(N,m_w)}.$$

Now, consider the covering $p : G(X,f) \to \Gamma(X,f)$, and, as usual, let n_w be the number of connected components of the Milnor fiber F in T_w (i.e. the number of vertices of G above $w \in \mathcal{W}(\Gamma)$). But this is exactly $|\text{coker}\,(arg_* \circ j)|$, in other words: $im(arg_* \circ j) = n_w\mathbf{Z}$. Since $arg_*((0,1)) = m_w \in \mathbf{Z}$, one has: $n_w | m_w$. Hence $|\text{coker}\,(p_w)| = (N, n_w)$. But this is exactly the number of vertices above w in $mod_N(G)$.

The above argument can be repeated in the case of the singular points of the divisor D. Let P be an intersection point of two irreducible divisor of D, and fix local coordinates (u,v) at P such that the local equation of D at P is $\{uv = 0\}$, and $\phi \circ f$ in a neighborhood of P is $u^{m_w}v^{m_v}$. Set $P' = (\pi')^{-1}(P)$. Then the local equation of X' at P' is $\{u^{m_w}v^{m_v} = z^N\}$, which has (N, m_w, m_v) irreducible components, hence $(\tilde{\pi}')^{-1}(P)$ contains exactly (N, m_w, m_v) points. Since $n_e = (m_w, m_v)$ is the number of edges of G above the edge $(v,w) \in \mathcal{E}(\Gamma)$, $(N, n_e) = (N, m_w, m_v)$ is the number of edges in $mod_N(G)$ above (v,w); hence it coincides with $\#(\tilde{\pi}')^{-1}(P)$. Moreover, the following diagram provides the adjacent relations in G:

$$\begin{array}{ccccccc}
\pi_1(T_w \setminus D) & \xrightarrow{arg_* \circ j} & \mathbf{Z} & \to & \mathbf{Z}_{n_v} & \to 0 \\
\uparrow & & \uparrow = & & \uparrow a & \\
\pi_1(T_w \cap T_v \setminus D) & \xrightarrow{arg_* \circ j} & \mathbf{Z} & \to & \mathbf{Z}_{n_e} & \to 0
\end{array}$$

The group $\mathbf{Z}_{n_w}$ (resp. $\mathbf{Z}_{n_e}$) is the index set of the vertices (resp. edges) of G above w (resp. above $e = (w,v)$). The edge in G indexed by $\hat{l} \in \mathbf{Z}_{n_e}$ has as one of its endpoints the vertex indexed by $a(\hat{l})$.

Now, in the commutative diagram:

$$\begin{array}{ccc}
\mathbf{Z}_{n_w} & \to & \mathbf{Z}_{(N,n_w)} \\
\uparrow a & & \uparrow a \\
\mathbf{Z}_{n_e} & \to & \mathbf{Z}_{(N,n_e)}
\end{array}$$

the left column codifies the adjacent relations in G, while the right column the adjacent relations in $mod_N(G)$ (in a similar way). Therefore, by the above discussion, it is clear that $mod_N(G)$ is the dual graph of the exceptional divisors of $\tilde{X}'$.

The space $\tilde{X}'$ has only Hirzebruch–Jung singularities. The resolution of these singularities do not change the multiplicities of z along the irreducible exceptional divisors of $\tilde{X}'$, nor the topology of them. So, already at the level of $\tilde{X}'$ we can compute these invariants. First notice that the irreducible components above E_w are cyclically permuted by the Galois action, so their multiplicities and genera are the same. The formula $(*)$ (of this proof) shows that at a generic point of $(\tilde{\pi}')^{-1}(E_w)$, the multiplicity of z is $m_w/(N, m_w)$, proving (Step 2,a).

Above the branch point $D_v \cap E_w$ of the covering $(\tilde{\pi}')^{-1}(E_w) \to E_w$, there are (N, m_w, m_v) points, therefore, by an Euler–characteristic argument:

$$\chi((\tilde{\pi}')^{-1}(E_w)) = (2 - 2g_w - \delta_w)(N, m_w) + \sum_{v \in \mathcal{V}_w} (N, m_w, m_v).$$

But this is $(N, n_w) \cdot (2 - 2\tilde{g})$, proving (Step 2,b).

Now, we have to resolve the Hirzebruch–Jung singularities of $\tilde{X}'$. The singular points $\{P_{\tilde{e}}\}$ are codified by the edges $\tilde{e}$ of $mod_N(G)$, and their local equations are $\{u^{m_w/t_e} v^{m_v/t_e} = z^{N/t_e}\}$, where $t_e = (N, m_w, m_v)$, $e = (v, w)$. Therefore, any edge $\tilde{e}$ of $mod_N(G)$ must be replaced by the string of the corresponding Hirzebruch–Jung singularity. Hence the last part follows from subsection 2§.II. $\diamondsuit$

III. Cyclic coverings and monodromy.

3.14. – The main goal of the present section is to compare the covering $p : G(X, f) \to \Gamma(X, f)$ with the algebraic monodromy operator associated with the Milnor fibration $arg : L_X \setminus L_f \to S^1$.

Start again with a normal surface singularity (X, x) and a germ $f : (X, x) \longrightarrow (\mathbf{C}, 0)$ of an analytic function. The characteristic map of the Milnor fibration $F \longrightarrow L_X \setminus L_f \xrightarrow{arg} S^1$ is called the geometric monodromy $h_{geom} : F \longrightarrow F$ (where F denotes the Milnor fiber).

At homology level h_{geom} induces the algebraic monodromies $h_q : H^q(F, \mathbf{Z}) \longrightarrow H^q(F, \mathbf{Z})$ $(q = 0, 1)$. If F has k connected components, then $H^0(F, \mathbf{Z}) \approx \mathbf{Z}^k$ and $h_0(x_1, ..., x_k) = (x_2, ..., x_k, x_1)$ (cf. 2.20.). In particular, h_0 is finite.

The monodromy h_1 is more complicated – in general it is not finite. But, by the Monodromy Theorem (see, e.g. [6, 18, 21, 1, 2], all the eigenvalues of h_1 are roots of unity, and its Jordan decomposition contains blocks only of size one and two.

In this section, we connect the Jordan decomposition of h_1 with the number of independent cycles in the cyclic coverings with branch locus $\{f = 0\}$. Finally, we show that the number of Jordan blocks of h_1 is completely determined by the universal cyclic covering graph $G(X, f) \longrightarrow \Gamma(X, f)$. Some results of this subsection can be compared with some results of Ph. Du Bois and F. Michel [8].

3.15. – The characteristic polynomial. For $q = 0, 1$ we define $\Delta_q(t) = \det(th_q - 1)$. Then, for $q = 0$, $\Delta_0(t) = t^k - 1$. The characteristic polynomial $\Delta_1(t)$ is determined by $\Gamma(X, f)$, via A'Campo's formula [1, 2]:

3.16.
$$\Delta_0(t)/\Delta_1(t) = \prod_{w \in \mathcal{W}(\Gamma)} (t^{m_w} - 1)^{2 - 2g_w - \delta_w}.$$

It is convenient to use the notation $H = H^1(F, \mathbf{C})$; and H_λ for the generalized eigenspace corresponding to the eigenvalue λ of h_1 (i.e. $H_\lambda = \{x \in H |\ (h_1 - \lambda I)^n x = 0$ for some sufficiently big $n\}$.) The dimension $\dim_{\mathbf{C}} H_\lambda$ is exactly the order of $t - \lambda$ in $\Delta_1(t)$.

3.17. – The Jordan blocks of h_1. Let $\#^l_\lambda(f)$ be the number of Jordan blocks of size l of the restriction $h_1|_{H_\lambda}$. Obviously, $\#^1_\lambda(f)+2\#^2_\lambda(f)=\dim H_\lambda$ and $\dim\ker(H_1-\lambda I)=\#^1_\lambda(f)+\#^2_\lambda(f)$.

First consider $\lambda=1$.

3.18. Proposition. (cf. [29]) *Let $c_{\Gamma(X)}$ be the number of independent cycles in $\Gamma(X)$, and $g=\sum_{w\in\mathcal{W}(\Gamma)} g_w$. Then:*

a) $\dim H_1 = 2g+2c_{\Gamma(X)}+\#\mathcal{A}(\Gamma)-1$;

b) $\dim\ker(h_1-1)=2g+c_{\Gamma(X)}+\#\mathcal{A}(\Gamma)-1$.

Therefore, $\#^2_1(f)=c_{\Gamma(X)}$. In particular, $\#^2_1(f)$ is independent of the germ f, and depends only on the topology of the link L_X.

Proof. The order of $t-1$ in Δ_0/Δ_1 is $1-\dim H_1$. Via A'Campo's formula this is $\sum_w(2-2g_w-\delta_w)$. Now, use $\sum_w\delta_w=\#\mathcal{A}+2\#\mathcal{E}$, and the "Euler-characteristic" identity $1-c_{\Gamma(X)}=\#\mathcal{W}-\#\mathcal{E}$, and (a) follows. By the Wang exact sequence associated with the Milnor fibration

$$H^0(F)\xrightarrow{h_0-I}H^0(F)\longrightarrow H^1(L_X\setminus L_f)\longrightarrow H^1(F)\xrightarrow{h_1-I}H^1(F)$$

one has that $\operatorname{coker}(h_0-I)\approx\mathbf{C}$, hence $\dim\ker(h_1-1)=\dim H^1(L_X\setminus L_f)-1$. But using (2.17.) and (2.18.) $\dim H^1(L_X\setminus L_f)=\dim H_1(E)+\#\mathcal{A}=2g+c_\Gamma+\#\mathcal{A}$. $\diamondsuit$

As a corollary of (3.18.), we obtain that if L_X has no cycles (e.g. (X,x) is smooth as in the case of plane curve singularities), then f has no Jordan block of size 2 with eigenvalue $\lambda=1$.

Now, assume that $f:(X,x)\longrightarrow(\mathbf{C},0)$ is the smoothing of an *isolated singularity* at x. Then $arg_*(f)$ is onto, hence for any N, $X_{f,N}$ is connected. Moreover, there is natural identification of the Milnor fibers of f and $z:(X_{f,N},x)\to(\mathbf{C},0)$ such that the monodromy of z is the N^{th}–power of the monodromy of f. Therefore, by (3.18.) $c_{\Gamma(X_{f,N})}=\#^2_1(z)=\sum_{\lambda^N=1}\#^2_\lambda(f)$. Thus we have the following:

3.19. Corollary. *If f defines an isolated singularity, then:*

$$\sum_{\lambda^N=1}\#^2_\lambda(f)=c_{\Gamma(X_{f,N})}.$$

Since any eigenvalue of h_1 is a root of unity, h_1 is of finite order if and only if it has no Jordan blocks of size 2. Therefore, the above corollary generalizes a theorem of A. Durfee, which says that for plane curve singularities $f:(\mathbf{C}^2,0)\longrightarrow(\mathbf{C},0)$, h has finite order if and only if the graphs of the cyclic coverings have no cycles [9].

3.20. Remark. In some particular cases $c_{\Gamma(X_{f,N})}$ can be computed from the multiplicity system of $\Gamma(X,f)$ alone. For example, if L_X is a rational homology sphere, then by (2.31.):

$$\begin{aligned}1-c_{\Gamma(X_{f,N})} &= \#\mathcal{W}(\Gamma(X_{f,N}))-\#\mathcal{E}(\Gamma(X_{f,N}))\\ &= \sum_{w\in\mathcal{W}(\Gamma(X,f))}(M_w,N)-\sum_{e\in\mathcal{E}_n(\Gamma(X,f))}(m_e,N)\end{aligned}$$

where $M_w = g.c.d.\{\{m_v\}_{v\in\mathcal{V}_w}, m_w\}$ and $m_e = g.c.d.(m_{v_1}, m_{v_2})$ with $e = (v_1, v_2)$, and $\mathcal{E}_n$ denotes the set of edges connecting two non–arrowheads. Then by (3.19.) and by an easy computation using $\#\mathcal{W} - \#\mathcal{E}_n = 1$ one has:

$$\sum_{\lambda^N=1} \#^2_\lambda(f) = \sum_{e\in\mathcal{E}_n(\Gamma(X,f))} ((m_e, N) - 1) - \sum_{w\in\mathcal{W}(\Gamma(X,f))} ((M_w, N) - 1)$$

This is nothing else than W. Neumann's formula for $\#^2_\lambda(f)$ [10, 32].

Now, we will show that, in order to determine the numbers $\#^2_\lambda(f)$, we don't have to consider all the cyclic coverings, but only $G(X, f)$, the universal cyclic covering graph of $\Gamma(X, f)$. First notice that the identity (3.19.) is valid even if $f^{-1}(0)$ has non-isolated singularities, but $G(X, f)$ is connected. Then $X_{f,N}$ is connected for any N.

3.21. Corollary. *Assume that the universal covering graph $G(X, f)$ is connected (e.g. f defines an isolated singularity). Then:*

(a)

$$\sum_{\lambda^N=1} \#^2_\lambda(f) = c_{mod_N(G(X,f))}.$$

(b)

$$\sum_\lambda \#^2_\lambda(f) = c_{G(X,f)};$$

i.e. the number of independent cycles of $G(X, f)$ is exactly the total number of Jordan blocks of size 2 of the algebraic monodromy h_1 of f.

(c) Let ξ_n be a primitive n^{th}–root of unity. Then:

$$\phi(n) \cdot \#^2_{\xi_n} = \sum_{k|n} \mu(n/k) \cdot c_{mod_k(G(X,f))}$$

where ϕ is the Euler function and μ is the Möbius function, namely: $\phi(n) = \#\{1 \le k \le n : (k, n) = 1\}$, and $\mu(k) = 1$ for $k = 1$, $\mu(k) = (-1)^t$ if k is a product of t different primes, and $\mu(k) = 0$ if $p^2|k$ for some prime p.

Proof. (a) follows from a similar argument as (3.19.) and the main theorem (3.7.). For (b), take an N which is multiple of all the integers $\{n_v\}_{v\in\mathcal{V}}$ and $\{n_e\}_{e\in\mathcal{E}}$ (the integers which characterize the type of the universal covering $G(X, f) \longrightarrow \Gamma(X, f)$), and also $\lambda^N = 1$ for any eigenvalue λ of h_1. For (c), first notice that $\#^2_{\xi_n}$ does not depend on the choice of ξ_n (because h_1 is defined over $\mathbf{Z}$). Then (c) follows from the Möbius Inversion Formula (see, e.g. [16], page 107). ♢

3.22. Example. Set (X, x) as in (2.36.) (cf. also (2.25.)). Then $h_1(f_1)$ is finite, but $h_1(f_2)$ has a Jordan block of size 2, with eigenvalue $\lambda = -1$. In this case $\Gamma(X, f_1) = \Gamma(X, f_2)$, but $arg_*(f_1) \neq arg_*(f_2)$. This shows a subtle connection between the Jordan block structure of $h_1(f)$ and the representation $arg_*(f)$.

3.23. Example. Set (X, x) as in (2.38.) (or 2.26.). Then for both $i = 1, 2$, $h_1(f_i)$ has a Jordan block of size 2 with eigenvalue $\lambda = 1$, because $\Gamma(X, f)$ has a cycle. Since $G(X, f)$ has two independent cycles, there is one more Jordan block of size 2 and this one has eigenvalue $\lambda = -1$.

3.24. – Actually, an even stronger connection can be established between the monodromy h_1 and the graph $G(X, f)$. Let $|G|$ be the topological realization of $G(X, f)$. The $\mathbf{Z}$–action of $G(X, f)$ induces a "geometric action" $h_{G,geom}$ on $|G|$ (by the identification $1_{\mathbf{Z}} = h_{G,geom}$). At homological level, this induces a finite morphism $h^*_{|G|} : H^1(|G|) \to H^1(|G|)$.

In the sequel, $H^*(Y)$ denotes the cohomology with complex coefficients. We invite the reader to review the notations and the results of subsection (2.27.). Recall that $\mathcal{V}(\Gamma)$ is the index set of the irreducible components of $D = \phi^{-1}(f^{-1}(0))$. Let T_v be a small tubular neighborhood of the irreducible component corresponding to $v \in \mathcal{V}(\Gamma)$. For any edge $e = (v_1, v_2)$ of Γ set $T_e = T_{v_1} \cap T_{v_2}$. If $F = f^{-1}(\delta)$ is the Milnor fiber, then for δ sufficiently small $F \subset \cup_v T_v$. Put $F_v = F \cap T_v$ and $F_e = F \cap T_e$ for any $v \in \mathcal{V}(\Gamma)$ and $e \in \mathcal{E}(\Gamma)$. Then by Mayer–Vietoris argument one has the following exact sequence:

$$0 \to H^0(F) \to \oplus_v H^0(F_v) \to \oplus_e H^0(F_e) \to H^1(F) \to \oplus_v H^1(F_v) \to \oplus_e H^1(F_e) \to 0.$$

Now, by the very definition of the graph $G = G(X, f)$, one has the exact sequence:

$$0 \to H^0(|G|) \to \oplus_v H^0(F_v) \to \oplus_e H^0(F_e) \to H^1(|G|) \to 0.$$

Now, we recall that there is a continuous map $\alpha : F \to |G|$ (cf. 2.27.) which is compatible with the above decomposition, in particular the above exact sequences (complexes) are connected by maps (i.e. by a morphism of complexes) induced by α.

This provides an exact sequence:

$$0 \to H^1(|G|) \xrightarrow{\alpha^*} H^1(F) \xrightarrow{\delta^0} \oplus_v H^1(F_v) \xrightarrow{\delta^1} \oplus_e H^1(F_e) \to 0.$$

The monodromy acts on this exact sequence, the operators on the corresponding groups are: $h^*_{|G|}$, h_1, $\oplus_w h_w$ and $\oplus_e h_e$. Notice that for any $w \in \mathcal{W}(\Gamma)$ the natural projection of T_w to E_w induces a m_w–covering of $F_w \to E_w$ with Galois group $\mathbf{Z}_{m_w}$. Moreover, the monodromy action h_w can be identified with the action of the generator $\hat{1}$ of this Galois group. In particular, h_w has finite order. Now, it is elementary to verify that the other monodromy operators h_a $(a \in \mathcal{A})$ and h_e $(e \in \mathcal{E})$ are also of finite order. On the other hand, $h^*_{|G|}$ is finite by its construction.

