Five Lectures on

Normal Surface Singularities

by

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with the assistance of

Ágnes Szilárd and Sándor Kovács

Preface

These are the notes of my lectures delivered at the Summer School in "Low dimensional topology" organized in Budapest, Hungary, 1998 August. The written version preserves the structure of the lectures: the content of the five lectures is grouped in five sections. During the Summer School, corresponding to each lecture, problem–solving sessions were organized in the afternoons. In these sessions I was helped by Ágnes Szilárd and Sándor Kovács. The corresponding exercises are inserted in the text in their proper places. But, needless to say, the written version is richer, contains complete proofs, and more examples and exercises. These exercises constitute an integral part of the notes. The first part is rather introductory, but the presentation gets more and more involved as we progress. As a general reference, I recommend the books of Barth, Peters and Van de Ven [5], D. Eisenbud [8], Griffiths and Harris [11], R. Hartshorne [12], H. Laufer [13], I. R. Shafarevich [28], and the Park City notes of M. Reid [25].

Most of the counterexamples and exercises are based on the computation of the resolution graph of hypersurface suspension singularities [22]. This algorithm is presented in Appendix 1. Starting from section 3 the basic machinery is the cohomology of sheaves. The necessary minimum is separated in Appendix 2. The material of sections 2 and 3 are based on the results of M. Artin [1, 2], H. Laufer [14] and Ph. Wagreich [29]. Section 4 contains some results of Ph. Wagreich [29], but basically I follow H. Laufer's paper [16]. The proof of (4.3) is original. In section 5, the definition of the elliptic sequence and some of its properties are borrowed from the papers of S. S.-T. Yau (see the list of references). The rest of the section is based on the results of the author, a part of them will be published in [23]. In all these last sections I was substantially influenced by the paper of M. Reid [25], already mentioned.

Obviously, the manuscript is not exhaustive. The all presentation is built around only one guiding question: when is possible to read the analytic invariants like the geometric genus and the Hilbert–Samuel function from the topology of the singularity, or equivalently, from the structure of the 3–manifold which is the link of the singularity.

This question connects these lectures with the program of the Summer School.

I am grateful to Ágnes Szilárd and Sándor Kovács for their help and support, and I would like to thank the Mathematical Institute of the Hungarian Academy, especially Károly (Karcsi) Böröczky, giving me the privilege to share a podium with my fellow lecturers Mike Davis, John Morgan and Walter Neumann.

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Contents

1§. Normal surface singularities.

Our basic objects of study are normal surface singularities. As a general reference, see [11], [12], [13].

1.1. Definition. A surface singularity $(X,0)$ is defined by:

$$(X,0)=(\{f_1=...=f_m=0\},0)\ \subset\ (\mathbf{C}^N,0),$$

where $f_i:(\mathbf{C}^N,0)\to(\mathbf{C},0)$ are germs of analytic functions with

$$r(p)=\mathrm{rank}\left[\frac{\partial f_i}{\partial z_j}(p)\right]_{i=1,...,m;\ j=1,...,N}=N-2$$

for any generic (or smooth) point p of X. If $r(0)=N-2$ then $(X,0)$ is a smooth germ, i.e. analytically isomorphic to $(\mathbf{C}^2,0)$. If $r(0)<N-2$, but $r(p)=N-2$ for all $p\in X-\{0\}$ then we say that $(X,0)$ has an <u>isolated singularity</u> at the origin.

1.2. Remark. In general $m>N-2$, for example:

$$(X,0)=\{uw=v^2,\ vt=w^2,\ ut=vw\}\ \subset\ (\mathbf{C}^4,0)$$

cannot be defined by two equations only. (Cf. Example (3.19.), part 3.)

If a surface singularity can be defined by $m=N-2$ equations, then it is called a <u>complete intersection</u>.

1.3. Definition. Let $\mathbf{C}\{z_1,...,z_N\}=\mathcal{O}_{\mathbf{C}^N,0}$ be the local ring of power series $\sum a_\alpha z^\alpha$ convergent in a small neighbourhood of the origin. Denote by $(f_1,...,f_m)$ the ideal generated by $f_1,...,f_m$. Then the local ring of analytic functions defined on $(X,0)$ is

$$\mathcal{O}_{X,0}\ =\ \mathcal{O}_{\mathbf{C}^N,0}/(f_1,...,f_m).$$

Its unique maximal ideal is $\mathtt{m}_{X,0}=(z_1,...,z_N)$. In the sequel, we will assume that $(X,0)$ is irreducible, i.e. $(X,0)$ is not the union of two proper analytic subsets. The irreducible singularity $(X,0)$ is called <u>normal</u> if $\mathcal{O}_{X,0}$ is an integrally closed ring in its field of fractions; or equivalently: if any bounded holomorphic function $f:X-\{0\}\to\mathbf{C}$ can be extended to a holomorphic function defined on X.

1.4. Exercise. Set $(X,0)=(\{x^2=y^3\},0)\ \subset\ (\mathbf{C}^3,0)$. Then $\mathcal{O}_{X,0}=\frac{\mathbf{C}\{x,y,z\}}{(x^2=y^3)}$ is not normal.

1.5. Normalization. If an irreducible germ $(X,0)$ is *not* normal, then it has a unique normalization $(\overline{X},0)\to(X,0)$ which corresponds to the embedding $\mathcal{O}_{X,0}\hookrightarrow\mathcal{O}_{\overline{X},0}:=\overline{\mathcal{O}_{X,0}}=$ integral closure of $\mathcal{O}_{X,0}$ in its field of fractions.

1.6. Exercise. The normalization of $\frac{\mathbf{C}\{x,y,z\}}{(x^2=y^3)}$ is smooth.

1.7. Examples of normal surface singularities:

(a) Quotient singularities. Consider the natural action of $\mathrm{GL}(2,\mathbf{C})$ on the coordinate functions of $(\mathbf{C}^2,0)$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

This induces a $\mathrm{GL}(2,\mathbf{C})$-action on $\mathcal{O}_{\mathbf{C}^2,0}$; and also a geometric action on $(\mathbf{C}^2,0)$. Consider a finite subgroup G of $\mathrm{GL}(2,\mathbf{C})$ that acts on $\mathbf{C}^2-\{0\}$ freely. Then $(X,0)=(\mathbf{C}^2,0)/G$ is called a quotient singularity. Its ring of analytic functions is $\mathcal{O}_{X,0}=(\mathcal{O}_{\mathbf{C}^2,0})^G=$ the algebra of functions invariant under the action of G.

Examples:

(i) $\mathbf{Z}_n$ acts on $\mathbf{C}^2$ by $\xi * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi x \\ \xi^{-1} y \end{pmatrix}$, where $G=\left\{\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi^n=1\right\}$. Then the invariant algebra $\mathcal{O}^G_{\mathbf{C}^2,0}=\langle x^n, xy, y^n\rangle \approx \mathbf{C}\{u,v,w\}/\langle uw=v^n\rangle$; hence $(X,0)=(\mathbf{C}^2,0)/\mathbf{Z}_n \simeq \{(u,v,w)\in(\mathbf{C}^3,0)\ :\ uw=v^n\}$.

(ii) $\mathbf{Z}_n$ acts on $\mathbf{C}^2$ by $\xi * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi x \\ \xi y \end{pmatrix}$. Then $\mathcal{O}^G_{\mathbf{C}^2,0}=\langle x^n, x^{n-1}y, ..., x^{n-k}y^k, ..., y^n\rangle$ $\approx \mathbf{C}\{z_1,...,z_{n+1}\}/(z_iz_j=z_kz_l)_{i+j=k+l}$. For $n=3$ we obtain the singularity described in Remark (1.2.).

(b) Hypersurface singularities. An isolated complete intersection singularity is always normal. This follows from Serre's Criterion ([26], or [8]). A weaker form of this criterion assures that for complete intersections the normality is equivalent with the regularity in codimension two. In particular, if a hypersurface singularity $f:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ satisfies $\partial f(x)\neq 0$ for x in a punctured neighbourhood of the origin, then $(X,0)=(\{f=0\},0)$ defines a normal singularity.

1.8. Remark. Actually, by Serre's Criterion, any normal surface singularity has an isolated singularity (at most). For a different proof of this fact and other properties of the normalization, see also [13] Chapter 3.

1.9. Hirzebruch–Jung singularities. The following two facts are equivalent. If a surface singularity $(X,0)$ satisfies them it is called a Hirzebruch-Jung singularity. (For more details, see e.g. [5].)

(a) There is a finite analytic map $f:(X,0)\longrightarrow(\mathbf{C}^2,0)$ such that the discriminant locus of f, in some local coordinates (u,v) of $(\mathbf{C}^2,0)$, is $\{uv=0\}$.

(This means that $f:f^{-1}\{uv\neq 0\}\longrightarrow\{uv\neq 0\}$ is a topological covering.)

(b) $(X,0)=A_{n,q}$, where $A_{n,q}=$ normalization of $(\{xy^{n-q}+z^n=0\},0)\subset(\mathbf{C}^3,0)$, with $(n,q)=1$ and $0<n<q$.

In the sequel, we would like to extract some numerical invariants from the local ring $\mathcal{O}_{X,0}$. Obviously, a trivial one is: $\dim\mathcal{O}_{X,0}=2$.

1.10. Convention. From now on, $(X, 0)$ will denote a normal surface singularity.

I. Analytical invariants.

1.11. The Hilbert-Samuel function is defined by:

$$f_{\mathrm{HS}}(k) = \dim_{\mathbf{C}} \mathcal{O}_{X,0} \,/\, \mathtt{m}_{X,0}^k \quad \text{for } k \geq 1.$$

It has the following properties:
(a) $f_{\mathrm{HS}}(1) = \dim_{\mathbf{C}} \mathbf{C} = 1$.
(b) $f_{\mathrm{HS}}(2) = \dim \mathcal{O}_{X,0} \,/\, \mathtt{m}_{X,0} \; + \; \dim \mathtt{m}_{X,0} \,/\, \mathtt{m}_{X,0}^2 = 1 + \dim \mathtt{m}_{X,0} \,/\, \mathtt{m}_{X,0}^2$.

It is not difficult to verify (we leave it as an exercise) that $\dim \mathtt{m}_{X,0} \,/\, \mathtt{m}_{X,0}^2 = \min\{N \mid (X,0) \subset (\mathbf{C}^N, 0)\}$. In particular, $f_{\mathrm{HS}}(2) = 3$ if and only if $(X, 0)$ is smooth.
(c) For $k >> 0$, the function $f_{\mathrm{HS}}(k)$ behaves like a polynomial function, i.e. there exists a polynomial P_{HS} (the Hilbert-Samuel polynomial) such that

$$f_{\mathrm{HS}}(k) = P_{\mathrm{HS}}(k) \quad \text{for } k >> 0.$$

The degree of P_{HS} equals the dimension of $(X, 0)$, hence for surface singularities we get

$$P_{\mathrm{HS}}(k) = \frac{m}{2!} k^2 + a_1 k + a_2.$$

1.12. The Multiplicity. The integer m in the above formula of P_{HS} is called the *multiplicity* of $(X, 0)$. We use the notation $m = mult(X, 0)$. Geometrically $mult(X, 0)$ can be defined as follows:
Consider an embedding $(X, 0) \subset (\mathbf{C}^N, 0)$ for some N. Take a *generic* affine space L of dimension $N - 2$ (close to the origin). Then $mult(X, 0) = \#X \cap L$.

1.13. Exercise. Show that the two definitions of the multiplicity agree for $(X, 0) = \{f = 0\} \subset (\mathbf{C}^3, 0)$, where

(a) f is homogeneous of degree m; or

(b) $f = f_m + f_{m+1} + \ldots$ $\quad$ (f_d homogeneous of degree d).

1.14. The Geometric Genus of $(X, 0)$. [14] The last analytic invariant (important from the point of view of these notes) is the geometric genus

$$p_g = \dim_{\mathbf{C}} \frac{H^0(X - \{0\}, \Omega^2)}{H^0_{L^2}(X - \{0\}, \Omega^2)},$$

where Ω^2 represents the holomorphic 2-forms and $H^0_{L^2}$ the global L^2-forms. (Here $H^0_{L^2}$ can be replaced by H^0_{bounded}.)
Remark: In (1.24.) we will give a different, sheaf-theoretical definition of p_g. (Cf. also with (2.47.).)