Let N_0 be an integer such that $\lambda^{N_0} = 1$ for any eigenvalue λ of h_1 and h_v. Then $\delta^0(im(h_1^{N_0} - 1)) = 0$, hence $H^1(|G|) \subset im(h_1^{N_0} - 1)$. But, by the above corollary (3.21.) (part b), these spaces have the same dimension. Therefore, we have:

3.25. Corollary. *Fix an N_0 such that $\lambda^{N_0} = 1$ for any eigenvalue λ of $h_1(f)$. Then the pairs $(H^1(|G|), h^*_{|G|})$ and $(im(h_1^{N_0} - 1), h_1)$ are isomorphic. This identification is compatible* *with the generalized eigenspace decomposition, in particular, for any eigenvalue λ there exists the following commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \to & H^1(|G|)_\lambda & \to & H^1(F)_\lambda & \to & coker(h_1^{N_0}-1)_\lambda & \to & 0 \\
 & & \Big\downarrow h^*_{|G|} & & \Big\downarrow h_1 & & \Big\downarrow h_1 & & \\
0 & \to & H^1(|G|)_\lambda & \to & H^1(F)_\lambda & \to & coker(h_1^{N_0}-1)_\lambda & \to & 0.
\end{array}$$

This shows that for any λ:

$$\#^2_\lambda(f) = H^1(|G|)_\lambda.$$

Actually, on this picture we can also see all the cohomology groups $H^1(mod_N(G)|)$ for arbitrary integer N. Indeed, the "orbit projection" $o_N : |G| \to |mod_N(G)|$ induces an injective morphism $o_N^* : H^1(|mod_N(G)|) \to H^1(|G|)$ (where $im(o_N^*)$ are the invariant cocycles), therefore o_N^* can be identified with the inclusion

$$H^1(|mod_N(G)|) = \ker(h^N_{|G|} - 1) \hookrightarrow H^1(|G|).$$

3.26. – The fact that the 2×2–Jordan blocks can be characterized by a graph with a $\mathbf{Z}$–action, has the following interesting consequence (cf. [29]):

3.27. Proposition. *Assume that $c_\Gamma = 0$ and $G(X,f)$ is connected. Assume that h_1 has a Jordan block of size 2 with an eigenvalue β with $\beta^k = 1$. Write $k = pq$ with $(p,q) = 1$. Then h_1 has a Jordan block of size 2 with some eigenvalue, say λ, such that either $\lambda^p = 1$ or $\lambda^q = 1$.*

Proof. Fix a vertex $w^* \in \mathcal{W}(\Gamma)$ such that there is an arrow $a \in \mathcal{A}(\Gamma)$ with $(a, w^*) \in \mathcal{E}(\Gamma)$. Let $\mathcal{E}_n$ be the set of edges of Γ connecting two non–arrowhead vertices. For any $e \in \mathcal{E}_n$ let $w(e)$ be the vertex of e which has larger distance in Γ from w^* (i.e. $w(e)$ and w^* are in two different components of $\Gamma \setminus \{e\}$). Then $e \mapsto w(e)$ defines a bijection $\mathcal{E}_w \to \mathcal{W} \setminus \{w^*\}$. By Euler–characteristic argument:

$$1 - c_{mod_k(G)} = \sum_{w\in\mathcal{W}} g.c.d.(k, n_w) - \sum_{e\in\mathcal{E}_n} g.c.d.(k, n_e).$$

But $n_{w^*} = 1$ (cf. 2.33.), hence:

$$c_{mod_k(G)} = \sum_{e\in\mathcal{E}_n} g.c.d.(k, n_e) - g.c.d.(k, n_{w(e)}),$$

where $n_{w(e)}|n_e$.

If $\#^2_\beta(f) \neq 0$ for some β with $\beta^k = 1$, then $c_{mod_k(G)} \neq 0$, hence there exists $e \in \mathcal{E}_n$ with $(k, n_e) > (k, n_{w(e)})$. But then, for the same e, a similar strict inequality holds either for p or for q (instead of k). This ends the proof. $\diamondsuit$

3.28. Remark. The above proof contains a generalization of the result (3.20.) of Neumann. Namely, even if $g \neq 0$ (but with $c_{\Gamma(X)} = 0$), there exist positive integers n_i and m_i ($i \in I$), with $m_i|n_i$, such that:

$$\sum_{\lambda^N=1} \#^2_\lambda(f) = \sum_i g.c.d.(N, n_i) - g.c.d.(N, m_i).$$

3.29. The finiteness of h_1 revisited. Above we proved the characterization $\sum_\lambda \#^2_\lambda(f) = c_{G(X,f)}$, but we still can ask: when is this number zero?

For example, Lê D. T. proved that the monodromy of an irreducible germ $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is always finite [20].

In Lê's result the irreducibility assumption is really important. Indeed, if we take $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ given by $f(x, y) = (x^2 + y^3)(x^3 + y^2)$, then by (2.38.), the resolution of the double covering $\mathbf{C}^2_{f,2}$ has a cycle, hence $h_1(f)$ is not finite (historically, this is the first germ with non–finite monodromy, found by A'Campo [1]).

On the other hand, if $f : (X, x) \to (\mathbf{C}, 0)$ has a finite monodromy $h_1(f)$, then $c_{\Gamma(X)}$ must be zero (cf. 3.18.).

So, this motivates to investigate the case when $c_{\Gamma(X)} = 0$ and $\#\mathcal{A}(\Gamma) = 1$. Under a more restrictive condition, when L_X is an *integer* homology sphere, W. Neumann (using a stronger version of the identity (3.20.)) proved that $h_1(f)$ is finite, provided that $\#\mathcal{A}(\Gamma) = 1$ (see [10] page 111, or [32]). The next result is a generalization of the results of Lê and Neumann.

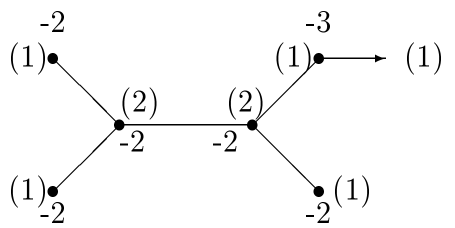
3.30. Theorem. *Assume that L_X is a rational homology sphere and $f : (X, x) \to (\mathbf{C}, 0)$ defines an isolated singularity with $\#\mathcal{A}(\Gamma) = 1$. Then if $\#^2_\lambda \neq 0$ for some λ of order k, then there exists a prime number $p \geq 2$ such that $p|k$ and $p^2|\ |H_1(L_X, \mathbf{Z})|$.*

In particular, if $|H_1(L_X, \mathbf{Z})|$ is square free, then $h_1(f)$ is finite for any f.

(Recall that the order $|H_1(L_X, \mathbf{Z})|$ of the torsion group $H_1(L_X, \mathbf{Z})$ is exactly $|\det(E_w \cdot E_v)|$.)

Proof. Assume that $\#^2_\lambda \neq 0$ with $\lambda^k = 1$. Then by (3.27.), there is a prime number p and eigenvalue η such that $p|k$, $\eta^{p^\alpha} = 1$ for some $\alpha \geq 1$, and $\#^2_\eta \neq 0$. Write the characteristic polynomial $\Delta(t) = \det(th_1 - 1)$ as a product of cyclotomic polynomials $\prod_l \phi_l(t)^{t_l}$. By Wang's exact sequence and Alexander duality, coker $(h_1 - 1) = H_1(L_X, \mathbf{Z})$. Then $\Delta(1) = \prod_l \phi_l(1)^{t_l} = |H_1(L_X, \mathbf{Z})|$. Now, if l is not a prime power, then $\phi_l(1) = 1$, whereas $\phi_l(1) = q$ if l is a power of the prime q. Since h_1 has a double eigenvalue of order a power of the prime p, we obtain the divisibility $p^2|\Delta(1)$. ♢

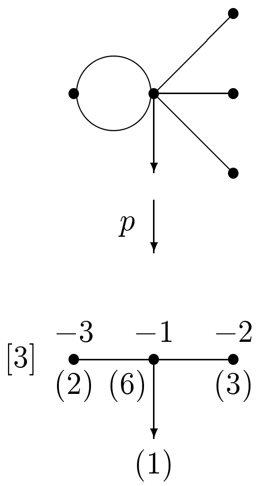
3.31. Example. [29] Consider the embedded resolution graph of $f : (X, x) \to (\mathbf{C}, 0)$, where $(X, 0) = (\{x^{10} + x^3y^2 + y^3 + z^2 = 0\}, 0) \subset (\mathbf{C}^3, 0)$ and $f(x, y, z) = x$:



Then $|\det(E_v \cdot E_w)| = |H_1(L_X, \mathbf{Z})| = 4$. Then G can be computed from 2.33., hence the monodromy h_1 contains exactly one Jordan block of size 2 and the eigenvalue of this block is $= -1$.

Finally, we ask: in the above result (3.30.), is it really important for L_X to be a rational homology sphere ? Is the monodromy finite, for example, if $c_{\Gamma(X)} = 0$, $\#\mathcal{A}(\Gamma(X, f)) = 1$ and $\mathrm{Tors}(H_1(L_X, \mathbf{Z})) = 0$ (i.e. the intersection matrix $(E_w \cdot E_v)$ is unimodular) ? The answer is negative ! A possible contraexample is the following:

3.32. Example. In theorem (3.30.) not only $c_\Gamma = 0$ is important but also $g = 0$. To see this, take $(X, x) = (\{x^2 + y^7 - z^{14} = 0\}, 0) \subset (\mathbf{C}^3, 0)$ and the function $f_2(x, y, z) = z^2 - y$ as in (2.25.) and (2.36.). Let P be the intersection point of the strict transform S_a of $\{f_2 = 0\}$ with the exceptional divisor E. Then, in some local coordinate system (u, v) of P, $\{u = 0\}$ represents E (in a neighborhood of P), $\{v = 0\}$ represents S_a, and $f_2 = u^2v^2$ (cf. 2.25.). Consider $g = z$. Since z in the neighborhood of P can be represented as $z = u$ (modulo a local invertible germ), $f_2 + g^k$ near P has the form $u^2v^2 + u^k$. For example, if $k = 3$, then one needs two more blowing ups in order to resolve $f_2 + g^k$. In this case, the graph $\Gamma(X, -y + z^2 + z^3)$ and its covering $G(X, -y + z^2 + z^3)$ are the following:



So, in this case, the intersection matrix of Γ is unimodular, hence $H_1(L_X, \mathbf{Z})$ is torsion free, $c_\Gamma = 0$, and $\#\mathcal{A} = 1$, but $\#^2_{-1} = 1$, hence h_1 is not finite. The reason is that $g \neq 0$.

This shows also that the part b) of Theorem B in [29] is true only with the additional assumption $g = 0$. (The other statements of [29] are correct; the error in (B,b) comes from the identity (1) on the page 591.)

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Resolution graphs of some surface singularities, II. (generalized Iomdin series)

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Abstract. In this article, we determine the resolution graph of the hypersurface singularity $(\{f + g^k = 0\}, 0)$, where $(f, g) : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ is an ICIS, and k is a sufficiently large integer. All these graphs are coordinated by an "universal bi-colored graph" $\Gamma_{\mathcal{C}}$ associated with the ICIS (f, g). Its definition is rather involved, and in concrete examples it is difficult to compute it. Nevertheless, we will present a large number of examples. This will be very helpful in the exemplification of its properties as well. Then we present our main construction which provides the dual resolution graph of the series $\{f + g^k = 0\}$ from the graph $\Gamma_{\mathcal{C}}$ and the integer k. This is formulated in a purely combinatorial algorithm. The result is a highly non-trivial generalization of the cyclic covering case of $(\mathbf{C}^2, 0)$.

Introduction.

The present article (in the sequel: Part II) is a natural continuation of [15] (called Part I) in such an extent that we will use the notations and results of Part I without rewieving them, and we even use a continuous numbering of the sections.

Consider an isolated complete intersection singularity (ICIS) $\Phi = (f, g) : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ such that f has an 1–dimensional singular locus. Then construct the so called "generalized Iomdin series" $f + g^k$ (for $k \geq k_0$) (see section 4, i.e. the first section of part II). Our goal is to describe the resolution graph $\Gamma(X_k)$ of $(X_k, 0) := (\{f + g^k = 0\}, 0)$.

The Iomdin series is a generalization of the cyclic covering case $f(x, y) + z^k$, where f is an isolated plane curve singularity. But the differences (both technical and of principle) are huge. For example, while in the case of the cyclic coverings, the relevant monodromy representations are easy (being representations of $\mathbf{Z}$), the monodromy representation of an ICIS is incomparably more complicated. Moreover, in the cyclic case there is a global Galois action, which disappears in the general case.

However, the construction of $\Gamma(X_k)$ will show strong similarities with the case of cyclic coverings: we will construct from the ICIS Φ a universal graph $\Gamma_{\mathcal{C}}$ which will coordinate the whole series of graphs $\{\Gamma(X_k)\}_k$. Already the existence of such a graph is really remarkable.

The construction of $\Gamma_{\mathcal{C}}$ is rather involved. We will show that all the geometric information needed for the construction of the graphs $\{\Gamma(X_k)\}_k$ is contained in a small neighborhood of a special curve arrangement $\mathcal{C}$, situated in the embedded resolution of the local divisor $(\{fg = 0\}, 0) \subset (\mathbf{C}^3, 0)$. $\Gamma_{\mathcal{C}}$ is exactly the dual graph of $\mathcal{C}$. We will decorate this graph with different weights, which codify the topology of the irreducible components of $\mathcal{C}$, and also the local behavior of the functions f and g near $\mathcal{C}$. In section 5 we present this construction, some examples and also some properties of $\Gamma_{\mathcal{C}}$. E.g., we will prove that from $\Gamma_{\mathcal{C}}$ one can recover the resolution graph of $(\{f = 0\}, 0)$, and also the equisingular type of the transversal singularities of $Sing\{f = 0\}$.

In section 6 we will present the algorithm which will provide, in a purely combinatorial way, the graph $\Gamma(X_k)$ from $\Gamma_{\mathcal{C}}$ and the integer k (for $k \geq k_0$). The validity of the algorithm is based on the following important fact. The graph $\Gamma(X_k)$ appears as a cyclic covering graph of $\Gamma_{\mathcal{C}}$ (modified with some Hirzebruch–Jung strings). The weights of $\Gamma_{\mathcal{C}}$ codify all the local data from a neighborhood of $\mathcal{C}$, but a priori it is not clear that from this data one can recover the necessary global information in order to identify the covering $\Gamma(X_k)$. But, using the classification results of section 1 (part I), and the properties of $\Gamma_{\mathcal{C}}$, we show that the covering data of the covering can be determined from the weights of $\Gamma_{\mathcal{C}}$, and with the determined covering data there is a unique graph covering. This basically guarantees the vanishing of the global invariants.

The graph $\Gamma_{\mathcal{C}}$ can also be connected with some other invariants of the Iomdin series (for details, see (4.10.) and (6.17.)).

4§. Preliminaries about generalized Iomdin series

I. Basics of ICIS.

4.1. – In this subsection we review the basic notions related to isolated complete intersection singularities (ICIS). For details, see [8]. In order to keep the notation uniform, we only present the case $(\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$.

Consider an analytic germ $\Phi = (f, g) : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ which defines an ICIS. This means that if $I \subset \mathcal{O}_{\mathbf{C}^3,0}$ denotes the ideal generated by f, g and the 2×2 minors of the Jacobian matrix $(d\Phi)$, then $\dim \mathcal{O}_{\mathbf{C}^3,0}/I < \infty$. In other words, the scheme-theoretical intersection $\Phi^{-1}(0) = \{f = 0\} \cap \{g = 0\}$ has only an isolated singularity at the origin. In particular, $(Sing\{f = 0\}) \cap \{g = 0\} = \{0\}$ and $\{f = 0\}$ intersects $\{g = 0\}$ in the complement of the origin transversally in a smooth (punctured) curve.

The *critical locus* $(C_\Phi, 0)$ of Φ is the set of points of $(\mathbf{C}^3, 0)$, where Φ is not a submersion. Its image $(\Delta_\Phi, 0) := \Phi(C_\Phi, 0) \subset (\mathbf{C}^2, 0)$ is called the *discriminant locus* of Φ. In the sequel, we denote by (c, d) the local coordinates of $(\mathbf{C}^2, 0)$.

4.2. The Milnor fibration. Fix a germ Φ as above. Then there exist a sufficiently small closed ball $B^3_\epsilon \subset \mathbf{C}^3$ (centered at the origin with radius $\epsilon > 0$) and $D^2_\eta \subset \mathbf{C}^2$ with radius $0 < \eta << \epsilon$ such that:

- the set $(\Phi^{-1}(0) \setminus \{0\}) \cap B^3_\epsilon$ is non-singular;
- $\partial B^3_{\epsilon'}$ intersects $\Phi^{-1}(0)$ transversally for all $0 < \epsilon' \leq \epsilon$.
- $C_\Phi \cap \Phi^{-1}(D^2_\eta) \cap \partial B^3_\epsilon = \emptyset$; and the restriction of Φ to $\Phi^{-1}(D^2_\eta) \cap \partial B^3_\epsilon$ is a submersion.

4.3. Definition. *The map $\Phi : \Phi^{-1}(D^2_\eta) \cap B^3_\epsilon \to D^2_\eta$ with the above properties is called a "good" representative of the ICIS Φ. For such a good representative, let Σ_Φ denote the intersection $C_\Phi \cap \Phi^{-1}(D^2_\eta)$ and $\Delta_\Phi = \Phi(\Sigma_\Phi)$.*

With these notations we have the following additional properties (cf. 2.8 in [8]):

4.4. Theorem. *(i) $\Phi : \overline{B^3_\epsilon} \cap \Phi^{-1}(D^2_\eta) \longrightarrow D^2_\eta$ is proper. The analytic sets Σ_Φ and Δ_Φ are 1–dimensional, and the restriction $\Phi|_{\Sigma_\Phi} : \Sigma_\Phi \to \Delta_\Phi$ is proper with finite fibres.*

(ii) $\Phi : (\Phi^{-1}(D^2_\eta - \Delta_\Phi) \cap B^3_\epsilon, \Phi^{-1}(D^2_\eta - \Delta_\Phi) \cap \partial B^3_\epsilon) \to D^2_\eta - \Delta_\Phi$ *is a locally trivial* C^∞*-fibration of a pair of spaces.*

4.5. Definition. *A fibre* $F_{c,d} = \Phi^{-1}(c,d) \cap B^3_\epsilon$*, for* $(c,d) \in D^2_\eta - \Delta_\Phi$*, is called "a Milnor fibre" and the fibration itself is referred to as "the Milnor fibration". For any fixed base point* $b_0 = (c_0, d_0) \subset D^2_\eta - \Delta_\Phi$*, one has the natural* geometric monodromy representation*:* $m_{geom} : \pi_1(D^2_\eta - \Delta_\Phi, b_0) \longrightarrow Diff^\infty(F_{b_0})/isotopy$*.*

In general, it is extremely difficult to determine this representation. Actually, even the induced *algebraic monodromy representation* $m_{alg} : \pi_1(D^2_\eta - \Delta_\Phi, b_0) \to Aut H_*(F_{b_0}, \mathbf{Z})$ is a very complicated object.

4.6. – Let $\Delta_\Phi = \Delta_1 \cup ... \cup \Delta_t$ denote the decomposition of Δ_Φ into irreducible components. Let $\Sigma_{i,1}, ..., \Sigma_{i,s_i}$ be the irreducible decomposition of $\Phi^{-1}(\Delta_i) \cap \Sigma_\Phi$ for $i = 1, ..., t$. Clearly, $\Sigma_\Phi = \cup_{i=1}^t \cup_{j=1}^{s_i} \Sigma_{ij}$ is a decomposition of Σ_Φ into irreducible components.

Assume that f has a 1-dimensional singular locus. Clearly, $Sing\{f = 0\} \subset \Sigma_\Phi$ and $\Phi(Sing\{f = 0\}) = \{c = 0\}$ is an irreducible component of the discriminant Δ_Φ. By convention, we denote this component by Δ_1. Then $\Sigma_{1,1}, ..., \Sigma_{1,s_1}$ are exactly the irreducible components of $(Sing\{f = 0\}, 0)$. Sometimes, we will use the simplified notations $\Sigma_{1,j} = \Sigma_j$ and $s_1 = s$. Part (i) of the above theorem guarantees that the restriction $\Phi : \Sigma_j \to \Delta_1$ is a branched covering for any $1 \leq j \leq s$. Let d_j denote its degree.

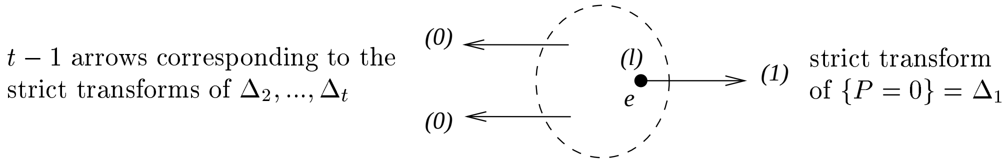
4.7. Definition. – Transversal singularity types. Fix an irreducible component Σ_j of $Sing\{f = 0\}$, $j \in \{1, ..., s\}$. Take a point $q \in \Sigma_j - \{0\}$ and the germ (H, q) of a generic smooth transversal slice to Σ_j at q. The intersection $(\{f = 0\} \cap H, q)$ determines an isolated plane curve singularity $(T\Sigma_j, q) \subset (H, q)$. Its topological (or equisingular) type does not depend on the choice of q and (H, q). *It is called the transversal singularity of* $Sing\{f = 0\}$ *corresponding to the branch* Σ_j*.*

II. Generalized Iomdin series.

The *generalized Iomdin series* is a special family of topological series of composed hypersurface singularities. For the general theory of the topological series, the interested reader can consult [18, 13, 9, 10]. In this subsection we restrict ourselves to the definition of the special case of the Iomdin series.