II. Topological invariants.

1.15. The Link of $(X,0)$. Consider an arbitrary embedding $(X,0) \subset (\mathbf{C}^N, 0)$. By [19], there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ the $(2N-1)$-dimensional real sphere $S^{2N-1}_\epsilon = \{z \in \mathbf{C}^N \ : \ |z| = \epsilon\}$ intersects $(X,0)$ transversally. In particular, for ϵ sufficiently small, the intersection $L_X = X \cap S^{2N-1}_\epsilon$ is an orientable 3-dimensional real manifold. L_X does not depend on the embedding $(X,0) \subset (\mathbf{C}^N,0)$ and on ϵ.
A different, but equivalent, definition is the following: Take a real analytic map $\rho : (X,0) \longrightarrow [0,\infty)$ with $\rho^{-1}(0) = \{x\}$. Then $L_X = \rho^{-1}(\epsilon)$ for ϵ sufficiently small.

1.16. Theorem. (see e.g. [19]) *Using the above notations, $\rho^{-1}([0,\epsilon])$, or $X \cap B_\epsilon$ (where $B_\epsilon = \{z \in \mathbf{C}^N \ : \ |z| \leq \epsilon\}$) is homeomorphic to $Cone(L_X)$; in particular, L_X contains all the topological information about the singularity $(X,0)$.*

In general, we write X for some $X \cap B_\epsilon$, and we say that it is a "nice" (Stein) representative of the germ $(X,0)$.

1.17. Examples of Links.

(a) $(X,0)$ is smooth if and only if $L_X \simeq S^3$.
(The fact that $L_X \simeq S^3$ implies that $(X,0)$ is smooth is a non-trivial result of D. Mumford [21], and it was generalized by W. Neumann [24].)

(b) Consider $(X,0) = (\{x^2+y^2+z^n = 0\}, 0)$. The link of $(X,0) \subset (\mathbf{C}^3,0)$ is $L_X =$ the lens space $L(n, n-1)$. (Use the construction in (1.7.)a(i).)

(c) Consider $(X,0) = (\{x^2+y^3+z^5=0\},0)$. The link L_X is the Poincaré sphere (with $H_1(L_X, \mathbf{Z}) = 0$, and $|\pi_1(L_X)| = 120$).

(d) Consider an $A_{n,q}$ singularity (cf. Example (1.9.)). Its link is the lens space $L_X = S^3/\mathbf{Z}_n = L(n,q)$, where $\mathbf{Z}_n = \{\xi \in \mathbf{C} \ : \ \xi^n = 1\}$, and $\xi * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi x \\ \xi^q y \end{pmatrix}$.

(e) Consider $(X,0) = (\{x^3+y^3+z^3=0\},0)$. Its link L_X is a fiber bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & L_X \\ & & \downarrow \\ & & S^1 \times S^1 \end{array} \qquad \text{with Euler-number } e = -3.$$

(f) If $(X,0) = (\{x^a+y^b+z^c=0\},0)$ with $(a,b) = (b,c) = (c,a) = 1$, then the link L_X is an integer homology sphere known as the Seifert 3-manifold $\Sigma(a,b,c)$.

1.18. The Main Question. Is it possible to recover the analytic invariants, such as f_{HS}, p_g or $mult(X,0)$, from topological information i.e. from the link L_X?

1.19. Definition. The resolution and the resolution graph of a singularity.
(a) A resolution of $(X,0)$ consists of a smooth complex 2-manifold $\widetilde{X}$ and a surjective proper analytic map $\pi : \widetilde{X} \longrightarrow X$ (where X is a fixed representative of the germ $(X,0)$) such that $\pi : \pi^{-1}(X-\{0\}) \longrightarrow X-\{0\}$ is an isomorphism. We say that the resolution is good if the exceptional divisor $\pi^{-1}(x) = E$ is a normal crossing divisor. This means that for the decomposition of the exceptional curve E into irreducible components $E = E_1 \cup ... \cup E_s$, each E_i is a smooth curve; the irreducible components intersect each other transversally and $E_i \cap E_j \cap E_k = \emptyset$ for distinct indices i, j, k. A good resolution always exists (and it is not unique). Moreover, from any resolution, by some extra blowing-ups, one can obtain a "good" resolution.

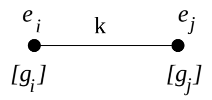
A resolution is minimal, if it does not contain any -1-rational curves (i.e. there is no rational irreducible exceptional divisor E_i with self intersection $E_i^2 = -1$). (Cf. with Castelnuovo's Criterion (3.19.1.).) A minimal resolution is unique and it is not necessarily good.

(b) The combinatorics of a resolution can be codified in the dual resolution graph denoted by $\Gamma_{\tilde{X}}$. First assume that the resolution is "good". Then $\Gamma_{\tilde{X}}$ is constructed as follows.
• Let a vertex v of the graph correspond to an irreducible component E_v of the exceptional divisor E. Let $\mathcal{V}$ denote the set of vertices of $\Gamma_{\tilde{X}}$.
• Connect two vertices v and w of the graph by k edges if the corresponding irreducible components E_v and E_w intersect each other in k distinct points.
• Each vertex $v \in \mathcal{V}$ is decorated by two weights: the genus of E_v (denoted by $[g_v]$) and the self-intersection of E_v (denoted by e_v). If $g = 0$, we omit $[0]$.

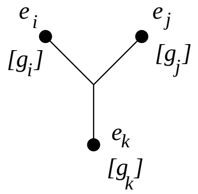
If the resolution is not "good" then we have to add some more information, and the combinatorial codification is more complicated. Actually, in this case $\Gamma_{\tilde{X}}$ is not really a graph in the classical sense. If the irreducible exceptional divisor E_v has some singular points, then we decorate the vertex v of $\Gamma_{\tilde{X}}$ with some additional data in which we codify the topological type of (the plane curve) singularities of E_v. Moreover, if $p \in E$ is a point on the exceptional divisor, then p can be contained in more then two irreducible component of E. Let $\mathcal{I}_p = \{i : p \in E_i\}$. Then corresponding to the intersection point p, we have to put on $\Gamma_{\tilde{X}}$ a "special edge" which "connects" all the vertices corresponding to $\mathcal{I}_p$, and it is decorated with all the intersection multiplicities $(E_i, E_j)_p$ for any $i \neq j$ from $\mathcal{I}_p$.

In this note we will consider only simple examples, so we do not enter in more details. But computing minimal resolutions, very often we meet the following situations:

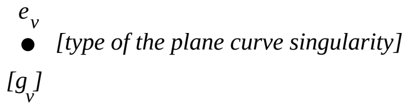
(a) Two smooth irreducible components E_i and E_j meet tangentially with intersection multiplicity k. Then we connect the two vertices by an edge and we decorate it by k:



(b) Three smooth irreducible components meet at the same point, such that the intersection of any two is transversal. This is codified by the "triangle":



(c) An irreducible exceptional curve has a singular point which is not contained by any other irreducible component. This is codified by:

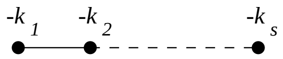


Clearly, in any situation, the graph $\Gamma_{\tilde{X}}$ codifies the intersection matrix $(E_v \cdot E_w)_{v,w}$ completely.

1.20. Examples of Minimal Resolution Graphs. The graphs in all these examples can be computed using the algorithm presented in Appendix 1.

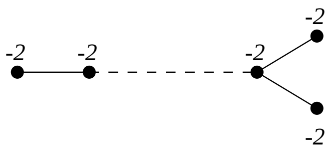
(a) If $(X, 0)$ is smooth, then $\Gamma_{\tilde{X}} = \emptyset$.

(b) For $(X, 0) = (\{x^2 + y^2 + z^n = 0\}, 0)$, where $n \geq 2$, the minimal resolution graph is $\Gamma_{\tilde{X}} = -A_{n-1}$:



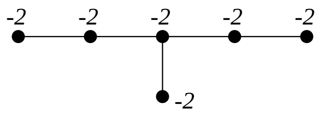
$(n-1$ vertices)

(c) For $(X, 0) = (\{x^2 y + y^n + z^2 = 0\}, 0)$, where $n \geq 3$, the minimal resolution graph is $\Gamma_{\tilde{X}} = -D_{n+1}$:

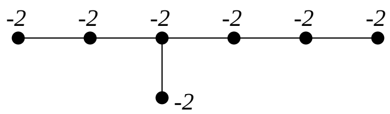


$(n+1$ vertices)

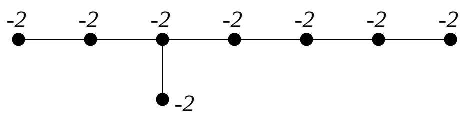
(d) For $(X, 0) = (\{x^3 + y^4 + z^2 = 0\}, 0)$ the minimal resolution graph is $\Gamma_{\tilde{X}} = -E_6$:



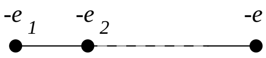
(e) For $(X, 0) = (\{x^3 + xy^3 + z^2 = 0\}, 0)$ the minimal resolution graph is $\Gamma_{\tilde{X}} = -E_7$:



(f) For $(X, 0) = (\{x^2 + y^3 + z^5 = 0\}, 0)$ the minimal resolution graph is $\Gamma_{\tilde{X}} = -E_8$:



(g) The minimal resolution graph of a $A_{n,q}$−singularity is:



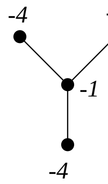
, where $\frac{n}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\cdots - \frac{1}{e_s}}}$ with $e_i \geq 2$ for any i.

(h) For $(X, 0) = (\{x^n + y^n + z^n = 0\}, 0)$ the minimal resolution graph $\Gamma_{\tilde{X}}$ consists of a single vertex with weights $[g = \frac{(n-1)(n-2)}{2}]$ and $e = -n$.

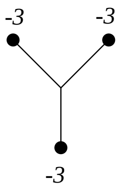
1.21. Remark. In general, the topological type of a singularity can carry a large family of analytic structures. This is happening for example in the case of (1.20. h) when $n \geq 3$. On the other hand, in some cases, the topological type carries exactly one analytic structure; this means that the minimal resolution graph determines completely the analytic structure of the singularity. These graphs (singularities) are called "taut". The $A - D - E$ and $A_{n,q}$ (i.e. Hirzebruch–Junq) singularities are taut.

1.22. More examples of resolution graphs.

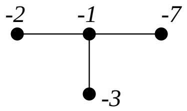
(a) For $(X, 0) = (\{x^3 + y^3 + z^4 = 0\}, 0)$ a good resolution graph is:



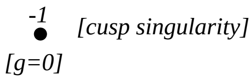
The minimal resolution graph $\Gamma_{\tilde{X}_{min}}$ is:



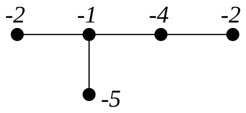
(b) For $(X, 0) = (\{x^2 + y^3 + z^7 = 0\}, 0)$ a good resolution graph is:



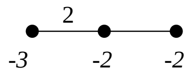
The minimal resolution graph $\Gamma_{\tilde{X}_{min}}$ is



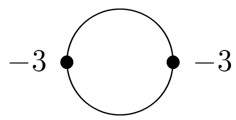
(c) For $(X, 0) = (\{x^2 + y^5 + z^7 = 0\}, 0)$ a good resolution graph is:



The minimal resolution graph $\Gamma_{\tilde{X}_{min}}$ is



(d) If $(X, 0) = (\{x^2 + y^3)(x^3 + y^2) + z^2 = 0\}, 0)$, then the minimal resolution graph is:



1.23. The Main Properties of $\Gamma_{\tilde{X}}$.
(a) The intersection matrix $(E_v \cdot E_w)_{v,w}$ is negative definite [21, 1, 2] (cf. also (2.13.)); and by a difficult result of Grauert [10], any decorated graph whose matrix $(E_v \cdot E_w)_{v,w}$ is negative definite can occur.
(b) The resolution graph $\Gamma_{\tilde{X}}$ determines and is determined by the link L_X, i.e. by the topology of $(X, 0)$. More precisely, the link can be recovered from the graph by plumbing (see below), and conversely, the link determines the graph by a deep result of W. Neumann [24]. Thus the resolution graph codifies exactly the topological information of the singularity $(X, 0)$! (All the possible resolution graphs of $(X, 0)$ contain the same amount of information.)

1.24. The Main Question Revisited. Is it possible to recover the analytic invariants f_{HS}, p_g and $mult(X, 0)$ from the resolution graph $\Gamma_{\tilde{X}}$?
<u>Remark:</u> For a fixed resolution $\pi : \widetilde{X} \longrightarrow X$ of a singularity $(X, 0)$ the geometric genus p_g can also be defined by the formula

$$p_g = \dim H^1(\widetilde{X}, \mathcal{O}_{\tilde{X}}).$$

Actually, this will be our working definition (cf. also with (2.47.)). Here we really see how deeply is involved the analytic structure in the definition of p_g. Nevertheless, at least in some "nice" cases, we hope that we can determine this analytic invariant purely topologically!