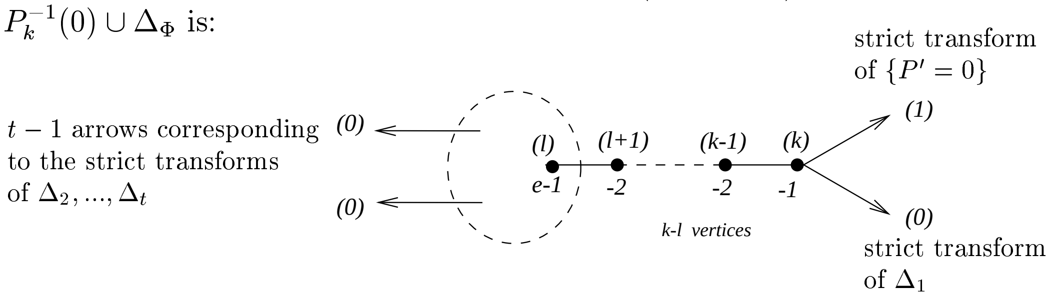
4.8. Definition. *Consider a germ* $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ *with a 1-dimensional singular locus. Then fix another germ* $g : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$*, such that the pair* $\Phi = (f, g) : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ *forms an ICIS. Then the family of singularities* $f + g^k : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ *constitutes the* generalized Iomdin series *of* f*, associated with* g*, where* k *is any integer larger than a bound* $k_0 = k_{0,\Phi}$*. (The bound depends only on* Φ*, and will be determined in the sequel).*

Now, we provide a more precise definition. First notice that the germs f and $f + g^k$ are composed singularities. Indeed, $f = \Phi \circ P$, where $P : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is the plane curve singularity $(c, d) \mapsto c$. Moreover, $f + g^k = \Phi \circ P_k$ with the same Φ, but P_k given by $(c, d) \mapsto c + d^k$. Similarly as above, let Δ_Φ be the discriminant locus of Φ with the distinguished component $\Delta_1 = \{c = 0\}$. A possible embedded resolution graph of the divisor $(P^{-1}(0) \cup \Delta_\Phi, 0) \subset (\mathbf{C}^2, 0)$ can be visualized schematically as:



Above, the multiplicities (m) are the vanishing orders of the (lift of the) germ P along the corresponding components of the total transform of $P^{-1}(0) \cup \Delta_\Phi$.

By definition, the germ $f + g^k$ belongs to the generalized Iomdin series of f, associated with g, if $k \geq l + 1$. This means that a possible (schematic) embedded resolution graph of $P_k^{-1}(0) \cup \Delta_\Phi$ is:

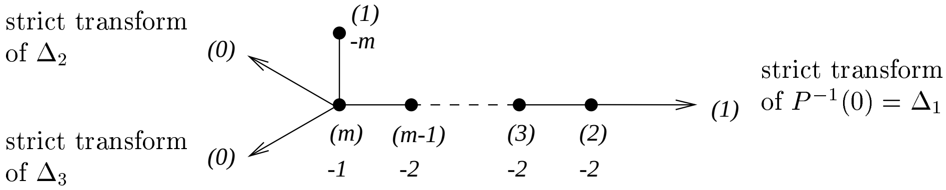


where the multiplicities (m) are the vanishing orders of the germ P_k along the corresponding components of the total transform of $P_k^{-1}(0) \cup \Delta_\Phi$. (Hence $k_{0,\Phi} = l + 1$.)

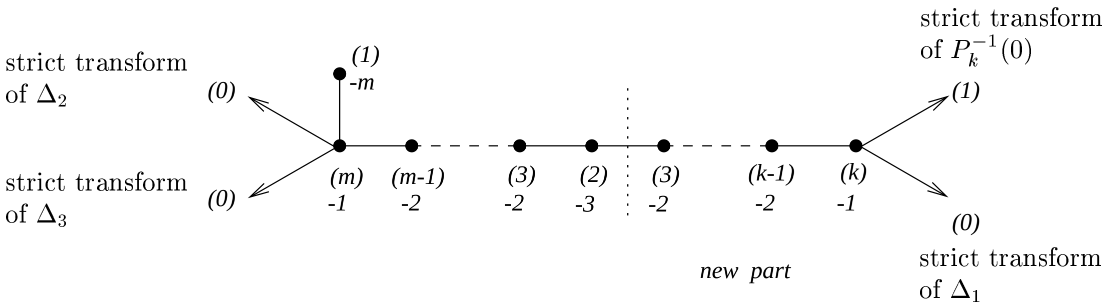
4.9. Example. Let $f = x^n + y^n + xyz$ with $n \geq 3$ and $g = z$. Then $\Phi = (f, g) : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ is an ICIS. The critical locus of Φ is $\Sigma_\Phi = \{x = y = 0\} \cup \cup_{j=1}^n \{x = \xi_0^j y,\ z = -n\xi_0^{j(n-1)} y^{n-2}\}$, where $\xi_0 = e^{2\pi i/n}$. Moreover, $\{x = y = 0\} = Sing(f^{-1}(0))$. Then $\Phi(\{x = y = 0\}) = \{c = 0\}$ and $\Phi(\{x = \xi_0^j y,\ z = -n\xi_0^{j(n-1)} y^{n-2}\}) = \{c = (2-n)y^n,\ d = -n\xi_0^{j(n-1)} y^{n-2}\}$ (for $j = 1, ..., n$), but some of these components may coincide. Indeed, let $l = g.c.d.(n, n-2)$. Note that $l = 1$ or $l = 2$.

If $l = 2$, i.e. if $n = 2m$ for some $m \geq 1$, then $\Delta_\Phi = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where $\Delta_1 = \{c = 0\}$, $\Delta_2 = \{c^{m-1} = Kd^m\}$, $\Delta_3 = \{c^{m-1} = -Kd^m\}$ (for some constant $K \neq 0$).

The resolution graph of $P^{-1}(0) \cup \Delta_\Phi = \Delta_1 \cup \Delta_2 \cup \Delta_3$ is



The resolution graph of $P_k^{-1}(0) \cup \Delta_\Phi = \{c = d^k\} \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$, for any $k \geq 3$, is:



The reader is invited to construct the corresponding diagrams in the case $l = 1$. In this second case Δ_Φ has two components: $\Delta_1 = \{c = 0\}$ and $\Delta_2 = \{c^{2m-1} = K'd^{2m+1}\}$ (for some constant $K' \neq 0$). Note that for both $l = 1$ and $l = 2$ one has $d_1 = 1$.

4.10. Remark. Notice that all the germs $f + g^k$ $(k \geq k_0)$ define *isolated* hypersurface singularities. The name of the series originates from the work of I. N. Iomdin [5], who studied the case when g was a generic linear form (called the "classical" case).

There is a rich literature dealing with the invariants of the germs of the (classical or generalized) Iomdin series. Some results compare the invariants of $f + g^k$ with some invariants of Φ. For example, using the polar curve method, one can determine the difference of the Euler-characteristic of the Milnor fibers of $f + g^k$ and of the ICIS Φ using the multiplicity of Δ_Φ (for details, see some of the papers of Lê Dũng Tráng and B. Teissier, e.g. [20]).

Other results compare some invariant of $f+g^k$ with the corresponding invariant of the non-isolated singularity f. Correction term formulae $i(f + g^k) - i(f)$ were obtained for many "additive" invariants i: in the classical case by Iomdin for the case when i the Euler characteristic of the corresponding Milnor fibers [5], by D. Siersma when i denotes the zeta function [19], by M. Saito when i is the spectrum [17], and by A. Némethi when i is the signature [11, 12]. In order to explain why it is more convenient to compute the *correction term* $i(f + g^k) - i(f)$, rather than the individual terms $i(f + g^k)$ and $i(f)$, we introduce the following object. Fix a good representative of Φ, and assume that $\Delta_1 = \{c = 0\}$.

4.11. Definition. *For any integer $M > 0$, we define the "wedge set" of $\Delta_1 \subset D^2_\eta$ by*

$$\mathsf{W}_{\eta,M} = \{(c, d) \in D^2_\eta \mid 0 < |c| < |d|^M\}.$$

Now, it is easy to verify that for a sufficiently large integer $M >> 0$ (and with η shrinked, if necessary) $\mathsf{W}_{\eta,M} \subset D^2_\eta \setminus \Delta_\Phi$. In particular, Φ is a locally trivial fibration over $\mathsf{W}_{\eta,M}$.

The point is that in order to determine the individual terms $i(f + g^k)$, one needs, in general, to understand the usually rather complicated monodromy representation associated with Φ. On the other hand, for some "nice" (in some sense "additive") invariants (such as the invariants listed above), the correction term $i(f+g^k)-i(f)$ depends *only* on the behavior of Φ above Δ_1 and on the representation associated with the fibration above a wedge set. But this representation is much simpler than the original monodromy representation, since $\pi_1(\mathsf{W}_{\eta,M}) = \mathbf{Z}^2$. In fact, this is exactly the main idea in the construction of the series: the geometry of the limit object is changed in a small tubular neighborhood of a knot. Here the knot is $\Delta_1 \cap \partial D^2_\eta$, and its tubular neighbourhood is $\overline{\mathsf{W}_{\eta,M}} \cap \partial D^2_\eta$.

Recall that the aim of this work is to compute another invariant: the resolution graph of the hypersurface singularities $(X_k, 0) = (\{f + g^k = 0\}, 0)$. First notice that this invariant is not (really) additive, which makes its study more difficult. Moreover, our aim is to obtain the "whole" invariant, not "just" a correction term. Nevertheless, the wedge set will still play a crucial role: the resolution graphs of all the germs $(X_k, 0)$ depend only on the behavior of Φ over the closure of a wedge set. Moreover, the construction will provide a "correction term" as well.

Our method is completely different (although obviously not absolutely independent) from the polar curve method, or from the usual constructions applied in the cases $i(f + g^k) - i(f)$. The discussion of some connections is the subject of another paper [16], where e.g. the relationship between the resolution graphs of the singularities $(X_k, 0)$ and the monodromy representation over $\mathsf{W}_{\eta,M}$ will be developed.

5§. A graph associated with the ICIS Φ.

I. A special curve arrangement.

5.1. – The embedded resolution of $f^{-1}(0) \cup g^{-1}(0) \subset \mathbf{C}^3$ and its stratification. Consider an ICIS $\Phi = (f, g) : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ and let $(D, 0) \subset (\mathbf{C}^3, 0)$ denote the local divisor $(D, 0) := ((fg)^{-1}(0), 0)$.

Fix an embedded resolution $r : V^{emb} \to U$ of $(D, 0) \subset (\mathbf{C}^3, 0)$. This means that U is a small representative of $(\mathbf{C}^3, 0)$, V^{emb} is a smooth analytic 3–manifold and r is a proper map such that:

(a) $r|_{V^{emb} \setminus r^{-1}(Sing(f^{-1}(0)) \cup Sing(g^{-1}(0)))}$ is a biholomorphic isomorphism onto its image;

(b) the total transform $\mathbf{D} := r^{-1}(D)$ is a normal crossing divisor.

Recall that by the general theory, r is isomorphic over the complement of $Sing(D)$. Above, $Sing(f^{-1}(0)) \cup Sing(g^{-1}(0))$ is a smaller set than $Sing(D)$, but the intersection $f^{-1}(0) \cap g^{-1}(0)$ is already transversal off the origin, hence the above assumption can always be satisfied for a convenient resolution.

Moreover, we can assume that $U = \Phi^{-1}(D^2_\eta) \cap B^3_\epsilon$ as well.

From (b), for any $p \in \mathbf{D}$, there is a coordinate neighbourhood U_p of p with local coordinates (u, v, w) such that $(fg \circ r)|_{U_p}(u, v, w) = u^l \cdot v^m \cdot w^n$, where $l, m, n \geq 0$.

Corresponding to these local equations, $\mathbf{D}$ has a natural stratification:

5.2. Definitions. *To each $p \in \mathbf{D}$ assign the integer $k \in \{1, 2, 3\}$, if in the above local equation there are exactly k non-zero integers among l, m, n. Denote the stratum corresponding to the number k by $\mathbf{D}^{(k)}$.*

Divide $\mathbf{D}$ into the following three disjoint subsets: $\mathbf{D}_c = r^{-1}(\{f = 0, g \neq 0\})$, $\mathbf{D}_d = r^{-1}(\{g = 0, f \neq 0\})$ and $\mathbf{D}_0 = r^{-1}(\{g = 0, f = 0\})$.

A point $p \in \mathbf{D}^{(1)}$ is said to be of type $*$ (where $* = c$ or d or 0), if $p \in \mathbf{D}_*$. Similarly, an irreducible component B of the total transform $\mathbf{D}$ is of type $*$ if its generic point is of *type* $*$.

A point $p \in \mathbf{D}^{(2)}$ *is said to be of type* $*_1 - *_2$*, if* p *is an intersection point of two local divisors of type* $*_1$ *and* $*_2$*. An irreducible curve* $C \subset \overline{\mathbf{D}^{(2)}}$ *is of type* $*_1 - *_2$ *if its generic point is of that type. There is a similar definition for the points of the strata* $\mathbf{D}^{(3)}$.

In the sequel Φ denotes a fixed good representative of the ICIS, r is an embedded resolution as above with the compatibility $U = \Phi^{-1}(D^2_\eta) \cap B^3_\epsilon$. Define the composition $\widetilde{\Phi} = \Phi \circ r$. Then the restriction $r^{-1}(\Phi^{-1}(D^2_\eta - \Delta_\Phi) \cap B^3_\epsilon) \longrightarrow D^2_\eta - \Delta_\Phi$ of $\widetilde{\Phi}$ is a locally trivial C^∞-fibration, which is obviously equivalent with the fibration induced by Φ over $D^2_\eta - \Delta_\Phi$. Its fibre $\widetilde{F_{c,d}} = \widetilde{\Phi}^{-1}(c,d)$ is called "a lifted Milnor fibre".

5.3. The curve $\mathcal{C}$. Notice that in general (and also in most of the concrete examples) it is rather difficult to find a resolution r. Moreover, the codification of its exceptional divisors and the corresponding intersections and normal bundles can be a rather difficult problem. Nevertheless, what we need from this resolution is only a 1–dimensional object, the special curve arrangement:

$$\mathcal{C} = (\mathbf{D}_c \cap \mathbf{D}_0) \cup (\mathbf{D}_c \cap \mathbf{D}_d).$$

In order to understand the special role $\mathcal{C}$ plays in the geometry of the series, notice the following (see also the comments in 4.10.–4.11.). Since $\{f + g^k = 0\} = \Phi^{-1}(\{c + d^k = 0\})$ and $\{c + d^k = 0\} \subset \overline{\mathsf{W}_{\eta,M}}$ (for all $k > M$), all the information that we need is in the space above $\overline{\mathsf{W}_{\eta,M}}$. But $\widetilde{\Phi}^{-1}(\overline{\mathsf{W}_{\eta,M}})$ contains the 2–dimensional exceptional divisor $\mathbf{D}_0 = \widetilde{\Phi}^{-1}(0)$ as well (fact which is not very encouraging). However, as it turns out, the relevant information is already contained in a smaller set, the closure of $\widetilde{\Phi}^{-1}(\mathsf{W}_{\eta,M})$. The intersection of this set with $\mathbf{D}_0$ is exactly $\mathcal{C}$ (see below). This shows that from the resolution r we only need a small tubular neighbourhood of $\mathcal{C}$.

5.4. – Let $\mathcal{C} = C_1 \cup \cdots \cup C_l$ be the decomposition of $\mathcal{C}$ into irreducible components. By construction, each C_i is either a smooth curve or it has some self–intersection (double point) singularities. Let $T(C_i)$ denote a fixed small open tubular neighbourhood of C_i in V^{emb} $(1 \le i \le l)$, and write $T(\mathcal{C}) = \cup_{i=1}^l T(C_i)$.

5.5. Theorem. – First characterization of $\mathcal{C}$. *For any open (tubular) neighbourhood* $T(\mathcal{C})$*, there exist a sufficiently large integer* M *and a sufficiently small* η *such that*

$$\widetilde{\Phi}^{-1}(\mathsf{W}_{\eta,M}) \;\subset\; T(\mathcal{C}).$$

This follows from the identity: $\overline{\widetilde{\Phi}^{-1}(\mathsf{W}_{\eta,M})} \cap \widetilde{\Phi}^{-1}(0) = \mathcal{C}$.

Proof. The identity follows by local verifications. As an exemplification, we show that if $p \in \mathbf{D}^{(1)}$ is of type 0, then $p \notin \overline{\widetilde{\Phi}^{-1}(\mathsf{W}_{\eta,M})}$. Indeed, in a local coordinate neighbourhood U_p of p, $\widetilde{\Phi}|_{U_p} = (u^m, u^\mu)$ for some integers $\mu, m > 0$. Assume that $p \in \overline{\widetilde{\Phi}^{-1}(\mathsf{W}_{\eta,M})}$. Then there exists a sequence $\{z_j\}_{j=1}^\infty$ in $\widetilde{\Phi}^{-1}(\mathsf{W}_{\eta,M}) \cap U_p$ for which $\lim_{j\to\infty} z_j = p$. Hence, if (u_j, v_j, w_j) are the local coordinates of z_j, then $(u_j, v_j, w_j) \to 0$. Moreover, since $z_j \in \widetilde{\Phi}^{-1}(\mathsf{W}_{\eta,M})$ it follows that $\widetilde{\Phi}(z_j) \in \mathsf{W}_{\eta,M}$, therefore $|u_j|^m < |u_j|^{M\mu}$. This contradicts $M\mu - m > 0$, for $M > 0$ sufficiently large. All the other local verifications are left to the reader.

The inclusion follows from the identity and the properness of $\widetilde{\Phi}$. $\diamondsuit$

5.6. Corollary. *a) For any open (tubular) neighbourhood $T(\mathcal{C}) \subset V^{emb}$, there exist a sufficiently large M and a sufficiently small η such that for any $(c,d) \in \mathsf{W}_{\eta,M}$ the "lifted Milnor fibre" $\widetilde{F_{c,d}}$ is in $T(\mathcal{C})$.*

b) For any $p \in \mathcal{C}$ and local neighbourhood U_p of p, there exist a sufficiently large M and a sufficiently small η such that for any $(c,d) \in \mathsf{W}_{\eta,M}$ one has $U_p \cap \widetilde{F_{c,d}} \neq \emptyset$.

5.7. Corollary. *The curve $\mathcal{C}$ is connected.*
Proof. Use (5.6.) and the fact that $F_{c,d}$ (or equivalently, $\widetilde{F_{c,d}}$) is connected. $\diamondsuit$

5.8. Theorem. – Second characterization of $\mathcal{C}$. *Consider the punctured disc $\mathcal{D}_\eta = \{c = 0, d \neq 0\} \cap D^2_\eta \subset \mathbf{C}^2$. Then*

$$\overline{\widetilde{\Phi}^{-1}(\mathcal{D}_\eta)} \cap \widetilde{\Phi}^{-1}(0) = \mathcal{C}.$$

Its proof is similar to that of (5.5.). The analog of (5.6.) is:

5.9. Corollary. *a) For any open tubular neighbourhood $T(\mathcal{C}) \subset V^{emb}$, if $|d|$ is sufficiently small then $\widetilde{\Phi}^{-1}((0,d)) \subset T(\mathcal{C})$.*

b) For any $p \in \mathcal{C}$ and neighbourhood U_p of p, if $|d|$ is sufficiently small then $U_p \cap \widetilde{\Phi}^{-1}((0,d)) \neq \emptyset$.