1.25. The Plumbing Construction. Fix a graph Γ with weights $\{[g_v],\ e_v\}_{v \in \mathcal{V}}$ ($g_v \geq 0$), i.e. Γ has the same type as the graph associated with a good resolution. We construct a $\mathcal{C}^\infty$ 4-manifold $P(\Gamma)$ with a boundary as follows. Fix a $\mathcal{C}^\infty$ disc-bundle T_v (i.e. a fiber bundle $p_v : T_v \longrightarrow B_v$ with the fibers being 2-dimensional real discs) with Euler-number e_v and base-space a Riemann-surface of genus g_v. If v is the endpoint of s_v edges, then fix s_v points in B_v, fix disc-neighbourhoods $\{D_{v,i}\}_{i=1,\dots,s_v}$ of these points, as well as local trivializations above the discs $D_{v,i}$, so that $p_v^{-1}(D_{v,i}) \simeq D_{v,i} \times D$.

Obviously, any edge (v, w) determines a pair of multidiscs $D_{v,i} \times D$ and $D_{w,j} \times D$ in T_v and T_w respectively. Identify them by $(x, y) \simeq (y, x)$. After these gluings we obtain a 4-manifold with corners, but these corners can be smoothed out.

In the case of a good resolution $\pi : \widetilde{X} \longrightarrow X$ with dual graph $\Gamma_{\tilde{X}}$, the real 4-manifold $\tilde{X}$ is diffeomorphic to $P(\Gamma_{\tilde{X}})$, in particular its boundary $L_X = \partial\tilde{X}$ is diffeomorphic to the space $\partial P(\Gamma_{\tilde{X}})$ constructed by plumbing.

1.26. Example. If the graph $\Gamma_{\tilde{X}}$ of a good resolution consists of a single vertex with weights $[g]$ and e, then the link L_X is an S^1-bundle over a surface with genus g with Euler number e.

1.27. Exercise. If Γ is a graph as above, let c_Γ denote the number of independent cycles of Γ i.e. $c_\Gamma = \mathrm{rank} H_1(|\Gamma|, \mathbf{Z})$. Prove that

$$\mathrm{rank}\, H_1(\partial P(\Gamma), \mathbf{Z}) = c_\Gamma + 2 \cdot \sum_{v \in \mathcal{V}} g_v$$

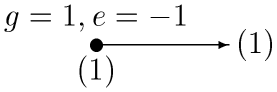
provided that the intersection matrix associated with Γ is non-degenerate.

1.28. The embedded resolution graph of an analytic germ. Fix a normal surface singularity $(X, 0)$. Sometimes we need to describe the behaviour of a germ of an analytic function $f : (X, 0) \to (\mathbf{C}, 0)$ as well. Its topological properties are codified in the embedded resolution graph associated with an embedded resolution π. The construction is the following. Consider a resolution $\pi : \tilde{X} \to X$ of X which has the additional property that $D := \{f \circ \pi = 0\}$ is a normal crossing divisor in $\tilde{X}$. (In particular, π is a "good" resolution of X.) Let $E = \pi^{-1}(0)$ be the exceptional divisor and $E = \cup_{w \in \mathcal{W}} E_w$ be its decomposition in irreducible divisors. Write $D = E \cup \cup_{a \in \mathcal{A}} S_a$, where $\{S_a\}_a$ are the irreducible components of the strict transform of $f^{-1}(0)$. Then the embedded resolution graph is constructed in the same way as the resolution graph associated with the (good) resolution π, but we add some new decorations: the vanishing order of the function $f \circ \pi$ along the irreducible components of D.

More precisely, the vertices $\mathcal{V} = \mathcal{W} \coprod \mathcal{A}$ of G_f consist of the nonarrowhead vertices $\mathcal{W}$ (corresponding to the irreducible exceptional divisors), and arrowhead vertices $\mathcal{A}$ (corresponding to the strict transform divisors of D). For simplicity, we will also assume that no irreducible exceptional divisor has a self intersection. If two irreducible divisors corresponding to $v_1, v_2 \in \mathcal{V}$ have an intersection point then (v_1, v_2) $(= (v_2, v_1))$ is an edge of G_f. The set of edges is denoted by $\mathcal{E}$.

The graph G_f is decorated by the self–intersections $e_w := E_w \cdot E_w$ and genera $[g_w]$ for any $w \in \mathcal{W}$. For any $v \in \mathcal{V}$ let m_v be the multiplicity of $f \circ \pi$ along the irreducible divisor corresponding to v. These are the decorations of the graph G_f.

1.29. Example. If $(X, 0) = (\{x^2 + y^3 + z^6 = 0\}, 0)$, then the minimal resolution graph of the germ induced by $(x, y, z) \mapsto z$ is:



III. Appendix 1. The resolution graph of suspension singularities

This combinatorial algorithm is from the author's paper [22]. The idea and the method is already in the book of H. Laufer [13].

We start with an isolated plane curve singularity $g : (\mathbf{C}^2, 0) \longrightarrow (\mathbf{C}, 0)$. We give a recipe obtaining the minimal resolution graph $\Gamma_{\tilde{X}}$ of the hypersurface singularity $(X, 0) = (\{g + z^n = 0\}, 0) \subset (\mathbf{C}^3, 0)$. The algorithm uses the *embedded* resolution graph G_g of the plane curve singularity $(\{g = 0\}, 0) \subset (\mathbf{C}^2, 0)$ and the integer n. We assume that the algorithm for finding the embedded resolution graph of a plane curve singularity is well-know. The details can be found in [6] or [9].

Actually, we will consider the analytic germ $z : (X, 0) \to (\mathbf{C}, 0)$ induced by the projection $(x, y, z) \mapsto z$, and first we will determine its embedded resolution graph G_z. A possible resolution graph $\Gamma_{\tilde{X}}$ of the singularity $(X, 0)$ can be obtained from the graph G_z by dropping the arrowheads (that represent the strict transforms of z) and the multiplicities of z along the irreducible components E_i of the exceptional divisor.

Finding G_z first has a big advantage: it is much easier to compute the multiplicities of the function z along the components of the exceptional divisor than the self intersections of these curves. Once the multiplicities are found the missing self intersections can be computed using the following relation that is valid for each vertex w that corresponds to an irreducible component of the exceptional divisor.

$$e_w m_w + \sum_{v \in \mathcal{V}_w} m_v = 0 \qquad (*).$$

For a proof of this formula see e.g. [13]; cf. also with (2.6.) Here m_i and e_i denote the multiplicities and self intersections of E_i respectively, and $\mathcal{V}_w$ denotes the set of all vertices and arrows adjacent to w.