II. The weighted dual graph of the curve arrangement $\mathcal{C}$.

5.10. – For each irreducible component C_i of $\mathcal{C}$, there are exactly two irreducible components B_1 and B_2 of the total transform $\mathbf{D}$ for which C_i is a component of $B_1 \cap B_2$. By the definition of $\mathcal{C}$, we can assume that B_1 is of type c and B_2 is of type 0 or d. Let m_{f,B_i} (respectively m_{g,B_i}) be the vanishing order (or multiplicity) of $f \circ r$ (respectively of $g \circ r$) along B_i ($i = 1, 2$). Then $m_{g,B_1} = 0$, $m_{f,B_1} > 0$, $m_{g,B_2} > 0$, and $m_{f,B_2} \geq 0$. Notice that from the ordered triple $(m_{f,B_1}; m_{f,B_2}, m_{g,B_2})$ one can recover the local equations for $f \circ r|_{U_p}$ and $g \circ r|_{U_p}$ in a small coordinate neighbourhood U_p of any point $p \in C_i \cap \mathbf{D}^{(2)}$.

The components C_i of $\mathcal{C}$ are either compact (projective) or non-compact. The compact components are exactly those which are contained in $r^{-1}(0)$.

The non-compact components form the strict transform of $\{f = g = 0\}$. In particular, they are of type $c - d$. Moreover, since (f, g) is an ICIS, the multiplicity of f (resp. of g) along the strict transform of $\{f = 0\}$ (resp. of $\{g = 0\}$) is one. Therefore, the ordered triple $(m_{f,B_1}; m_{f,B_2}, m_{g,B_2})$ assigned to such an irreducible curve is $(1; 0, 1)$.

Now we are ready to define the weighted dual graph $\Gamma_\mathcal{C}$ of $\mathcal{C}$. Similarly to the embedded resolution graphs of germs defined on normal surface singularities (cf. 2.1.–2.4.), the graph $\Gamma_\mathcal{C}$ has vertices $\mathcal{V}$ and edges $\mathcal{E}$.

5.11. Definition. The set of vertices $\mathcal{V}$ consists of two disjoint subsets $\mathcal{W}$ and $\mathcal{A}$, where $\mathcal{W}$ denotes the non-arrowhead vertices, and $\mathcal{A}$ denotes the arrowhead vertices. Let the non-arrowhead vertices correspond to the compact irreducible curves, and the arrowhead vertices to the non-compact irreducible curves of $\mathcal{C}$.

For any two curves $C_i, C_j \subset \mathcal{C}$, which correspond to vertices $v_i, v_j \in \mathcal{V}$, if C_i and C_j intersect in l points, then let the vertices v_i and v_j be connected by l edges $e_k = \{(v_i, v_j)_k\}_{k=1,\dots,l}$ in $\Gamma_{\mathcal{C}}$. Moreover, if a compact component $C_i \subset \mathcal{C}$, corresponding to a vertex $v_i \in \mathcal{W}$, intersects itself, then let each self-intersection point determine a loop (as a special edge) (v_i, v_i) in the graph $\Gamma_{\mathcal{C}}$. The edges are not directed, i.e. $(v_i, v_j)_k = (v_j, v_i)_k$.

The the graph $\Gamma_{\mathcal{C}}$ be decorated as follows:

• Let each non-arrowhead vertex $v_i \in \mathcal{W}$ have two weights assigned to it: the ordered triple of integers $(m_{f,B_1}; m_{f,B_2}, m_{g,B_2})$ assigned to the irreducible component C_i corresponding to the vertex v_i, and the genus g_i of (the normalization of) C_i.

• Let each arrowhead vertex have a single weight, the ordered triple $(1; 0, 1)$.

• Let each edge have a weight $\in \{1, 2\}$ determined as follows. By definition, any edge e corresponds to an intersection point $p \in \mathbf{D}^{(3)}$, hence it is the intersection point of three local irreducible components of $\mathbf{D}$. The corresponding types of these components are $c - * - **$ or $c - c - *$, with $* \neq c$. In the first case let the weight on the edge e be 1, in the second case 2.

5.12. Summary of notation for $\Gamma_{\mathcal{C}}$, and local equations.

Vertices. $(m; n, \nu)$

Case 1. For all $w \in \mathcal{W}$, $[g]$ means that the corresponding irreducible curve $C_w \subset \mathcal{C}$ is compact and its genus is g. When C_w is rational, $[g]$ will be omitted.

Furthermore, there is a local coordinate neighbourhood U_p of any point $p \in C_w \cap \mathbf{D}^{(2)}$ such that the local irreducible components $U_p \cap \mathbf{D} = B_1^l \cup B_2^l$ of $\mathbf{D}$ have representation $B_1^l = \{u = 0\}$, $B_2^l = \{v = 0\}$, and

$$f \circ r|_{U_p} = u^m v^n, \qquad g \circ r|_{U_p} = v^\nu \quad \text{with } m, \nu > 0;\ n \geq 0.$$

Obviously, if C_w is of type $c - d$ then $n = 0$, and if C_w is of type $c - 0$ then $n > 0$.

Case 2. $\to$ (1;0,1) denotes an arrowhead $v \in \mathcal{A}$.

The curve C_v corresponding to v is non-compact. In a neighbourhood U_p of a generic point $p \in C_v \cap \mathbf{D}^{(2)}$ we have $U_p \cap \mathbf{D} = B_1^l \cup B_2^l$ with $B_1^l = \{u = 0\}$, $B_2^l = \{v = 0\}$ and

$$f \circ r|_{U_p} = u, \qquad g \circ r|_{U_p} = v.$$

Edges. An edge e corresponds to an intersection point $p \in C_i \cap C_j$ (or a self-intersection of C_i if $i = j$). In a local coordinate neighbourhood of p, the irreducible components of $U_p \cap \mathbf{D} = B_1^l \cup B_2^l \cup B_3^l$ are given by $B_1^l = \{u = 0\}$, $B_2^l = \{v = 0\}$, $B_3^l = \{w = 0\}$.

Case 1.

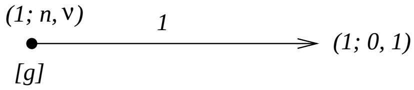
(m; n,ν) 1 (m; l,λ)

[g] [g']

v_1 v_2 means that the unique component of type c is B_1^l, and

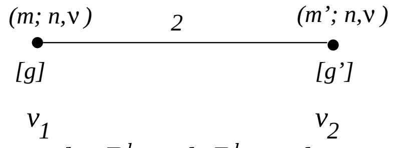
$$f \circ r|_{U_p} = u^m v^n w^l, \qquad g \circ r|_{U_p} = v^\nu w^\lambda.$$

In fact, $n = l = 0$ means a $(c-d-d)$– ; $n > 0, l = 0$ a $(c-0-d)$– ; $n = 0,\ l > 0$ a $(c-d-0)$– ; and $n > 0, l > 0$ a $(c-0-0)$–type intersection.

If the vertex v_2 is replaced by an arrowhead, then the edge is 

The local equations for $f\circ r|_{U_p}$ and $g\circ r|_{U_p}$ are as above, but with $m = \lambda = 1$ and $l = 0$.

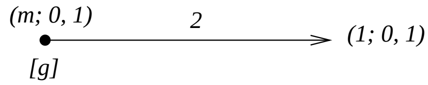
Case 2.



means that there are two components of type c, namely B_1^l and B_2^l, and

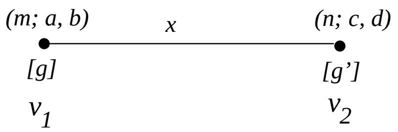
$$f\circ r|_{U_p} = u^m v^{m'} w^n, \qquad g\circ r|_{U_p} = w^\nu.$$

where $l > 0$ gives a $(c-c-0)$–type intersection, while $l = 0$ a $(c-c-d)$–type.

If v_2 is replaced by an arrowhead, then the edge is 

The local equations for $f\circ r|_{U_p}$ and $g\circ r|_{U_p}$ are the same, but with $m' = \nu = 1$ and $n = 0$.

5.13. A compatibility property of the decorations. Consider the following edge.



a.) if $m \neq n$, then $(a,b) = (c,d)$ and $x = 2$;
b.) if $(a,b) \neq (c,d)$, then $m = n$ and $x = 1$.

In particular, in the cases (a)-(b) above, the weight of the edge is determined by the weights of the vertices. In all these cases we omit the corresponding edge–decoration.

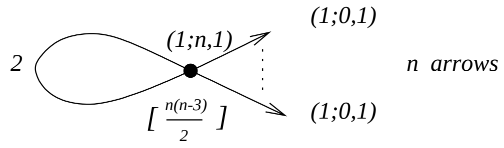
III. Examples.

5.14. Preliminary remarks.

1. We emphasize again that in general it is rather long and difficult to find a resolution r. In the literature there are very explicit resolution algorithms, but they are rather involved, and in general very slow. Therefore, in almost all of our examples, we used a sequence of some ad hoc blow ups (following the naive principle: "blow up the worst singular locus") with the hope to obtain a more or less small configuration. The computations are extremely long, and are not given here. (They were, in fact, done with the help of *Mathematica*.) Hence, we admit that for the reader the verification of some of the examples listed in the body of the paper can be a really difficult job.

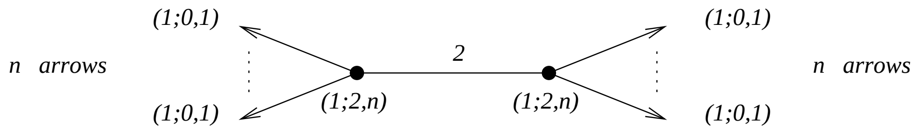
2. Since the resolution r is not unique, different resolutions r might result in different graphs $\Gamma_{\mathcal{C}}$. Moreover, contrary to the case of resolution graphs of normal surface singularities, in the case of $\Gamma_{\mathcal{C}}$ we cannot claim (at this moment) the existence of a unique minimal graph. On the other hand, the properties of the graph $\Gamma_{\mathcal{C}}$ (see the next subsection) suggest that $\Gamma_{\mathcal{C}}$ inherits a lot of properties of the resolution graphs of normal surface singularities. In particular, we expect the existence of some kind of calculus of graphs, which would provide a method to identify all the possible graphs associated with the same ICIS Φ.

5.15. Let $f(x,y,z) = x^n + y^n + xyz^{n-2}$ and $g(x,y,z) = z$. Then a possible $\Gamma_{\mathcal{C}}$ is:

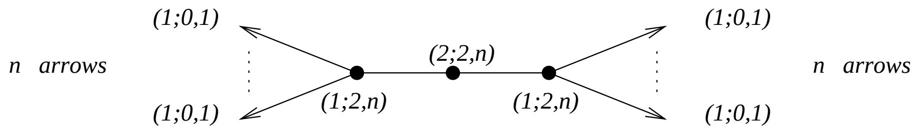


More generally, if f is given by a homogeneous polynomial, and g is a generic linear function, then the construction of $\Gamma_{\mathcal{C}}$ is not difficult. Indeed, first blow up the origin. Then consider the intersection curve C of the strict transform St of $\{f = 0\}$ with the exceptional divisor E. In the neighbourhood of a singular point p of C, (St, p) has a natural product structure $(C, p) \times (D, 0)$, where $(D, 0)$ denotes the disc. Then any embedded resolution of $(C, p) \subset (E, p)$ provides an embedded resolution of (St, p) if the quadratic transformations of the plane are replaced by the corresponding blow ups along 1–dimensional axes in the direction $(D, 0)$.

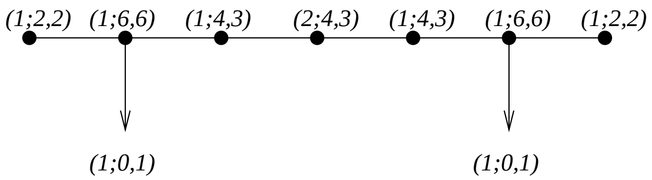
5.16. Let $f(x,y,z) = x^2 + y^2$ and $g(x,y,z) = x^n + y^n + z^n$. A possible $\Gamma_{\mathcal{C}}$ is:



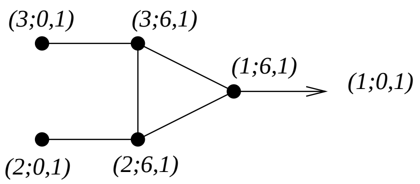
The graph $\Gamma_{\mathcal{C}}$ is not unique, but depends on the embedded resolution r. Another valid graph $\Gamma_{\mathcal{C}}$ for the same pair (f, g) is (cf. also with 5.22.):



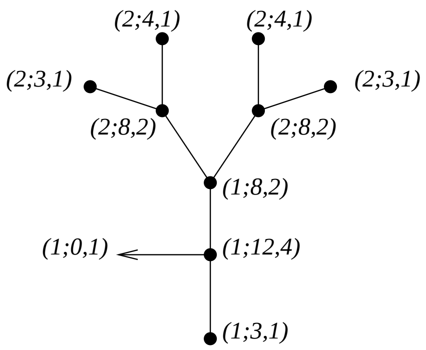
5.17. Let $f(x,y,z) = x^2 - y^2$ and $g(x,y,z) = y^2 + z^3$. A possible $\Gamma_{\mathcal{C}}$ is:



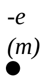
5.18. Consider $f(x,y,z) = x^3 + y^2 + xyz$ and $g(x,y,z) = z$. Then a possible $\Gamma_{\mathcal{C}}$ is:

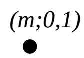


5.19. Consider $f(x,y,z) = x^2y^2 + z^2(x+y)$ and $g(x,y,z) = x+y+z$. Then a possible $\Gamma_{\mathcal{C}}$ is:

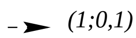


5.20. Consider $f(x,y,z) = f'(x,y)$ and $g(x,y,z) = z$, where $f' : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is an isolated plane curve singularity. Replacing the quadratic transformations of the plane by blowing ups along 1–dimensional axis (the local z–axis), one obtains the following. Let $\Gamma(\mathbf{C}^2, f')$ denote the minimal embedded resolution graph of the plane curve singularity $f' : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$. Then a possible dual graph $\Gamma_{\mathcal{C}}$ can be obtained from $\Gamma(\mathbf{C}^2, f')$ using the following conversion:

a non-arrowhead vertex 

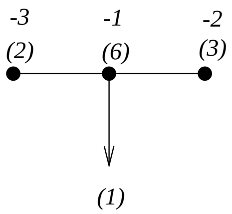
should be replaced by 

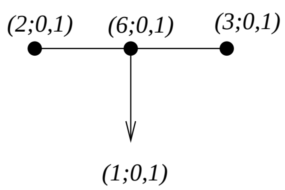
an arrowhead vertex 

should be replaced by 

All edges in $\Gamma_{\mathcal{C}}$ have weight 2. All vertices $w \in \mathcal{W}$ of $\Gamma_{\mathcal{C}}$ have genus 0.

For example, let $f(x,y,z) = f'(x,y) = x^2 - y^3$ and $g(x,y,z) = z$. Then:

$\Gamma(\mathbf{C}^2, f')$: 

and $\Gamma_{\mathcal{C}}$: 

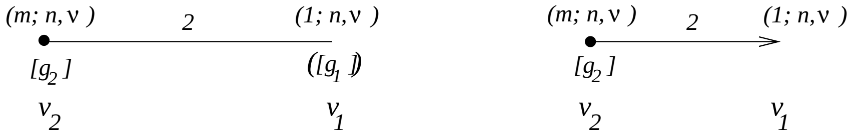
IV. Some properties of the weighted dual graph $\Gamma_{\mathcal{C}}$.

5.21. – The first partition of $\Gamma_{\mathcal{C}}$. The vertices of the graph $\Gamma_{\mathcal{C}}$ can be divided into two disjoint sets $\mathcal{V}(\Gamma_{\mathcal{C}}) = \mathcal{V}^1(\Gamma_{\mathcal{C}}) \cup \mathcal{V}^2(\Gamma_{\mathcal{C}})$, where $\mathcal{V}^1(\Gamma_{\mathcal{C}})$ (respectively $\mathcal{V}^2(\Gamma_{\mathcal{C}})$) consists of the vertices with weight $(m; n, \nu)$ where $m = 1$ (respectively $m \geq 2$). We will use similar notations for $\mathcal{W}(\Gamma_{\mathcal{C}})$ and $\mathcal{A}(\Gamma_{\mathcal{C}})$.

The graph $\Gamma^1_{\mathcal{C}}$. Consider the maximal subgraph $\Gamma^1_{\mathcal{C}}$ of $\Gamma_{\mathcal{C}}$ which is spanned by the vertices $v \in \mathcal{V}^1(\Gamma_{\mathcal{C}})$ and has no edges of weight 2 (cf. 5.22.). Notice that all the arrowheads of $\Gamma_{\mathcal{C}}$ are arrowheads of $\Gamma^1_{\mathcal{C}}$ as well, i.e. $\mathcal{W}(\Gamma^1_{\mathcal{C}}) = \mathcal{W}^1(\Gamma_{\mathcal{C}})$ and $\mathcal{A}(\Gamma^1_{\mathcal{C}}) = \mathcal{A}(\Gamma_{\mathcal{C}})$.

If an arrowhead v is supported by an edge of weight 2, then v becomes an arrowhead of $\Gamma^1_{\mathcal{C}}$ without any supporting edge. This arrowhead vertex v (as a completely isolated vertex) forms a connected component of $\Gamma^1_{\mathcal{C}}$. For example, in the case of (5.20.), the graph $\Gamma^1_{\mathcal{C}}$ has no edges, and has only one vertex which is an arrowhead.

The graph $\Gamma^2_{\mathcal{C}}$. The "complementary" subgraph $\Gamma^2_{\mathcal{C}}$ is constructed in three steps. First, consider the maximal subgraph ${}'\Gamma^2_{\mathcal{C}}$ of $\Gamma_{\mathcal{C}}$ spanned by the vertices $v \in \mathcal{V}^2(\Gamma_{\mathcal{C}})$. Second, in order to obtain ${}''\Gamma^2_{\mathcal{C}}$, we add some arrowheads (and edges which support these arrowheads) as follows. Consider an edge e of weight 2 with endpoints $v_2 \in \mathcal{W}^2(\Gamma_{\mathcal{C}})$ and $v_1 \in \mathcal{V}^1(\Gamma_{\mathcal{C}})$. For such an edge, regardless of v_1 being an arrowhead or not, perform the following transformation:



${}'\Gamma^2_{\mathcal{C}}$ ${}''\Gamma^2_{\mathcal{C}}$

Thus $\mathcal{E}({}''\Gamma^2_{\mathcal{C}}) = \mathcal{E}({}'\Gamma^2_{\mathcal{C}}) \cup \{\text{edges of type } e\}$, $\mathcal{W}({}''\Gamma^2_{\mathcal{C}}) = \mathcal{V}^2(\Gamma_{\mathcal{C}}) = \mathcal{W}^2(\Gamma_{\mathcal{C}})$ and $\#\mathcal{A}({}''\Gamma^2_{\mathcal{C}}) = \#\{\text{edges of type } e\})$.