We will also need the notion of <u>a string of vertices</u> $G(u, v, a)$ corresponding to the integers u, v and a, with $(u, v, a) = 1$.
Consider the congruence:

$$v + x \cdot \frac{u}{(u,a)} \equiv 0 \pmod{\frac{a}{(u,a)}}.$$

Let $0 \leq x_1 < a/(u, a)$ be its solution, and take:

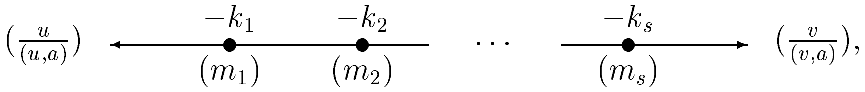
$$v + x_1 \cdot \frac{u}{(u,a)} = m_1 \cdot \frac{a}{(u,a)} \qquad (**).$$

If $x_1 \neq 0$, then consider the continuous fraction:

$$\frac{a/(u,a)}{x_1} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_s}}}, \quad k_1, \ldots, k_s \geq 2.$$

Then the string of vertices $G(u,v,a)$ is the following:

$G(u,v,a)$:



where all vertices have genus $g=0$ (i.e. they represent rational irreducible exceptional divisors), their self intersection numbers are $-k_1,\dots,-k_s$ respectively. The arrowheads have multiplicities $u/(u,a)$ and $v/(v,a)$, and the first vertex has multiplicity m_1 given by $(**)$. Therefore, $m_2,\dots,m_s$ can be easily computed using $(*)$, namely

$$-k_1m_1+\frac{u}{(u,a)}+m_2=0, \text{ and } -k_im_i+m_{i-1}+m_{i+1}=0 \text{ for } i\geq 2.$$

If $x_1=0$, then the string $G(u,v,a)$ has no vertices, it is only an edge.

Now we will present the algorithm for the computation of the graph G_z.

First recall that for a vertex w of G_g, $\mathcal{V}_w$ denotes the set of all vertices and arrows adjacent to w, set $s_w=\#\mathcal{V}_w$. Suppose that $\mathcal{V}_w=\{w_1,...,w_s\}$ some of which may be arrows. Let $d_w=\gcd(m_w,m_{w_1},...,m_{w_s})$.

STEP 1. Consider the embedded resolution graph G_g of g. It is a tree. The graph G_z can be considered as a "covering" $q: G_z\to G_g$.

(a) Above the vertex w of G_g put $\gcd(d_w,n)$ vertices for G_z, each with multiplicity $m_w/(m_w,n)$ and genus $\tilde{g}$, where:

$$2-2\tilde{g}=\frac{(2-s_w)\gcd(m_w,n)+\sum_{v\in\mathcal{V}_w}\gcd(m_w,m_v,n)}{\gcd(d_w,n)}.$$

The vertices in $q^{-1}(w)$ can be indexed by the group $\mathbf{Z}_{(d_w,n)}$.

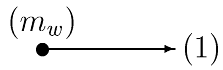
(b) Consider an edge $e=(w,v)$ of G_g

Above this edge, in G_z, put $d_e=\gcd(m_w,m_w,n)$ copies of strings of type

$$G(\frac{m_w}{\gcd(d_e,n)},\frac{m_v}{\gcd(d_e,n)},\frac{n}{\gcd(d_e,n)}).$$

These strings can be indexed by the group $\mathbf{Z}_{(d_e,n)}$. The arrowheads of the strings are identified with the vertices $q^{-1}(w)$ and $q^{-1}(v)$ respectively, via the natural morphisms $\mathbf{Z}_{(d_e,n)}\to\mathbf{Z}_{(d_w,n)}$ and $\mathbf{Z}_{(d_e,n)}\to\mathbf{Z}_{(d_v,n)}$ respectively.

(c) Consider an arrowhead of G_g:



Above this arrow put one string of type $G(m_w, 1, n)$ in G_z, whose "right end" is an arrowhead of G_z with multiplicity 1, and whose "left end" is identified with the unique vertex above w.

STEP 2. After Step 1. we obtain all the vertices and edges of G_z, and all the multiplicities of the germ z. Moreover, we also obtain all the self intersections of the vertices which are situated on the new strings. Any missing self intersections can be computed using $(*)$.

STEP 3. Drop the arrowheads and multiplicities of G_z to obtain (a possible) $\Gamma_{\tilde{X}}$. This graph, in general, is not minimal. To obtain the minimal resolution graph blow down the (-1)–rational curves.

1.30. Exercise. Using the algorithm verify the minimal resolution graphs of Exercise (1.20.).

2§. Cycles and divisors

2.1. Fix a resolution $\pi : \widetilde{X} \longrightarrow X$ of the normal surface singularity $(X, 0)$. The group of divisors $Div(\widetilde{X})$ of $\widetilde{X}$ is defined by

$$Div(\widetilde{X}) = \{\sum n_i D_i \mid D_i \text{ irreducible curve on } \widetilde{X},\ n_i \in \mathbf{Z}\}.$$

Notation. If D is a divisor $\sum n_i D_i$, then its <u>support</u> is $|D| = \bigcup\limits_{n_i \neq 0} D_i$.
For a divisor D, we define the line bundle $\mathcal{O}_{\tilde{X}}(D)$ (or $\mathcal{L}(D)$) by its local sections: if on the open set U the divisor D is given by $\{f^{-1} = 0\}$, then $\Gamma_U(\mathcal{O}_{\tilde{X}}(D)) = \{f \cdot \mathcal{O}_U\}$.
For example: the ideal sheaf of the reduced curve C is $\mathcal{O}_{\tilde{X}}(-C)$.

2.2. Definition. The <u>cycles</u> are divisors supported by the exceptional divisor E, i.e. cycles are divisors of the form $D = \sum n_i E_i$ with $n_i \in \mathbf{Z}$, where $E_i \subset E$ are irreducible exceptional curves. <u>Rational cycles</u> are of the form $D = \sum r_i E_i$ with $r_i \in \mathbf{Q}$. An integer or rational cycle is called <u>effective</u>, if $r_i \geq 0$ for all i.

There is a natural ordering of the cycles: $Z' = \sum n_i' E_i \leq Z'' = \sum n_i'' E_i$ if and only if $n_i' \leq n_i''$ for all i.