Finally, consider the set of edges in $\Gamma_{\mathcal{C}}$ of weight 2 with both endpoints in $\mathcal{V}^1(\Gamma_{\mathcal{C}})$ (these edges were deleted in the construction of $\Gamma^1_{\mathcal{C}}$). Any edge of this type is transformed into a double arrow:

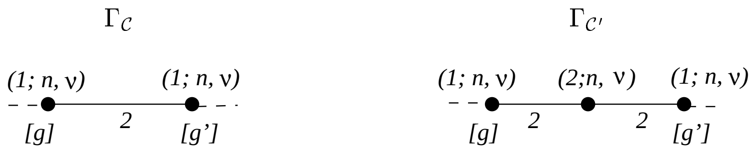
$$(1; n,\nu) \xleftrightarrow{\ 2\ } (1; n,\nu)$$

(In general, this type of graph codifies the minimal embedded resolution of a plane curve singularity of type A_1.) (Cf. also with 5.22.)

Then, by definition, the graph $\Gamma^2_{\mathcal{C}}$ is the union of ${}''\Gamma^2_{\mathcal{C}}$ and the collection of the double arrows considered above. Obviously, each double arrow forms a connected component of $\Gamma^2_{\mathcal{C}}$.

5.22. Example. – The case of 2–edges (first case): the double arrows. Consider an edge of $\Gamma_{\mathcal{C}}$ with weight 2 and both endpoints weighted by $(1, n, \nu)$ (and arbitrary second weights $[g]$ and $[g']$). This edge corresponds to a point $p \in \mathbf{D}^{(3)}$ of type $c - c - *$, with $* \in \{0, d\}$. Moreover, since the first multiplicity of the triplet $(1; n, \nu)$ is 1, both c–type local irreducible components B^l_1 and B^l_2 of $\mathbf{D}$ at p are part of the strict transform of $\{f = 0\}$ (cf. 5.12.). Their intersection corresponds to the strict transform of an irreducible component Σ_i of $Sing\{f=0\}$ with transversal type A_1, which has *not* been blown up during the resolution procedure r.

Performing an additional blow-up along this intersection, we obtain a new embedded resolution r', whose special curve arrangement $\mathcal{C}'$ will have an additional rational curve. The dual graph $\Gamma_{\mathcal{C}}$ is changed to the dual graph $\Gamma_{\mathcal{C}'}$ via the transformation:



Therefore, it may be assumed that $\Gamma_{\mathcal{C}}$ has no edges of this type.

Now, we discuss the properties of $\Gamma_{\mathcal{C}}$ and its subgraphs $\Gamma^1_{\mathcal{C}}$ and $\Gamma^2_{\mathcal{C}}$.

An immediate consequence of corollary (5.7.) is the following.

5.23. Proposition. *The graph $\Gamma_{\mathcal{C}}$ is connected.*

5.24. Remark. Neither $\Gamma^1_{\mathcal{C}}$ nor $\Gamma^2_{\mathcal{C}}$ is connected, in general. In fact, the number of connected components of $\Gamma^1_{\mathcal{C}}$ is exactly the number of irreducible components of $\{f=0\}$ (cf. 5.27.), and the number of connected components of $\Gamma^2_{\mathcal{C}}$ is exactly the number of irreducible components of $Sing\{f=0\}$ (cf. 5.32.). The reader can easily check these two facts using our examples. E.g.: in the case of (5.16.), the graph $\Gamma^1_{\mathcal{C}}$ has two components; and in the case of (5.19.) $\Gamma^2_{\mathcal{C}}$ has two components.

5.25. – The garph $\Gamma^1_{\mathcal{C}}$. In order to characterize the subgraph $\Gamma^1_{\mathcal{C}}$ one more notation is needed. Consider the germ $(\{f=0\},0)$ and let $n:\{f=0\}^{norm}\to\{f=0\}$ be its normalization. Notice that $\{f=0\}^{norm}$ is the disjoint union of some germs of normal surface singularities, each germ being the normalization of one of the irreducible components of $(\{f=0\},0)$. In particular, their number is exactly the number of irreducible components of $(\{f=0\},0)$ (see part I, 2.6.).

5.26. Definition. *Consider the subgraph $\Gamma^1_{\mathcal{C}}$ of $\Gamma_{\mathcal{C}}$. For any vertex $v\in\mathcal{V}(\Gamma^1_{\mathcal{C}})$, replace its weight $(1;n,\nu)$ by the weight (ν). For any non–arrowhead vertex $w\in\mathcal{W}(\Gamma^1_{\mathcal{C}})$, keep its weight $[g]$. Think about this modified weighted graph as a weighted embedded resolution graph of a germ of an analytic function defined on a normal surface singularity, and about the weights (ν) as the multiplicities of the corresponding function (cf. part I. 2.2.). Calculate "self-intersection numbers" e_w $(w\in\mathcal{W})$ for each non-arrowhead vertex via the relation (part I, 2.3.2). Let the resulting weighted graph be denoted by $G^1_{\mathcal{C}}$.*

By convention, a dual resolution graph, which consists of only one non–arrowhead vertex with weight (1) (i.e. it has no other vertices, and no edges), codifies the dual embedded resolution graph of a smooth germ defined on a smooth surface, which has not been blown up during the resolution.

5.27. Theorem. *The graph $G^1_{\mathcal{C}}$ is a possible embedded resolution graph of*

$$\{f=0\}^{norm}\xrightarrow{g\circ n}(\mathbf{C},0).$$

In particular, the number of connected components of $G^1_{\mathcal{C}}$ (which is the same as the number of connected components of $\Gamma^1_{\mathcal{C}}$) coincides with the number of irreducible components of the germ $(\{f=0\},0)$.

Via the above convention, a connected component of $G^1_{\mathcal{C}}$ which consists of only one non–arrowhead vertex corresponds to a smooth component of $\{f=0\}^{norm}$ one which $g\circ n$ is smooth (and which has not been modified during the resolution).

(Obviously, the subgraph $\Gamma^1_{\mathcal{C}}$ might contain cycles and/or non–arrowhead vertices with $g>0$.)

Proof. Consider the embedded resolution r and denote by St the strict transform of $\{f=0\}$. Then each irreducible component St_i of St is smooth and the restriction $r|_{St_i}: St_i \to \{f=0\}_i$ is a resolution of the corresponding component $\{f=0\}_i$ of $\{f=0\}$, which factors through the normalization of $\{f=0\}_i$. Moreover, $g\circ r: St_i\to \mathbf{C}$ defines a normal crossing divisor on St_i.

Finally, notice that the components $\{St_i\}_i$ are the only c–type irreducible components of $\mathbf{D}$ on which the vanishing order m_f of f is one. The remaining part of the proof consists of verifying some local equations and compatibilities of the corresponding graphs, which is left to the reader. $\diamondsuit$

Now we start to analyze the properties of the subgraph $\Gamma^2_{\mathcal{C}}$.

5.28. – Finer partitions. Corresponding to the partition $\mathcal{V}^1(\Gamma_{\mathcal{C}})\cup\mathcal{V}^2(\Gamma_{\mathcal{C}})$ of $\mathcal{V}(\Gamma_{\mathcal{C}})$, define $\mathcal{C}^k$ as the union of those irreducible components of $\mathcal{C}$ which correspond to vertices from $\mathcal{V}^k(\Gamma_{\mathcal{C}})$ $(k=1,2)$. Obviously, $\mathcal{C}=\mathcal{C}^1\cup\mathcal{C}^2$ and $\mathcal{C}^1\cap\mathcal{C}^2$ is a set of points.

Now, we define a finer partition of $\mathcal{V}^2(\Gamma_{\mathcal{C}})$. To any irreducible component Σ_j of $Sing\{f=0\}$, $1\leq j\leq s$ (cf. 4.6.), assign the set $\mathcal{B}_{\Sigma_j}$ containing those irreducible components B of the total transform $\mathbf{D}$ which are of type c and are projected via r *onto* Σ_j. Then define $\mathcal{C}_{\Sigma_j}\subset\mathcal{C}$ as the union of those irreducible components C of $\mathcal{C}$ for which $C\subset B$ for some $B\in\mathcal{B}_{\Sigma_j}$.

5.29. Lemma. $\cup_{j=1}^s\mathcal{C}_{\Sigma_j}=\mathcal{C}^2$ *and* $\mathcal{C}_{\Sigma_i}\cap\mathcal{C}_{\Sigma_j}=\emptyset$, *for any* $i\neq j$, $i,j\in\{1,...,s\}$.
Proof. The first identity follows from the fact that the set of c–type irreducible components of $\mathbf{D}$, along which f is vanishing with order strictly greater than 1, are exactly those components which project via r onto some component of $Sing\{f=0\}$. For the second part, take a point $p\in\mathcal{C}_{\Sigma_i}\cap\mathcal{C}_{\Sigma_j}$. Clearly, $p\in\mathbf{D}^{(3)}$ and it is of type $c-c-*$ with $*\neq c$. Consider the three local coordinate axes (as in the local description 5.12.). One of them is contained in a component $C\subset\mathcal{C}_{\Sigma_i}$, another in a component $C'\subset\mathcal{C}_{\Sigma_j}$. Denote the third local coordinate axis by C^*. Since it is contained in a component $B\in\mathcal{B}_{\Sigma_i}$, it projects via r onto Σ_i. But this is true also for j, hence $r(C^*)\subset\Sigma_i\cap\Sigma_j=\{0\}$. But along C^* the function g is not constant, which contradicts $r(C^*)=\{0\}$. $\diamondsuit$

In the next lemma, $T(\mathcal{C}_{\Sigma_j})$ denotes the tubular neighbourhood $\cup T(C_i)$ (cf. 5.4.), where the union is taken over the components $C_i\subset\mathcal{C}_{\Sigma_j}$.

5.30. Lemma. *a) For any open (tubular) neighbourhood $T(\mathcal{C}_{\Sigma_j})\subset V^{emb}$, there exist a sufficiently small $\gamma>0$ such that for any point $q\in\Sigma_j-\{0\}\subset\mathbf{C}^3$ with $|q|<\gamma$, $r^{-1}(q)\subset T(\mathcal{C}_{\Sigma_j})$.*

b) For any $p \in \mathcal{C}_{\Sigma_j}$ and local neighbourhood U_p of p, there exist a sufficiently small $\gamma > 0$ such that for any point $q \in \Sigma_j - \{0\}$ with $|q| < \gamma$, $U_p \cap r^{-1}(q) \neq \emptyset$.
Proof. The proof is similar to the proof of (5.6.) and (5.9.) (actually, it can also be deduced from (5.9.)). ♢

5.31. Corollary. *$\mathcal{C}_{\Sigma_j}$ is connected for all $j \in \{1, ..., s\}$.*
Proof. For any $q \in \Sigma_j - \{0\}$, the fiber $r^{-1}(q)$ is connected by Zariski's Connectivity Theorem (see e.g. [4]). Then use (5.30.). ♢

Therefore, $\mathcal{C}^2$ can be written as a disjoint union of the *connected* curves $\mathcal{C}_{\Sigma_j}$ $(1 \leq j \leq s)$. An immediate consequence of this is the following.

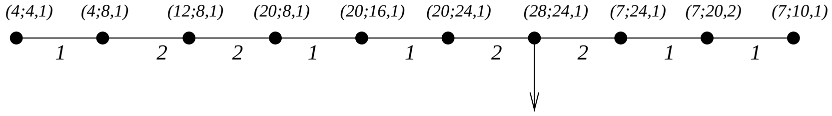
5.32. Proposition. *There is a one-to-one correspondence between the connected components of $\Gamma^2_{\mathcal{C}}$ and the irreducible components of $(Sing\{f = 0\}, 0)$. The set of non–arrowhead vertices in a connected component of $\Gamma^2_{\mathcal{C}}$ corresponds exactly to the irreducible components of one of the curves $\mathcal{C}_{\Sigma_j}$.*
(*Remark.* Notice that in the case of "double arrow components" of $\Gamma^2_{\mathcal{C}}$, corresponding to the component Σ_j (cf. 5.22.), one has $\mathcal{C}_{\Sigma_j} = \emptyset$.)

Later we will show that the minimal embedded resolution graph of the transversal singularity of a connected component Σ_j can be reconstructed from the corresponding connected components of $\Gamma^2_{\mathcal{C}}$ (cf. 5.43. and 5.51.). But for this we need some further preparation.

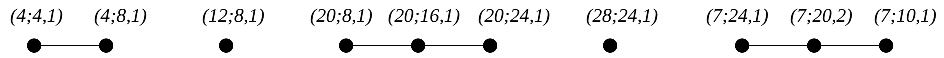
5.33. – Even finer partitions. Fix a connected component $\Gamma^2_{\mathcal{C},j}$ of $\Gamma^2_{\mathcal{C}}$ corresponding to the irreducible component Σ_j $(1 \leq j \leq s)$. Its non–arrowhead vertices are denoted by $\mathcal{W}(\Gamma^2_{\mathcal{C},j})$. We define on $\mathcal{W}(\Gamma^2_{\mathcal{C},j})$ the following equivalence relation. First, we say that $w_1 \sim w_2$ if w_1 and w_2 are connected by an edge of weight 1, then we extend $\sim$ to an equivalence relation. The equivalence classes $\{K_l\}_l = \{K_l(\Gamma^2_{\mathcal{C},j})\}_l$ determine a partition of $\mathcal{W}(\Gamma^2_{\mathcal{C},j})$.

Let $\Gamma(K_l)$ be the maximal subgraph of $\Gamma^2_{\mathcal{C},j}$ supported by the non–arrowhead vertices K_l. For the moment, we keep all the decorations of the corresponding vertices of $\Gamma(K_l)$. This graph is connected, it has no arrowheads, and its edges are all weighted by 1. (If from $\Gamma^2_{\mathcal{C},j}$ all the arrowheads and the edges of weight 2 are deleted, then the disjoint union $\cup_l \Gamma(K_l)$ is obtained.)

5.34. Example. Consider the example given in (5.56.). The next figure illustrates the partition of $\Gamma^2_{\mathcal{C},1}$:



The graphs $\Gamma(K_l)$ are:



5.35. – For a fixed equivalence class $K_l = \{w_{i_1}, w_{i_2}, \dots, w_{i_t}\}$, consider the corresponding irreducible curves $\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \dots, \mathcal{C}_{i_t}$ of $\mathcal{C}$. By construction, there exists an irreducible component $B(K_l) \in \mathcal{B}_{\Sigma_j}$ which contains all of them. Moreover, the union $\mathcal{C}(K_l) := \mathcal{C}_{i_1} \cup \cdots \cup \mathcal{C}_{i_t}$ is a connected curve. Let $T(K_l)$ denote a small tubular neighbourhood of $\mathcal{C}(K_l)$ in $B(K_l)$.

Finally, consider the weights $(m_{i_k}; n_{i_k}, \nu_{i_k})$ associated with w_{i_k} (or with $\mathcal{C}_{i_k}$) $(1 \leq k \leq t)$. Again, by the definition of the partition, $m_{i_1} = \cdots = m_{i_t}$. We denote this integer by $m(K_l)$; it is exactly the vanishing order of $f \circ r$ along $B(K_l)$. On the other hand, the local equations show that the restriction $g \circ r|_{T(K_l)} : T(K_l) \to \mathbf{C}$ provides the principal divisor $(g \circ r)^{-1}(0) = \sum_k \nu_{i_k} \mathcal{C}_{i_k}$ in $T(K_l)$.

Therefore, the divisor (multiple curve) $\sum_k \nu_{i_k} \mathcal{C}_{i_k}$ in $T(K_l)$ can be interpreted as a central fiber of a proper analytic map (here $T(K_l)$ can be changed to $(g \circ r|B(K_l))^{-1}$(small disc)). It is convenient to introduce the integer $\nu(K_l) := g.c.d.(\nu_{i_1}, \dots, \nu_{i_k})$.

5.36. Lemma. *The generic fiber of $(g \circ r)|T(K_l)$ is a disjoint union of rational curves.*
Proof. Fix $d \neq 0$ with $|d|$ sufficiently small. Then $g^{-1}(d)$ intersects Σ_j in d_j points $\{q_i\}_i$ (cf. 4.6.). Then $((g \circ r)|T(K_l))^{-1}(d) = \cup_i (r^{-1}(q_i) \cap T(K_l))$. But $r^{-1}(q_i)$ is a fiber of an embedded resolution of the transversal singularity associated with Σ_j, hence each irreducible curve in $r^{-1}(q_i)$ is rational. $\diamondsuit$

5.37. – General properties of morphisms whose generic fiber is rational. First we recall the following fact (cf. [3], page 554):

If S is a minimal smooth surface, D a disc in $\mathbf{C}$, and $\pi : S \to D$ any proper holomorphic map whose generic fiber $\pi^{-1}(d)$ $(0 \neq d \in D)$ is irreducible and rational, then π is a (trivial) $\mathbf{P}^1$-bundle over D.

Now, assume that S is not minimal, and the generic fiber of $\pi : S \to D$ is a disjoint union of (say, N) rational curves. Then by the Stein Factorization theorem (see e.g. [4], page 280), (and possible after shrinking D), there exists a map $b : D' \to D$ given by $z \mapsto z^N$, and $\pi' : S \to D'$ such that $\pi = b \circ \pi'$, and the generic fiber of π' is irreducible and rational. Since the central fibers of π and π' are the same, it follows from the above fact that the central fiber of π can be blown down successively until an irreducible rational curve is obtained.

This discussion has the following consequences:

5.38. Proposition. – Properties of the graph $\Gamma(K_l)$.

a) The graph $\Gamma(K_l)$ is a tree with all $g_{w_k} = 0$. In particular, all the irreducible components of $\mathcal{C}^2$ are rational curves.

b) From the integers $\{\nu_{i_k}\}_{k=1}^t$ one can deduce the self–intersections $C_{i_k}^2$ of the curves C_{i_k} in $B(K_l)$ as follows. First notice that the intersection matrix $(C_{i_k} \cdot C_{i_{k'}})_{k,k'}$ (where the intersections are considered in $B(K_l)$) is a negative semi–definite matrix with rank $t-1$, and the "central divisor" $\sum_k \nu_{i_k} C_{i_k}$ is one element of its kernel (cf. [5], page 90). The intersections $C_{i_k} \cdot C_{i_{k'}}$ for $i_k \neq i_{k'}$ can be read from the graph $\Gamma(K_l)$ considered as a dual *graph. The self–intersections can be determined from the relations* $(\sum_k \nu_{i_k} C_{i_k}) \cdot C_{i_{k'}} = 0$ *(cf. also with part I, 2.3.2). In particular, if* $t = 1$, *then* $C_{i_1}^2 = 0$. *If* $t \geq 2$, *then the graph (curve configuration) is not minimal; if we blow down succesively all the* (-1)*–curves we obtain a rational curve with self–intersection zero. This gives a complete classification of the shapes of the possible graphs* $\Gamma(K_l)$ *(decorated with the set of integers* $\{\nu_{i_k}\}_k$*).*

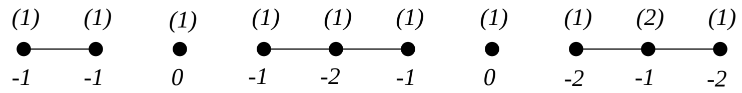
c) The number of irreducible (equivalently, connected) components of the generic fiber of $(g \circ r)|T(K_l)$ *is* $\nu(K_l) = g.c.d.(\nu_{i_1}, \ldots, \nu_{i_k})$.

The fact that each irreducible component of the generic fiber is rational can be translated into the relation:

$$2 \cdot \nu(K_l) = \sum_{k=1}^{t} \nu_{i_k}(2 - \delta_{w_{i_k}}),$$

where $\delta_{w_{i_k}}$ *is the number of vertices adjacent to* w_{i_k} *in* $\Gamma(K_l)$.

5.39. Example. Consider the example (5.34.) (based on 5.56.). Then the partition $\{\Gamma(K_l)\}_l$ of $\Gamma^2_{\mathcal{C},1}$ decorated with the integers $(\nu_{i_k})_k$ and the self–intersection numbers $C_{i_k}^2$ is the following:



5.40. – Now we return to the connected component $\Gamma^2_{\mathcal{C},j}$ of $\Gamma^2_{\mathcal{C}}$ corresponding to Σ_j $(1 \leq j \leq s)$. In (5.33.–5.39.) we introduced the partition $\{\Gamma(K_l)\}_l$. The arrowheads of $\Gamma^2_{\mathcal{C},j}$ as well as the edges connecting different classes $\{\Gamma(K_l)\}_l$ in $\Gamma^2_{\mathcal{C}}$ are 2–edges. Their detailed presentation is the subject of the next paragraph.

5.41. Example. – The case of 2–edges (second case). Consider an edge e of $\Gamma_{\mathcal{C}}$ of weight 2 having endpoints v_1 and v_2 such that v_2 is weighted with the triple $(m'; n, \nu)$ with $m' \geq 2$. Consider the local coordinates (u, v, w) as in (5.12.), with $B_1^l = \{u = 0\}$, $B_2^l = \{v = 0\}$ and $B_3^l = \{w = 0\}$.

First assume that the first entry of the weight $(m; n, \nu)$ of v_1 also satisfies $m \geq 2$, i.e. both v_1 and v_2 are non–arrowheads of $\Gamma^2_{\mathcal{C},j}$. Then $C_{v_1} \cap U_p$ and $C_{v_2} \cap U_p$ are the intersections $B_1^l \cap B_3^l$ and $B_2^l \cap B_3^l$. Their intersection point p is codified in the edge e. From the local equation $g \circ r|U_p = w^\nu$, we obtain that in the deformation $g \circ r|B_1^l \cup B_2^l \to \mathbf{C}$ the intersection point p splits into ν points.

If the first entry m of the weight $(m; n, \nu)$ of v_1 is $m = 1$, then the discussion is similar.

In both cases, set $C_e^* := B_1^l \cap B_2^l$. Then $r|C_e^* : C_e^* \to \Sigma_j$ is finite. Denote its degree by $d(e)$. Obviously $deg(r|C_e^*) \cdot deg(g|\Sigma_j) = deg(g \circ r|C_e^*) = \nu$. Sometimes it is more convenient (and precise) to use the notation $\nu(e)$ for ν. With these notations:

$$d(e) \cdot d_j = \nu(e).$$

Now we are able to characterize the structure of the subgraph $\Gamma^2_{\mathcal{C},j}$. The final goal is to reconstruct the embedded resolution graph of the transversal singularity associated with the branch Σ_j from the weighted graph $\Gamma^2_{\mathcal{C},j}$. First we start with a construction. The reader is invited to review the theory of cyclic covering of graphs (first section of part I).

5.42. – A covering construction from $\Gamma^2_{\mathcal{C},j}$. Fix an irreducible component Σ_j of $Sing\{f=0\}$ $(1\leq j\leq s)$, let $T\Sigma_j$ be the (equisingular type of the) transversal singularity associated with Σ_j (cf. 4.7.). Recall that $deg(g|\Sigma_j)=d_j$.

If (H,q) is a transversal slice as in (4.7.), then r above (H,q) determines a resolution of the transversal plane curve singularity $(H\cap\{f=0\},q)\subset(H,q)$. Its weighted dual embedded resolution graph (for a definition, see part I, 2.2.) is denoted by $G(T\Sigma_j)$. Since in local coordinates it is easier to work with the pullback of g, it is convenient to replace the single point $q\in\Sigma_j-\{0\}$ by the collection of d_j points $g^{-1}(d)\cap\Sigma_j$ (where $|d|$ is small and non-zero). The dual weighted graph associated with the curves situated above these points consists of exactly d_j identical copies of $G(T\Sigma_j)$, and it is denoted by $d_j\cdot G(T\Sigma_j)$.

Comparing the curves $r^{-1}(g^{-1}(d)\cap\Sigma_j)$ and $\mathcal{C}_{\Sigma_j}$ by the corresponding local equations, and using the results of (5.38.), especially part (c), and (5.41.), we obtain a cyclic covering of graphs

$$p: d_j\cdot G(T\Sigma_j)\to\{\text{a base graph}\}_j,$$

where the base graph and the covering data can be determined from $\Gamma^2_{\mathcal{C},j}$. This is given in the next paragraphs.

The base graph will be denoted by $\Gamma^2_{\mathcal{C},j}/\sim$. It is obtained from $\Gamma^2_{\mathcal{C},j}$ by collapsing it along edges of weight 1.

More precisely, each subgraph $\Gamma(K_l)$ is replaced by a non–arrowhead vertex. If two subgraphs $\Gamma(K_l)$ and $\Gamma(K_{l'})$ are connected by k 2–edges in $\Gamma^2_{\mathcal{C},j}$, then the corresponding vertices of $\Gamma^2_{\mathcal{C},j}/\sim$ are connected by k edges. (In fact, from 5.50. will follow that $k\leq 1$.) If the non–arrowhead vertices of $\Gamma(K_l)$ support k arrowheads altogether, then on the corresponding non–arrowhead vertex of $\Gamma^2_{\mathcal{C},j}/\sim$ one has exactly k arrowheads.

Since $\Gamma^2_{\mathcal{C},j}$ is connected, it is obvious that $\Gamma^2_{\mathcal{C},j}/\sim$ is connected as well.

The covering data of $p: d_j\cdot G(T\Sigma_j)\to\Gamma^2_{\mathcal{C},j}/\sim$.

First we recall that the *covering data* of a graph Γ is a collection of positive integers $\{n_v\}_{v\in\mathcal{V}(\Gamma)}$ and $\{n_e\}_{e\in\mathcal{E}(\Gamma)}$, such that for each edge $e\in\mathcal{E}(\Gamma)$ with endpoints v_1 and v_2, $n_e=d_e\cdot l.c.m(n_{v_1},n_{v_2})$ for some integer d_e. For details, see section 1 of part I, or at least (1.8.).

Now, we define a covering data for $\Gamma^2_{\mathcal{C},j}/\sim$. It is provided by the third entries ν of the weights $(m;n,\nu)$ of the vertices of $\Gamma^2_{\mathcal{C},j}$.

For any non–arrowhead vertex w of $\Gamma^2_{\mathcal{C},j}/\sim$, which corresponds to K_l in the above construction, set $n_w:=\nu(K_l)$. For any arrowhead vertex v of $\Gamma^2_{\mathcal{C},j}/\sim$, which corresponds to an arrowhead of $\Gamma^2_{\mathcal{C},j}$ with weight $(1;n,\nu)$, set $n_v:=\nu$. For any edge of $\Gamma^2_{\mathcal{C},j}/\sim$, which comes from a 2–edge e of $\Gamma^2_{\mathcal{C},j}$ with endpoints with weight $(*;n,\nu)$, set $n_e:=\nu\,(=\nu(e))$.

For simplicity (and suggestively) the covering data (which in section 1 of Part I is denoted by $(\mathbf{n},\mathbf{d})$), will here be denoted by ν.

The degeneration of $r^{-1}(g^{-1}(d)\cap\Sigma_j)$ into $\mathcal{C}_{\Sigma_j}$ provides the (already announced) result:

5.43. Theorem. – Characterization of the transversal singularities (first part). *For any $1\leq j\leq s$, there exists an embedded resolution graph $G(T\Sigma_j)$ of the transversal* *singularity associated with the branch Σ_j, and a cyclic covering of graphs*

$$p : d_j \cdot G(T\Sigma_j) \to \Gamma^2_{\mathcal{C},j}/\sim$$

with covering data ν (described above) and with the compatibility of the arrowheads: $\mathcal{A}(d_j \cdot G(T\Sigma_j)) = p^{-1}(\mathcal{A}(\Gamma^2_{\mathcal{C},j}/\sim))$ (cf. 1.25. (1)).

In particular, by (part I, 1.23.), (and from the fact that $G(T\Sigma_j)$ is a tree, being the graph of a plane curve singularity), one obtains that $\Gamma^2_{\mathcal{C},j}/\sim$ is a connected tree.

5.44. Remark. It is not easy to find concrete examples, when the covering data ν is not trivial (i.e. when the integers n_v and n_e are not all the same). Example (5.19.) shows such an example (see 5.55. for the detailed discussion).

5.45. – The above theorem (5.43.) has some important consequences. Before we formulate the first, we invite the reader to verify the following fact.
If Γ is a tree, and $p : G \to \Gamma$ is a cyclic covering of Γ with covering data $\{n_v\}_{v\in\mathcal{V}(\Gamma)}$, then the number of connected components of G is g.c.d.$\{n_v | v \in \mathcal{V}(\Gamma)\}$.

This implied in (5.43.), where $G(T\Sigma_j)$ and $\Gamma^2_{\mathcal{C},j}/\sim$ are connected, gives:

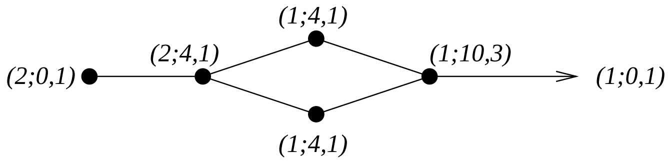
5.46. Corollary.
(a) $d_j =$ g.c.d.$\{\nu_w | w \in \mathcal{W}(\Gamma^2_{\mathcal{C},j})$, where $(m_w; n_w, \nu_w)$ is the weight of $C_w\}$.

(b) Let $\mathcal{E}_j$ be the set of 2–edges of $\Gamma_\mathcal{C}$ whose endpoints are non–arrowhead vertices such that one of the endpoints is in $\mathcal{W}(\Gamma^1_\mathcal{C})$, the other in $\mathcal{W}(\Gamma^2_{\mathcal{C},j})$. (By construction, $\mathcal{E}_j$ is exactly the index set of the arrowhead vertices of $\Gamma^2_{\mathcal{C},j}$.) Let $\#(T\Sigma_j)$ be the number of irreducible components of the transversal singularity $T\Sigma_j$. Then:

$$d_j \cdot \#(T\Sigma_j) = \sum_{e\in\mathcal{E}_j} \nu(e).$$

5.47. Example. In the example (5.19.), $Sing\{f = 0\}$ has two components. For both values $j \in \{1, 2\}$, the transversal type of Σ_j is A_1, $\#(T\Sigma_j) = 2$, $d_j = 1$, $\#\mathcal{E}_j = 1$, and $\nu(e) = 2$ for $e \in \mathcal{E}_j$.

5.48. Example. If $\#(T\Sigma_j) \neq 1$, then even if $\Gamma^1_\mathcal{C}$ and all $\Gamma^2_{\mathcal{C},j}$'s are trees, it is possible that $\Gamma_\mathcal{C}$ has some cycles. For example, if $f = x^5+y^2+xyz$ and $g = z$, then $Sing\{f = 0\} = \{x = y = 0\}$ and has transversal type A_1. $\Gamma^1_\mathcal{C}$ is a tree (representing a surface singularity of type A_2), $\Gamma^2_{\mathcal{C},j}$ is a tree, but $\Gamma_\mathcal{C}$ is:



5.49. Remark. During the construction of $\Gamma^1_{\mathcal{C}}$ we deleted all 2–edges (with at least one end–point in $\mathcal{W}(\Gamma^1_{\mathcal{C}})$), but these 2–edges have a geometric meaning in the case of $\Gamma^1_{\mathcal{C}}$ as well. Take such an edge e with endpoints v_1 and v_2. If $v_1 \in \mathcal{W}^1(\Gamma_{\mathcal{C}})$ but $v_2 \notin \mathcal{W}^1(\Gamma_{\mathcal{C}})$, then add to $\Gamma^1_{\mathcal{C}}$ an arrowhead supported by v_1. If both v_1 and v_2 are in $\mathcal{W}^1(\Gamma_{\mathcal{C}})$, then add to $\Gamma^1_{\mathcal{C}}$ two arrowheads, one of them supported by v_1, the other by v_2. Let $\mathcal{S}$ denote the collection of these arrowheads. Then $\mathcal{S}$ represents (in a similar way as the arrows of any dual graph) the strict transforms of the singular locus $Sing\{f=0\}$ in the resolution $r|_{\coprod St_i} : \coprod St_i \to \{f=0\}$ (cf. 5.27.). In particular, $\sum_j \#(T\Sigma_j) \geq \#\mathcal{S}$.

In general the inequality can be strict. For example, in the case (5.19.) (cf. also with 5.47.), $\sum_j \#(T\Sigma_j) = 4$ but $\#\mathcal{S} = 2$. In this case, the strict transform C^*_j of Σ_j is irreducible, even if the transversal type of Σ_j has two components (but $deg(C^*_j \to \Sigma_j) = 2$).

Now, we continue the list of consequences of theorem (5.43.). By theorem (5.38.) $\Gamma(K_l)$ is a tree for each l. Since $\Gamma^2_{\mathcal{C},j}/\sim$ is a tree as well, one gets:

5.50. Corollary. *a) $\Gamma^2_{\mathcal{C},j}$ is a connected tree.*

b) Moreover, since $\Gamma^2_{\mathcal{C},j}/\sim$ is a tree, corresponding to the/any covering data ν, (by part I, 1.19.) there is only one cyclic graph covering $p: G \to \Gamma^2_{\mathcal{C},j}/\sim$ (up to isomorphism). Therefore, the shape of the graph $G(T\Sigma_j)$ is completely determined by the weighted graph $\Gamma^2_{\mathcal{C},j}$.

In the sequel we discuss the compatibility of the decorations: we will show that one can recover all the decorations of $G(T\Sigma_j)$ from the decorations of $\Gamma^2_{\mathcal{C},j}$.

First we recall (cf. part I, 2.2.) that the embedded resolution graph $G(T\Sigma_j)$ has three type of decorations: (a) each vertex $v \in \mathcal{V}(G(T\Sigma_j))$ has a *multiplicity* (m_v); (b) each vertex $w \in \mathcal{W}(G(T\Sigma_j))$ has a *self–intersection* e_w; (c) each vertex $w \in \mathcal{W}(G(T\Sigma_j))$ has a *genus* $[g_w]$.

5.51. Theorem. – Characterization of the transversal singularities (second part). *For any $w \in \mathcal{W}(G(T\Sigma_j))$ corresponding to K_l, m_w is exactly $m(K_l)$; and for any arrowhead $v \in \mathcal{A}(G(T\Sigma_j))$: $m_v = 1$.*

By (part I, 2.3.2), each self-intersection number e_w can be determined from the set of multiplicities $\{m_v\}_{v\in\mathcal{V}}$. Finally, $g_w = 0$ for each w.

In particular, the weighted dual (embedded) resolution graph $G(T\Sigma_j)$ of the transversal singularity associated with Σ_j can be completely determined from the weighted graph $\Gamma^2_{\mathcal{C},j}$.

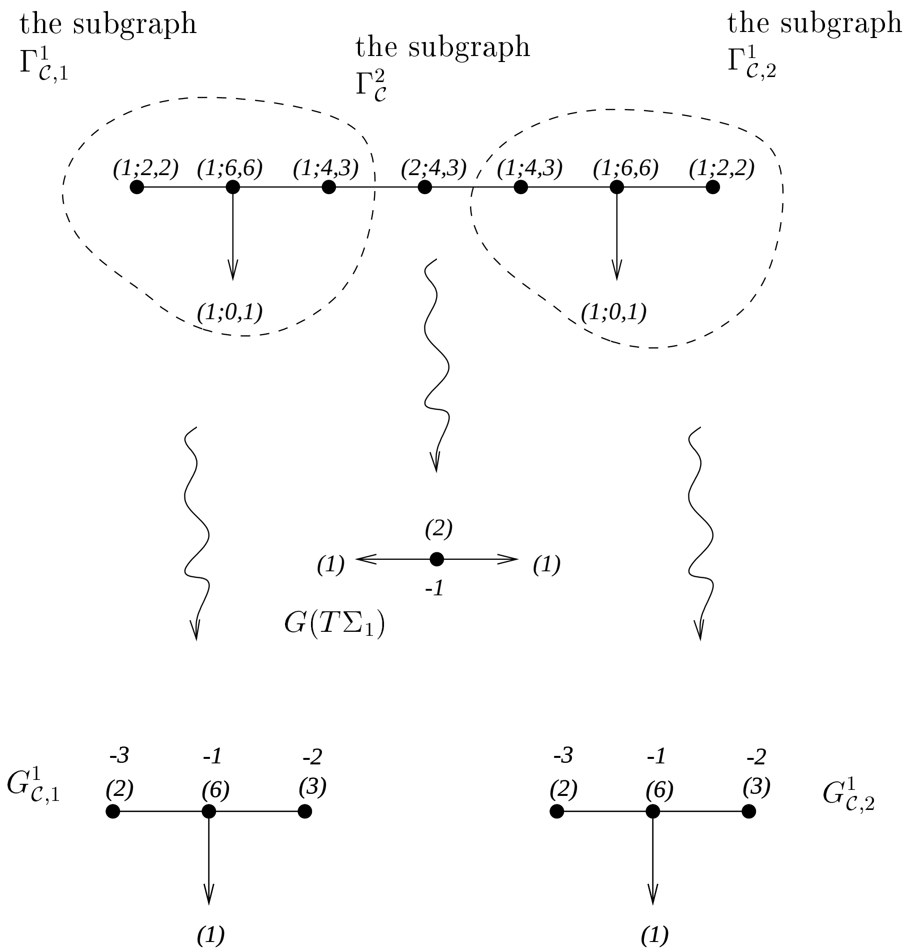
5.52. Remark. As we already emphasized in (5.38. b–c), the collection of integers $\{\nu_w\}$ $(w \in \mathcal{W}(\Gamma^2_{\mathcal{C},j}))$ satisfies serious compatibility restrictions. Moreover, since in the cyclic covering $d_j \cdot G(T\Sigma_j) \to \Gamma^2_{\mathcal{C},j}/\sim$ the covering graph $d_j \cdot G(T\Sigma_j)$ has no cycles, this imposes some additional restrictions for the integers $\{\nu_w\}$ $(w \in \mathcal{W}(\Gamma^2_{\mathcal{C},j}))$.

The results of this section and theorem (part I, 1.20.) culminate in the following corollary which is crucial for the algorithm presented in the next section.