2.3. Intersection of Cycles. The intersection of two cycles is defined by:

$$(\sum_i n_i E_i) \cdot (\sum_j m_j E_j) = \sum_{i,j} n_i m_j (E_i \cdot E_j).$$

Since the intersections $(E_i \cdot E_j)_{i,j}$ are determined by the resolution graph $\Gamma_{\tilde{X}}$, the intersection of any two cycles is determined by $\Gamma_{\tilde{X}}$ as well.

2.4. Principal Divisors. Any meromorphic function g defined on $\widetilde{X}$ determines a <u>principal divisor</u> $(g) = \sum n_i D_i$, where the D_i's are irreducible components of the zeros and the poles of g and $n_i \in \mathbf{Z}$ is the order of vanishing (resp. order of the pole) of g along D_i. (The integers n_i are called the "multiplicity of g along D_i".)

In particular, given an *analytic* function $f : (X, 0) \longrightarrow (\mathbf{C}, 0)$, its pullback $\pi \circ f$ determines an *effective* principal divisor $(\pi \circ f)$. Let $m_{E_i}(f)$ denote the multiplicity of $\pi \circ f$ along E_i. Note that in fact $m_{E_i}(f) > 0$ for all i. Then:

$$(\pi \circ f) = \sum_i m_{E_i}(f)\, E_i \ + \ St(f),$$

where $St(f)$ is supported by the strict transform of $\{f = 0\}$, i.e. by $\overline{\pi^{-1}(\{f = 0\} - \{0\})}$. The cycle $\sum\limits_i m_{E_i}(f)\, E_i$ will be denoted by $(f)_\Gamma$. Thus, with this notation, $(\pi \circ f) = (f)_\Gamma + St(f)$.

2.5. Exercise. Compute some examples!

2.6. Exercise. Prove that $(\pi \circ f) \cdot E_i = 0$ for all i. (For a proof, see e.g. [13].)
[A possible topological proof is the following. We can translate the problem in the language of the intersection of homological cycles (cf. with the discussion after (2.33.)). More precisely, consider $[(\pi \circ f)] \in H_2(\tilde{X}, \partial\tilde{X}, \mathbf{Z})$ and $[E_i] \in H_2(\tilde{X}, \mathbf{Z})$, then the above intersection is exactly the intersection of these cycles. But the homology class of $(\pi \circ f)$ is zero, because $(\pi \circ f)^{-1}(0)$ is the boundary of $(\pi \circ f)^{-1}([0, \infty))$.]

I. The maximal ideal cycle and the minimal cycle.

2.7. Motivation. The maximal ideal cycle and the minimal cycle. Our main goal is to recover information about the structure of analytic functions $f : (X, 0) \longrightarrow (\mathbf{C}, 0)$ (i.e. information about the local ring $\mathcal{O}_{X,0}$) from the resolution graph $\Gamma_{\tilde{X}}$ (i.e. from the topology of $(X, 0)$).

Consider the following set of "analytic" cycles:

$$\mathcal{Z}_{an} = \{(f)_\Gamma \ : \ f \in \mathtt{m}_{X,0}\}.$$

Obviously, if $Z_1 = \sum n_i E_i$ and $Z_2 = \sum m_i E_i$ are both in $\mathcal{Z}_{an}$, then:
a.) $Z_1 + Z_2 \in \mathcal{Z}_{an}$ (take $f_1 f_2$);
b.) $\min(Z_1, Z_2) := \sum \min(n_i, m_i) E_i \in \mathcal{Z}_{an}$ (take a generic linear combination $\lambda_1 f_1 + \lambda_2 f_2$).

The set $(\mathcal{Z}_{an}, \leq)$ is ordered, and (b) guarantees that there is a unique minimal element in $\mathcal{Z}_{an}$. (It is not difficult to prove that it corresponds to the generic linear function $l_{gen}|_X : X \longrightarrow \mathbf{C}$ for some/any embedding $X \subset \mathbf{C}^N$.) Following S. S.-T. Yau, (see any of his papers in the list of references) this cycle is called the <u>maximal ideal cycle</u> of the singularity $(X, 0)$, associated with the fixed resolution π. (In [25] it is called *fiber cycle*.) In this note it will be denoted by Z_{max}. Its name is motivated by the following fact: $\pi^*\mathtt{m}_{X,0} = \mathcal{I}\mathcal{O}_{\tilde{X}}(-Z_{max})$, where $\mathcal{I}$ is supported by a finite number of points. Therefore, Z_{max} is the maximal effective cycle with the property that $\pi^*\mathtt{m}_{X,0} \subset \mathcal{O}_{\tilde{X}}(-Z)$.

Using examples we will show that the maximal ideal cycle depends on the analytic structure of the singularity $(X, 0)$, i.e. in general, it *cannot* be recovered from the link L_X or the resolution graph $\Gamma_{\tilde{X}}$. (See Exercise (2.15.).)

However, there are some cases, when Z_{max} (or $\mathcal{Z}_{an}$) can be recovered from the topology of X; and our aim is to find such cases.

Notice that for any $f \in \mathtt{m}_{X,0}$ one has $(\pi \circ f) \cdot E_i = 0$ for all i (cf. Exercise (2.6.)). This means that $[(f)_\Gamma + St(f)] \cdot E_i = 0$. Since $St(f) \cdot E_i \geq 0$ for all i, we have $(f)_\Gamma \cdot E_i \leq 0$, for all i.

Thus the topological candidate for $\mathcal{Z}_{an}$ is

$$\mathcal{Z}_{top} = \{Z \neq 0 \text{ effective cycle} \mid Z \cdot E_i \leq 0 \ \forall i\}.$$

2.8. Exercise. If $Z = \sum n_i E_i \in \mathcal{Z}_{top}$, then, by definition, there is an i such that $n_i > 0$. Show that, in fact, $n_i > 0$ for all i.