5.53. Proposition. *Up to isomorphism of cyclic covering of graphs (with a fixed covering data), there is only one cyclic covering of the graph $\Gamma_{\mathcal{C}}$ provided that the covering data satisfies $n_v = 1$ for any $v \in \mathcal{V}^1(\Gamma_{\mathcal{C}})$.*

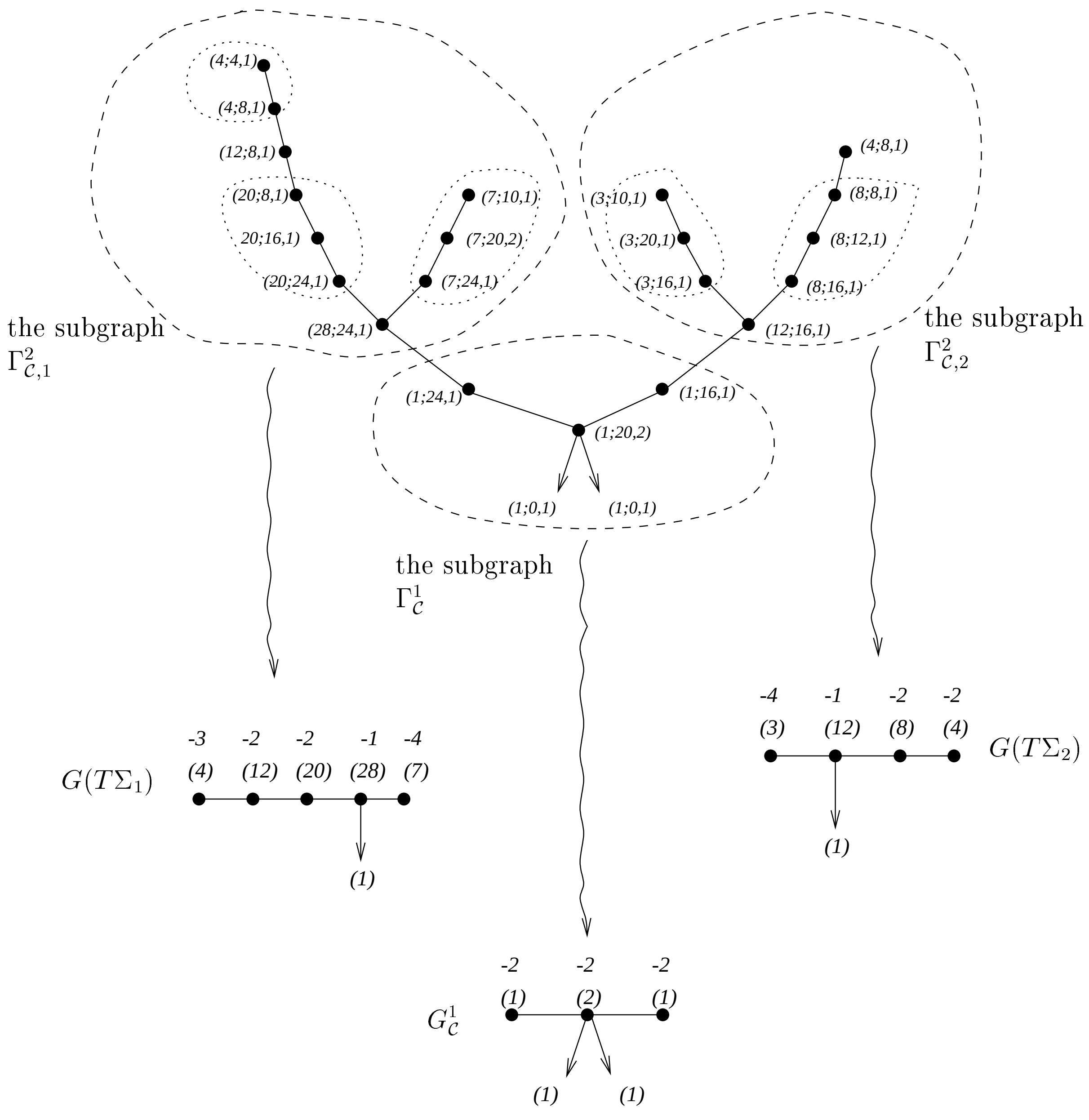
V. Illustrations of properties of the graph $\Gamma_{\mathcal{C}}$.

5.54. Example. Consider the ICIS (f, g), where $f = x^2 - y^2$ and $g = y^2 + z^3$ (cf. 5.17.). The next figure shows the decomposition of $\Gamma_{\mathcal{C}}$ into $\Gamma^1_{\mathcal{C}}$ and $\Gamma^2_{\mathcal{C}}$. The graph $\Gamma^1_{\mathcal{C}}$ has two components $\Gamma^1_{\mathcal{C},1}$ and $\Gamma^1_{\mathcal{C},2}$ corresponding to the two smooth components of the (normalization) of $\{f = 0\}$. The corresponding resolution graphs $G^1_{\mathcal{C},1}$ and $G^1_{\mathcal{C},2}$ provided by (5.26.) represent both an A_2 plane curve singularity corresponding to the restriction of g to the irreducible components of $\{f = 0\}$. The transversal singularity of $Sing\{f = 0\}$ is of type A_1 with graph $G(T\Sigma_1)$.

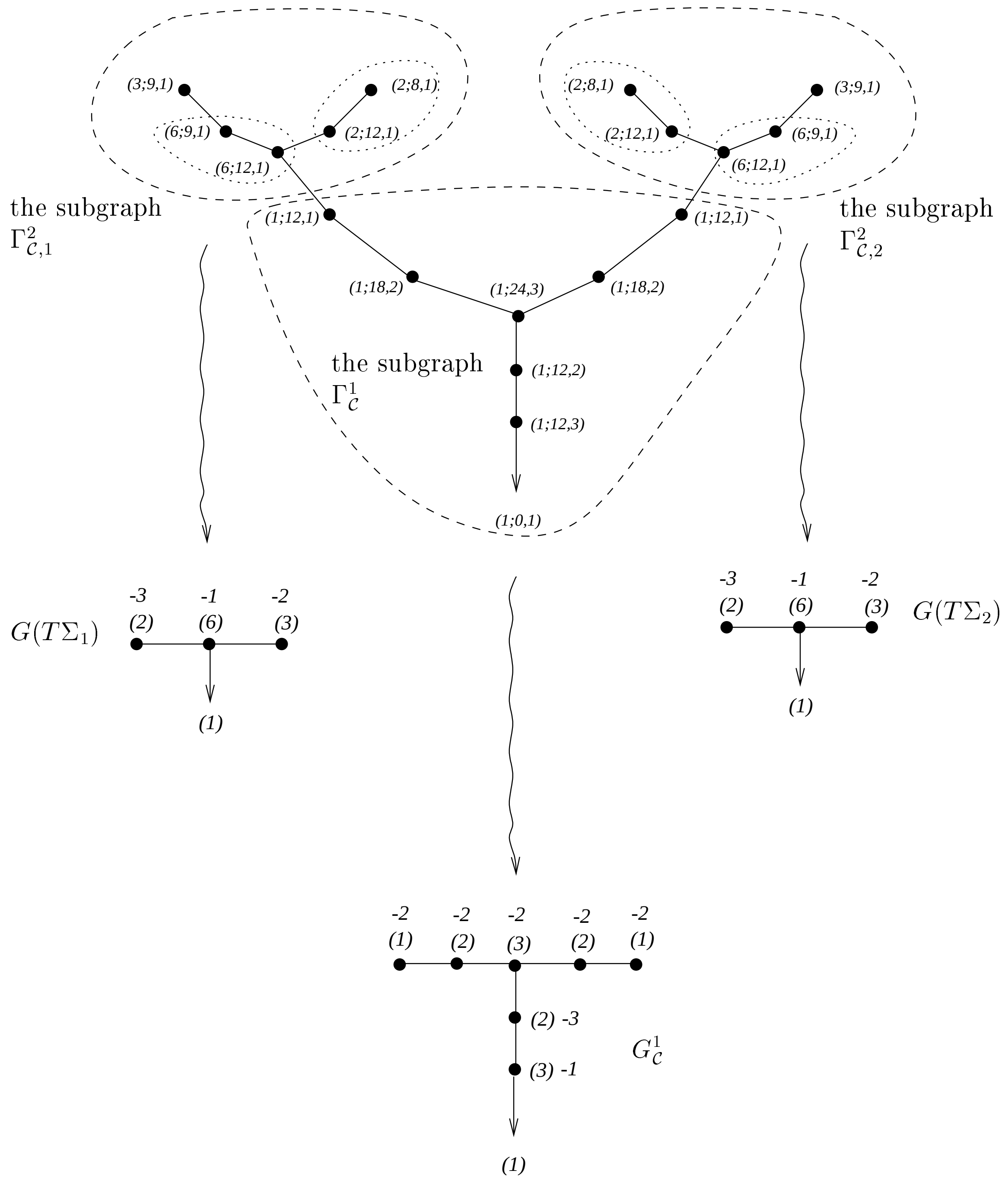


5.55. Example. Consider the example (5.19.) (cf. also 5.47. and 5.49.). Then $Sing\{f = 0\}$ has two components. For both $j \in \{1, 2\}$ fixed, $d_j = 1$ and there is only one equivalence class K_l. Therefore, $\Gamma^2_{\mathcal{C},j}/\sim$ consists of only one non-arrowhead which supports exactly one arrowhead. But $G(T\Sigma_j)$ has one non–arrowhead which supports two arrowheads (representing an A_1) singularity. Therefore, in this case the covering $G(T\Sigma_j) \to \Gamma^2_{\mathcal{C},j}/\sim$ is not a product (or identity).

5.56. Example. Consider the ICIS (f, g), where $f = x^3y^7 - z^4$ and $g = x + y + z$. In this example, the reader can verify that the example provides non–trivial partitions $\{K_l\}_l$. The case $j = 1$ is explained in (5.34.) and (5.39.). The picture (with obvious notations) is the following:



5.57. Example. With the same notations, if $f = y^3 + (x^2 - z^4)^2$ and $g = z$, then the picture is:



6§. The resolution graph of $\{\mathbf{f}+\mathbf{g}^{\mathbf{k}}=\mathbf{0}\}$.

I. The algorithm.

Consider a pair of germs $f, g : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$, such that f has a 1-dimensional singular locus and the pair $\Phi = (f, g)$ forms an ICIS. Let $k > 0$ denote a sufficiently large integer (cf. 4.8.). Then

$$(X_k, 0) := (\{f + g^k = 0\}, 0) \subset (\mathbf{C}^3, 0)$$

is an *isolated* hypersurface singularity. The algorithm presented in this section provides the minimal resolution graph $\Gamma(X_k)$ of $(X_k, 0)$.

Actually, our algorithm gives even more: we will construct the embedded resolution graph $\Gamma(X_k, g)$ of $g|X_k : (X_k, 0) \to (\mathbf{C}, 0)$. In particular, the graph will have three types of decorations: multiplicities, genera and self–intersections, and also arrowheads representing the strict transforms of $\{g|X_k = 0\}$ decorated with multiplicities as well. If we delete from the weighted graph $\Gamma(X_k, g)$ the arrowheads and multiplicities then we obtain the wanted resolution graph $\Gamma(X_k)$ of $(X_k, 0)$.

6.1. – A short preview of the algorithm. The algorithm starts with the the construction of the weighted dual graph $\Gamma_{\mathcal{C}}$ associated with an embedded resolution r of $(D, 0) := (\{fg = 0\}, 0) \subset (\mathbf{C}^3, 0)$ (cf. 5.1.). The graph $\Gamma_{\mathcal{C}}$ is not unique, it depends on the embedded resolution r. However, the algorithm works for *any* dual graph $\Gamma_{\mathcal{C}}$.

Obviously, in order to obtain $\Gamma_{\mathcal{C}}$, the optimal is to have a detailed description of the resolution r. But, let us emphasize again, in order to identify the necessary information for $\Gamma_{\mathcal{C}}$ from the resolution, only a small part of the resolution is needed: some information from a small neighborhood of the special curve arrangement $\mathcal{C}$.

This construction is clarified completely in (5.3. and 5.11.). In the sequel we assume that a weighted graph $\Gamma_{\mathcal{C}}$ have already been constructed. Then our algorithm provides, in a *purely combinatorial way*, the weighted dual embedded resolution graph $\Gamma(X_k, g)$ from the weighted dual graph $\Gamma_{\mathcal{C}}$ and the integer k.

The reader is invited to review the definitions regarding cyclic covering of graphs (part I, section 1), and its different "variations" (part I, 1.25.). In the construction, the resolution graph (string) of a Hirzebruch-Jung singularity plays a distinguished role. For the definition of these strings and their decorations, see (part I, 2.11.).

The algorithm states that the graph $\Gamma(X_k, g)$ is a cyclic graph–covering of $\Gamma_{\mathcal{C}}$ with the additional modification which replaces each edge by a string (cf. part I, section 1, and 1.25.; and compare also with the construction of cyclic coverings: part I, section 3, subsection II). The point is that for the covering data of this graph–covering, one can apply proposition (5.53.), which guarantees that up to an isomorphism, there is only one covering graph (with this covering data). In particular, the graph–covering is completely determined from the covering data, and from the information about the inserted strings. Having this in mind, the steps of the algorithm are self–explanatory:

Step 1. provides the covering data of the vertices, and also determines some of the (equivariant) decorations of these vertices.

Step 2. provides the covering data of edges, and the description of the inserted strings with their complete decorations.

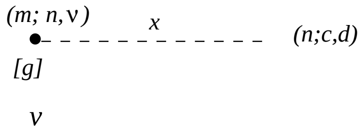
Step 3. provides the missing decorations of the vertices constructed in Step 1. After this step we obtain a possible embedded resolution graph $\Gamma(X_k, g)$ of $g|X_k : (X_k, 0) \to (\mathbf{C}, 0)$.

Step 4. is the procedure of deleting all the arrowheads and multiplicities of $\Gamma(X_k, g)$ (and successively blowing down all the rational (-1)–curves) in order to obtain the (minimal) resolution graph of $(X_k, 0)$.

6.2. Definition. – "Legs" and "stars". Before we start the detailed presentation of the algorithm, we introduce another notion, which helps the presentation of the algorithm.

Fix a non–arrowhead vertex v of $\Gamma_{\mathcal{C}}$ with weights $(m; n, \nu)$ and $[g]$. We define the notion of *star of* v, which keeps track of all the edges adjacent to v, preserving the weights $(n; c, d)$ of the other endpoint of the edges, and unifying the different cases represented by loops and edges connecting non–arrowheads or arrowheads.

The "star of v" consists of the vertex v and different "legs". A leg has the following form:



Here $x \in \{1, 2\}$ and the decorations satisfy the same compatibility conditions as the edges (5.13.). Sometimes we say that a leg as above "is supported by the vertex v".

Now, consider again the graph $\Gamma_{\mathcal{C}}$ and the fixed non–arrowhead vertex v. The collection of legs supported by v is defined as follows.

Any edge, with weight 1 and decoration as in (5.12. Edges, Case 1), with one of its endpoints $v_1 = v$, provides a leg supported by v with decorations $x = 1$ and $(n; c, d) = (m; l, \lambda)$. If the corresponding edge supports an arrowhead in $\Gamma_{\mathcal{C}}$ then automatically $(n; c, d) = (m; l, \lambda) = (1; 0, 1)$ (but otherwise, at the level of legs, we do not distinguish between the other end of the edge being an arrowhead or not).

Similarly, any edge, with weight 2 and decoration as in (5.12. Edges, Case 2), with one of its endpoints $v_1 = v$, provides a leg supported by v with decorations $x = 2$ and $(n; c, d) = (m'; n, \nu)$. Again, if the corresponding edge supports an arrowhead in $\Gamma_{\mathcal{C}}$ then automatically $(n; c, d) = (m'; n, \nu) = (1; 0, 1)$.

Any loop supported by v and weighted by x, provides *two* legs supported by v, both decorated by the same x and $(n; c, d) = (m; n, \nu)$.

The collection of all the legs constructed in this way forms the *star of* v. Obviously, if a non–arrowhead vertex of $\Gamma_{\mathcal{C}}$ is connected to v by more than one edge then each edge contributes a leg. For a general picture of a star, see the next paragraph.

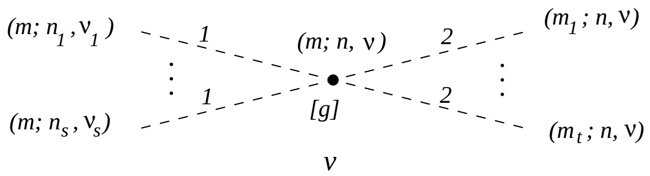
Another way to define the star of v is the following. Consider the topological realization of $\Gamma_{\mathcal{C}}$. Then a small neighborhood of the point which represents v is the *star of* v. (Obviously, we have to add also the natural decorations.)

6.3. – The algorithm.

6.4. Step 1. – The covering data of the vertices.

In order to construct the graph $\Gamma(X_k, g)$ as a covering graph of $\Gamma_{\mathcal{C}}$ (modified with strings, and decorated as an embedded resolution graph, see the discussion above) we need the covering data. The first step provides the integers $\{n_v\}_{v\in\mathcal{V}(\Gamma_{\mathcal{C}})}$.

Case 1. Consider a non-arrowhead vertex v of $\Gamma_{\mathcal{C}}$ decorated by $(m; n, \nu)$ and $[g]$. Consider its star (see the previous paragraph):



Let s (respectively t) be the number of legs weighted by $x = 1$ (respectively by $x = 2$). Above the vertex v of $\Gamma_{\mathcal{C}}$ put n_v non-arrowhead vertices, where

$$n_v = g.c.d.(m, k\nu - n, k\nu_1 - n_1, ..., k\nu_s - n_s, m_1, ..., m_t).$$

Put on each of these non–arrowhead vertices the same decoration: "the multiplicity" $(\tilde{m})$:

$$\tilde{m} = \frac{m\nu}{g.c.d.(m, \nu k - n)},$$

and the genus $[\tilde{g}]$ determined by the formula:

$$\begin{aligned} n_v \cdot (2 - 2\tilde{g}) &= (2 - 2g - s - t) \cdot g.c.d.(m, k\nu - n) \\ &+ \sum_{i=1}^{s} g.c.d.(m, k\nu - n, k\nu_i - n_i) + \sum_{j=1}^{t} g.c.d.(m, k\nu - n, m_j). \end{aligned}$$

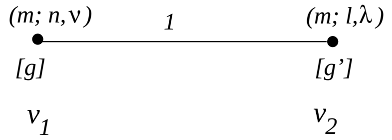
(In Step 3, the "self–intersection" of each vertex above v will also be provided.)

Case 2. Consider an arrowhead vertex v $\dashrightarrow$ (1;0,1) of $\Gamma_{\mathcal{C}}$. Above the vertex v, in the graph $\Gamma(X_k, g)$, put exactly one arrowhead vertex; and let $n_v = 1$. Let the "multiplicity" of this arrowhead be 1, i.e. the arrowhead of $\Gamma(X_k, g)$ is: $\dashrightarrow$ (1) .

6.5. Step 2. – The covering data of edges and the type of the inserted strings.

This step provides the integers $\{n_e\}_{e\in\mathcal{E}(\Gamma_{\mathcal{C}})}$.

Case 1. Consider an edge e in $\Gamma_{\mathcal{C}}$, with weight 1 (cf. with 5.12. Edges, Case 1):



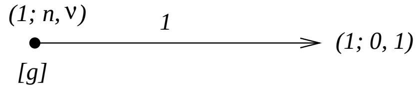
Define:

$$n_e = g.c.d.(m, k\nu - n, k\lambda - l).$$

Notice that Step 1 guarantees that in $\Gamma(X_k, g)$ there is a column of n_{v_1} (resp. n_{v_2}) vertices above the vertex v_1 (resp. v_2) of $\Gamma_{\mathcal{C}}$. Moreover, n_{v_1} and n_{v_2} both divide n_e.
Then, above the edge e insert (cyclically) in $\Gamma(X_k, g)$ exactly n_e strings of type

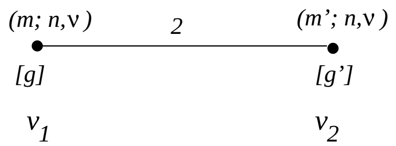
$$Str\left(\frac{k\nu - n}{n_e}, \frac{k\lambda - l}{n_e}; \frac{m}{n_e} \;\Big|\; \nu, \lambda; 0\right).$$

If the right vertex v_2 is replaced by an arrowhead, then the edge e is:



Therefore, in the above formulae $m = 1$, $n_e = 1$ and the string represents a nonsingular surface, i.e. it is the empty graph. Therefore, above such an edge e put a single edge (which supports that arrowhead of $\Gamma(X_k, g)$ which covers the corresponding arrowhead of $\Gamma_{\mathcal{C}}$).

Case 2. Consider an edge e in $\Gamma_{\mathcal{C}}$, with weight 2 (cf. 5.12. Edges, Case 2):



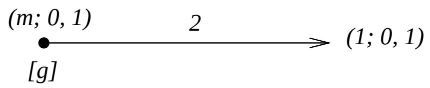
For such an edge, define:

$$n_e = g.c.d.(m, m', k\nu - n).$$

Similarly as above, Step 1 guarantees that in $\Gamma(X_k, g)$ there is a column of n_{v_1} (resp. n_{v_2}) vertices above the vertex v_1 (resp. v_2) of $\Gamma_{\mathcal{C}}$, and n_{v_1} and n_{v_2} both divide n_e.
Then, above the edge e insert (cyclically) in $\Gamma(X_k, g)$ exactly n_e strings of type

$$Str\left(\frac{m}{n_e}, \frac{m'}{n_e}; \frac{k\nu - n}{n_e} \;\Big|\; 0, 0; \nu\right).$$

If the right vertex v_2 is replaced by an arrowhead, then the edge e is:



Therefore, in the above formulae $m' = 1$, $n = 0$, $\nu = 1$ and $n_e = 1$. In this case, in general, the string is not empty, it is of type $Str(m, 1; k \,|\, 0, 0; 1) = Str(m, 1; k)$. In particular, above the edge e (which supports an arrowhead of $\Gamma_{\mathcal{C}}$) insert a string of type $Str(m, 1; k)$ (whose right end supports that arrowhead of $\Gamma(X_k, g)$ which covers the corresponding arrowhead of $\Gamma_{\mathcal{C}}$).

The reader is invited to consult (part I, 1.25.) and Step 3 of the algorithm presented in (part I, section 3, subsection II) where the insertion of the strings is explained in more details.

6.6. – The covering data revisited. Notice that the set of integers $\{n_v\}_{v\in\mathcal{V}(\Gamma_{\mathcal{C}})}$ and $\{n_e\}_{e\in\mathcal{E}(\Gamma_{\mathcal{C}})}$ satisfies the axioms of a covering data. Moreover, if $v \in \mathcal{V}^1(\Gamma_{\mathcal{C}})$ then $m = 1$ hence $n_v = 1$. Therefore, by proposition (5.53.), there is only one cyclic graph–covering of $\Gamma_{\mathcal{C}}$ with this covering data. If in this unique graph we replace the edges with the corresponding strings (as explained above) we obtain $\Gamma(X_k, g)$.

6.7. Step 3. – The missing decorations. The first two steps provide a graph with some decorations: all the multiplicities, all the genera, and some of the self–intersections. More precisely, the self–intersections of the vertices constructed in Step 1 are missing. In this step compute all these numbers using (part I, 2.3.2).

This finishes completely the construction of (a possible) embedded resolution graph $\Gamma(X_k, g)$ of the germ $g|X_k : (X_k, 0) \to (\mathbf{C}, 0)$.

6.8. Step 4. In order to obtain the resolution graph $\Gamma(X_k)$ of $(X_k, 0)$, the arrows and multiplicities of $\Gamma(X_k, g)$ have to be deleted. If the resulting graph is not minimal, then blowing down successively the rational curves with self–intersection -1 will provide a minimal one.

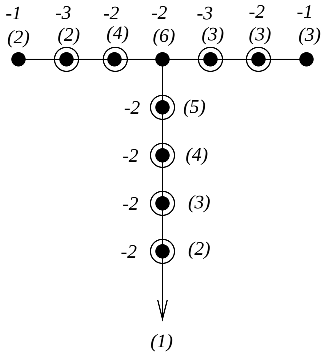
II. Examples.

6.9. Notation. In order to help the reader distinguish the vertices of $\Gamma(X_k, g)$ which cover some vertices of $\Gamma_{\mathcal{C}}$ (i.e. vertices constructed in Step 1) from the vertices of the strings inserted in Step 2, we will indicate the latter by $\odot$. Obviously, in the final graph we will make no distinction between them.

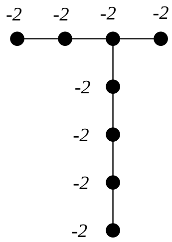
In the next examples, we give two graphs, the first is the result of steps 1-3, the second is the result of step 4.

6.10. Example. Let $f(x, y, z) = x^2 - y^3$ and $g(x, y, z) = z$. A possible dual graph $\Gamma_{\mathcal{C}}$ is shown in Example 5.20.

Set $k = 5$. The algorithm provides the minimal resolution graph of $(\{x^2 + y^3 + z^5 = 0\}, 0)$. Applying Steps 1-3, one obtains:



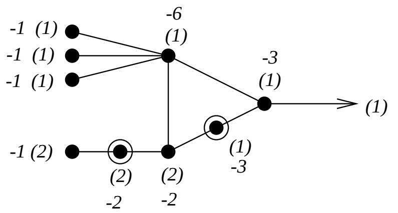
Step 4 gives:

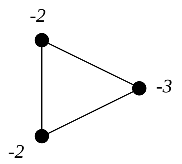


This is exactly the expected graph of the surface singularity E_8.

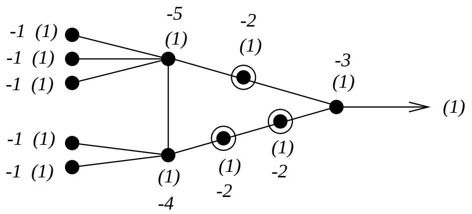
More generally, the above algorithm contains as a particular case the construction of the resolution graph of surface singularities of type $(\{f(x,y)+z^N=0\},0)$ (cf. part I, 3.12.), via the construction (5.20.) of $\Gamma_{\mathcal{C}}$.

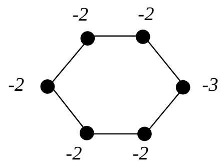
6.11. Example. Let $f(x,y,z)=x^3+y^2+xyz$ and $g(x,y,z)=z$. A possible dual graph $\Gamma_{\mathcal{C}}$ is shown in Example 5.18. Let $k=9$, then the algorithm provides the resolution of the cusp singularity $(\{x^3+y^2+z^9+xyz\},0)$.





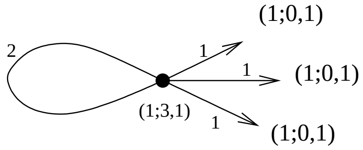
6.12. Example. Consider again $f(x,y,z)=x^3+y^2+xyz$ and $g(x,y,z)=z$, as in the previous example, but take $k=12$. Then the algorithm gives the minimal resolution graph of the cusp singularity $(\{x^3+y^2+z^{12}+xyz\},0)$.





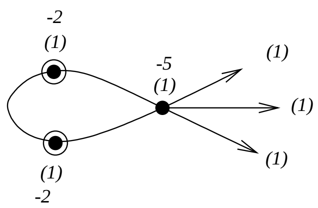
The interested reader can verify that for arbitrary k (and the same Φ and $\Gamma_{\mathcal{C}}$ as above), the graph $\Gamma(X_k)$ is a cyclic graph as above with one -3 curve and $k-7$ curves with self–intersection -2.

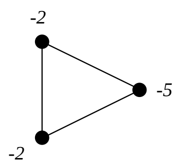
6.13. Example. Let $f(x,y,z)=x^3+y^3+xyz$ and $g(x,y,z)=z$. A possible dual graph $\Gamma_{\mathcal{C}}$ is:



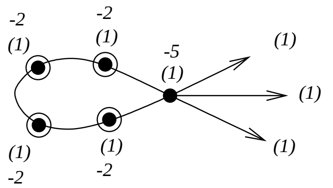
(cf. Example 5.15. with $n=3$.)

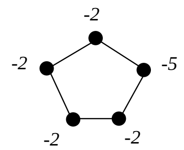
Set $k=6$. Then the minimal resolution graph of the cusp singularity $(\{x^3+y^3+z^6+xyz\},0)$, obtained in the same steps, is:





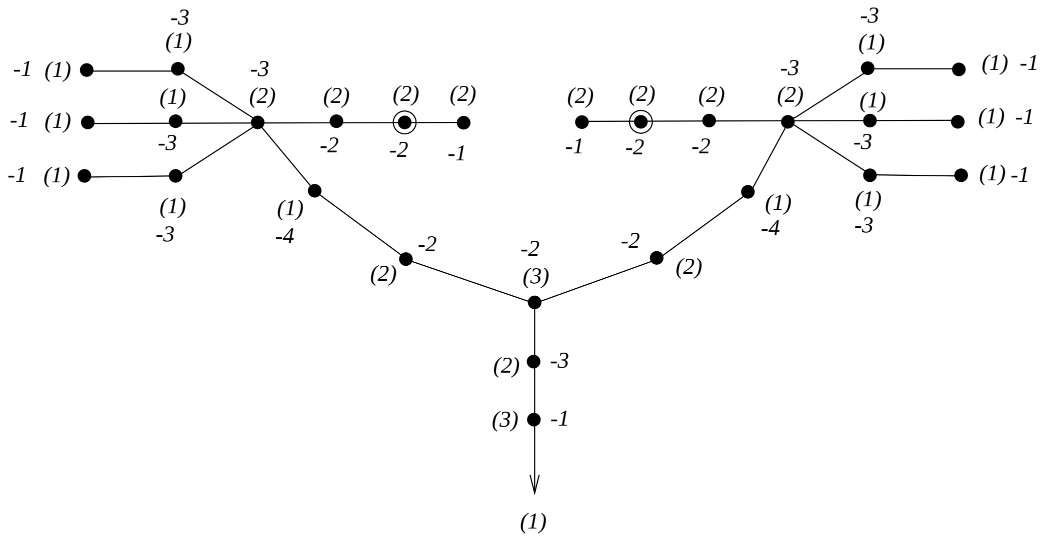
6.14. Example. Consider again $f(x,y,z)=x^3+y^3+xyz$ and $g(x,y,z)=z$ as in the previous example. Set $k=8$, then the graph of $(\{x^3+y^3+z^8+xyz\},0)$ is:



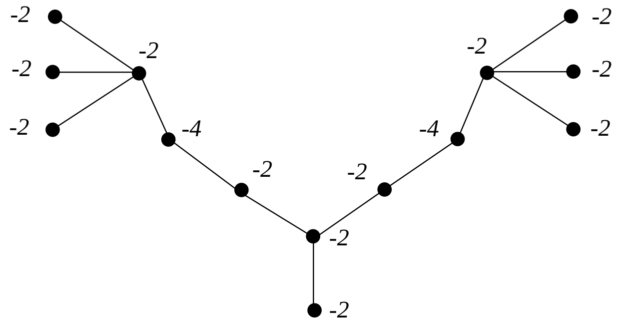


Again, one can verify that for arbitrary k (and the same Φ and $\Gamma_{\mathcal{C}}$ as above), the graph $\Gamma(X_k)$ is a cyclic graph as above with one -5 curve and $k-4$ curves with self–intersection -2.

6.15. Example. Let $f(x,y,z) = y^3 + (x^2 - z^4)^2$ and $g(x,y,z) = z$. A possible dual graph $\Gamma_{\mathcal{C}}$ is shown in Example (5.57.). Set $k = 15$. The embedded resolution graph of z defined on $(\{y^3 + (x^2 - z^4)^2 + z^{15} = 0\}, 0)$ is:



Applying Step 4, one gets the resolution graph of $(\{y^3 + (x^2 - z^4)^2 + z^{15} = 0\}, 0)$:



Note that this result can be checked by using the algorithm described in (part I, 3.12.), with $f(x,y) = (x^2 - y^4)^2 + y^{15}$ and $N = 3$.

III. Proof of the algorithm and final remarks.

6.16. – Outline of the proof. In this subsection we present the main steps of the proof of the algorithm, we emphasize the key points, and leave all the details to the reader.

The fundamental fact is that the lift of the singularity $(X_k, 0)$ in V^{emb}, via the embedded resolution $r : (V^{emb}, \mathbf{D}) \to (\mathbf{C}^3, D)$, has a distinguished position. Namely, let the strict transform of X_k be denoted by X_k^{st}. Then for sufficiently large k

$$X_k^{st} \cap \mathbf{D} = \mathcal{C} \quad \text{and} \quad Sing(X_k^{st}) \subset \mathcal{C}.$$

Moreover, along an irreducible component of the intersection $X_k^{st} \cap \mathbf{D}^{(2)}$, the surface X_k^{st} is either smooth or forms an equisingular family of plane curve singularities. Therefore, if X_k^{st} is normalized, the singularities along the intersection $X_k^{st} \cap \mathbf{D}^{(2)}$ are smoothed. The resulting normal surface $\widetilde{X_k^{st}}$ has singularities only above points of $X_k^{st} \cap \mathbf{D}^{(3)}$. Moreover, the singularities of the strict transform X_k^{st}, above the points $X_k^{st} \cap \mathbf{D}^{(3)}$, have local equations of type $\{w^\alpha = u^\beta v^\gamma\}$, with $\beta, \gamma \geq 1$ and $\alpha > 1$ (in order to find these integers, use the local equations (5.12.)). Thus, after normalization, these singularities are of Hirzebruch-Jung type and their resolution is well-known (cf. part I, 2.11.). A smooth surface X_k^{res}, which provides a resolution of the original isolated surface singularity $(X_k, 0)$, can be obtained by resolving these Hirzebruch–Jung singularities.

The construction is summarized in the next diagram:

$$X_k^{res} \xrightarrow{R} \widetilde{X_k^{st}} \xrightarrow{n} X_k^{st} \xrightarrow{r|} (X_k, 0),$$

where $r|$ is the restriction of the fixed embedded resolution r; n is the normalization of X_k^{st} and R is the map resolving the Hirzebruch-Jung singularities of the surface $\widetilde{X_k^{st}}$.

The graph constructed using the algorithm of the previous section is exactly the dual resolution graph of the resolution $(X_k^{res},\ r| \circ n \circ R)$. Note that by the above discussion the exceptional curves of the resolution X_k^{res} are of two type. Some exceptional curves are irreducible components of the lift $n^{-1}(\mathcal{C})$ and these are lifted further isomorphically by R. Other exceptional curves appear as a result of resolving Hirzebruch-Jung strings.

The crucial fact in the proof is that all the information necessary to recover the dual embedded resolution graph can be obtained by local multiplicity (vanishing order) computations. This claim needs some explanation.

The first technical difficulty (the "easy one") arises because of the fact that the dual resolution graph includes the self-intersection numbers of the exceptional curves as weights. These numbers are global invariants (characteristic classes) and in general, their computation is difficult. But if we consider the germ of an analytic function defined on X_k, then in any embedded resolution graph of it, all the self–intersections can be computed from the multiplicities of the germ using (part I, 2.3.2). This is exactly the motivation to consider the *embedded resolution* of the restriction of g to $(X_k, 0)$ instead of the resolution of $(X_k, 0)$. It is not difficult to verify that the procedure outlined above in fact gives the embedded resolution of $g|X_k : (X_k, 0) \to (\mathbf{C}, 0)$, and all the vanishing orders can be determined by local computations.

The second (globalization) problem is more serious. First we would like to remind the reader about the result regarding the resolution graph of cyclic covering. Fix a normal surface singularity $(X,0)$ and a germ f defined on $(X,0)$, and consider the N–cyclic covering $X_{f,N}$ of $(X,0)$ branched along $\{f=0\}$. Then in general, it is impossible to recover the resolution graph of $X_{f,N}$ from the embedded resolution graph of f and the integer N. Even if the embedded resolution graph of f and the integer N carry all the needed *local* information about the covering and multiplicities, the *global* information about monodromies, which determines the type of the graph–covering (i.e. the covering data) and also the shape of the covering graph, is missing. In this case of cyclic coverings, this global information can be codified in the universal covering graph of the embedded resolution graph of f. On the other hand, if the embedded resolution graph of f has some additional properties (e.g. if it is a tree and $g_w=0$ for all vertices $w\in\mathcal{W}$), then the necessary global information can be recovered from the local one. (For more details, see part I of the paper.)

In the case of the generalized Iomdin series we can ask the same question: is the global information (i.e. the number of irreducible components of $n^{-1}(C)$ in $\widetilde{X_k^{st}}$ situated above an irreducible component C of $\mathcal{C}$ in X_k^{st}, or the global configuration of the intersection of these components) codified in the local data? The answer is *yes*! This follows from the results of section 5, where it is proven that the covering data can be determined from $\Gamma_{\mathcal{C}}$. Moreover, proposition (5.53.) (cf. also part I, 1.20.) guarantees that with this covering data there is only one graph–covering. The fact which makes these claims true is that that part of $\mathcal{C}$ (namely $\mathcal{C}^2$), above which the covering is not trivial, contains only rational curves, and the dual graph of these curves has no cycles.

Once it is clarified that no global information is needed, everything is reduced to local computations. They are left to the reader. $\diamondsuit$

6.17. Final remarks.

1. If F_h denotes the Milnor fiber of a hypersurface singularity $h:(\mathbf{C}^3,0)\to(\mathbf{C},0)$, and χ the topological Euler–characteristic, then one has the following formula:

$$\chi(F_{f+g^k})-\chi(F_f)=k\cdot\sum_{j=1}^{s}d_j\mu_j,$$

where μ_j denoted the Milnor number of the transversal singularity $T\Sigma_j$ associated with Σ_j $(1\leq j\leq s)$. This formula for the "classical case" (when g is a generic linear form) was proved by Iomdin [5], for the general case, see [10, 9].

Notice that both integers d_j and ν_j can be determined from the weighted graph $\Gamma_{\mathcal{C}}$. Indeed, d_j is given in (5.46.). On the other hand, theorems (5.43.) and (5.51.) provide the embedded resolution graph of $T\Sigma_j$ which via a result of A'Campo's [1] provides μ_j.

In particular, $\Gamma_{\mathcal{C}}$ determines completely the correction term $i(f+g^k)-i(f)$ in the case of $i(h)=\chi(F_h)$ (cf. 4.10.).

2. Notice that for any vertex $w\in\mathcal{W}(\Gamma^1_{\mathcal{C}})$ the covering data is $n_w=1$, and also for any edge $e\in\mathcal{E}(\Gamma^1_{\mathcal{C}})$ one has $n_e=1$. Moreover, the type of the inserted string corresponding to e is trivial. Therefore, above $\Gamma^1_{\mathcal{C}}$ the covering is trivial, i.e. for any k, the graph (above) $\Gamma^1_{\mathcal{C}}$ stays as a stable part of the resulting graph $\Gamma(X_k, g)$ or $\Gamma(X_k)$.

On the other hand, the graph above $\Gamma^2_{\mathcal{C}}$ is increasing as k becomes larger. More precisely, the number of vertices above a vertex $w \in \mathcal{W}(\Gamma^2_{\mathcal{C}})$ is a periodical function in k. Similarly, the 1–edges of $\Gamma^2_{\mathcal{C}}$ are replaced by strings which behave periodically when k runs over the integers $k \geq k_0$. But the 2-edges of $\Gamma^2_{\mathcal{C}}$ are replaced by strings whose "size" increases infinitely as $k \to \infty$. Basically, the union of these increasing spots, associated with the subgraphs $\Gamma^2_{\mathcal{C},j}$ $(1 \leq j \leq s)$, correspond to the "correction term" $i(f+g^k) - i(f)$, where i denotes the resolution graph (cf. also with 4.10.).

3. Not all the decorations of the weighted graph $\Gamma_{\mathcal{C}}$ were used in the above algorithm. Nevertheless, we believe that all of them have important geometrical significance (we will explain this in the forthcoming paper [16], where we plan to discuss the relationship of $\Gamma_{\mathcal{C}}$ with other invariants).

4. Any fault–finder reader can object the following. We constructed our resolution graph $\Gamma(X_k, g)$ from the graph $\Gamma_{\mathcal{C}}$ whose construction is based on some properties of the embedded resolution r, which is a more complicated object than any resolution of $(X_k, 0)$. This is perfectly true!

But, if we consider our results as qualitative results, then already the existence of a graph $\Gamma_{\mathcal{C}}$ which coordinates all the resolutions graphs $\{\Gamma(X_k, g)\}_k$ in a purely combinatorial way (as it is explained in the algorithm), and has all the properties listed in section 5, (and conjecturally provides all the "correction terms" $i(f+g^k) - i(f)$ for different invariants i such as the zeta function, signature, spectral pairs, etc.), is really remarkable.

5. In this paper the case of the Iomdin series was discussed, but all the results can be generalized to the case of an arbitrary series of composed singularities.

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