2.9. Lemma. *If $Z_1,\ Z_2\ \in \mathcal{Z}_{top}$, then*
(a) $Z_1 + Z_2\ \in\ \mathcal{Z}_{top}$*;*
(b) $\min(Z_1,\ Z_2)\ \in\ \mathcal{Z}_{top}$*.*
Proof of (b): Set $Z = \min(Z_1,\ Z_2)$. By Exercise (2.8.) $Z \neq 0$. Write $Z_j = Z + A_j$ for $j = 1, 2$. Then obviously $|A_1| \cap |A_2| = \emptyset$. Then for any irreducible component E_i, either $E_i \not\subset |A_1|$ or $E_i \not\subset |A_2|$. Fix an E_i and assume $E_i \not\subset |A_1|$. Then $ZE_i = (Z_1 - A_1)E_i = Z_1E_i - A_1E_i \leq 0$, since $Z_1\ \in\ \mathcal{Z}_{top}$ implies $Z_1E_i \leq 0$ and $E_i \not\subset |A_1|$ implies $A_1E_i \geq 0$. □

By the above lemma, we get that $\mathcal{Z}_{top}$ has a unique minimal element. First it was introduced by Artin in [1, 2]. In this note we will call it minimal cycle, and we denote it by Z_{min}. (Other names used in the literature are "Artin's fundamental cycle" or "numerical cycle".) It is the topological candidate for $Z_{max} = (l_{gen})_\Gamma$. Since $\mathcal{Z}_{an} \subset \mathcal{Z}_{top}$, clearly $Z_{min} \leq Z_{max}$.

2.10. Laufer's Algorithm for finding Z_{min}. [14]

(a) Start with any of the cycles E_i. Denote it by Z_1.

(b) If Z_l is already constructed for some integer $l > 0$ and there is an irreducible component E_{j_l} for which $Z_l \cdot E_{j_l} > 0$, then let $Z_{l+1} = Z_l + E_{j_l}$.

(c) If $Z_l \cdot E_j \leq 0$ for all j, then STOP, and $Z_l = Z_{min}$.

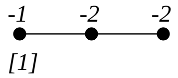
2.11. Exercise. Show that Laufer's Algorithm provides Z_{min} indeed.

2.12. Exercise. Compute the minimal cycle Z_{min} for the resolution graphs in Example (1.20.).

2.13. Exercise. ([1, 2]) Let $\{C_i\}_i$ be a connected bunch of curves on a regular surface.
(i) Suppose that their intersection matrix $(C_i \cdot C_j)_{i,j}$ is negative definite. Show that there exists an effective cycle $Z = \sum n_iC_i \neq 0$ with $Z \cdot C_i \leq 0$ for all i.
(ii) Conversely, if there exists an effective cycle $Z \neq 0$ with $Z \cdot C_i \leq 0$ for all i, then the intersection matrix $(C_i \cdot C_j)_{i,j}$ is negative semi-definite. Moreover, if $Z^2 < 0$, then $(C_i \cdot C_j)_{i,j}$ is negative definite.

2.14. (A part of) the Main Question revisited.
When does the equality $Z_{min} = Z_{max}$ hold?

2.15. Exercise. In general, $Z_{min} \neq Z_{max}$. Indeed, consider the normal hypersurface singularities $X_1 = \{x^2 + y^3 + z^{18} = 0\}$ and $X_2 = \{z^2 + y(x^4 + y^6) = 0\}$.
(a) Using the algorithm found in Appendix 1, show that the minimal resolution graph for both singularities is $\Gamma_{\tilde{X}_1} = \Gamma_{\tilde{X}_2}$:



(b) Using Laufer's Algorithm find Z_{min} for this graph.
(c) Show that in the case of $(X_1, 0)$ the equality $Z_{min} = Z_{max}$ holds. (Hint: Using the algorithm of Appendix 1 show that $Z_{min} = (z)_\Gamma$, where $z : (X_1, 0) \subset (\mathbf{C}^3, 0) \longrightarrow (\mathbf{C}, 0)$ is the projection on the third coordinate.)
(d) Show that in the case of $(X_2, 0)$, the strict inequality $Z_{min} < Z_{max}$ holds.

2.16. Remark. This example shows that the topology of a normal surface singularity, in general, does not determine the analytic information $\mathcal{Z}_{an}$!

2.17. The Connection Between Z_{max} and the Multiplicity of $(X, 0)$. Consider the geometric definition of the multiplicity $mult(X, 0)$ of $(X, 0)$ (see Definition (1.12.)). Notice that if l_1 and l_2 are generic linear functions on $(X, 0)$, then $mult(X, 0)$ is just the intersection multiplicity of l_1, l_2 and X at the point 0 (i.e. $mult(X, 0) = \#X \cap \{l_1 = \epsilon'\} \cap \{l_2 = \epsilon''\}$, where ϵ' and ϵ'' are generic and small).

Let $Z_{max} = \sum n_i E_i$ and fix a point $p \in E_i - (\bigcup\limits_{j \neq i} E_j)$ for some i. Then in a small neighbourhood U of p there exist local coordinates (x, y) such that $E_i \cap U = \{x = 0\}$.

By definition $\mathcal{O}_{\tilde{X}}(-Z_{max})|_U = x^{n_i}\mathcal{O}_U$. If there is a generic linear function $l \in \mathtt{m}_{X,0}$ such that $\pi \circ l|_U = x^{n_i} \cdot u$, where $u \in \mathcal{O}_{U,p}$ is invertible, then

$$\pi^*(\mathtt{m}_{X,0})|_U = \mathtt{m}_{X,0}|_U \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-Z_{max})|_U.$$

Actually, for any $l \in \mathtt{m}_{X,0}$ one has $\pi \circ l|_U = x^{n_i} \cdot g_{U,l}$ with some function $g_{U,l} \in \mathcal{O}_U$. If $g_{U,l}(p) = 0$ for <u>all</u> $l \in \mathtt{m}_{X,0}$, then we say that p is a <u>basepoint</u> of $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}}$. In this case $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}}|_U = \mathcal{I}_U \cdot \mathcal{O}_{\tilde{X}}(-Z_{max})$ for some (proper) ideal $\mathcal{I}_U$ of $\mathcal{O}_U$ (which is generated by all the germs $g_{U,l}$ for $l \in \mathtt{m}_{X,0}$).

The discussion is similar for the "intersection" points $p \in E_i \cap E_j$.

2.18. Theorem: [29]

a) If $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-Z_{max})$, i.e. $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}}$ has <u>no</u> basepoints, then $mult(X, 0) = -Z^2_{max}$.

b) If $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}} = \mathcal{I}\mathcal{O}_{\tilde{X}}(-Z_{max})$ for some proper ideal $\mathcal{I} \neq \mathcal{O}_{\tilde{X}}$, then $mult(X, 0) > -Z^2_{max}$.

2.19. Remark. Obviously, Z_{max} and the above discussion depends on the choice of the resolution π. If $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}}$ has basepoints, by some blow-ups in these points, the resolution π can be replaced with another resolution that has no basepoints. However, if the minimal resolution does have basepoints, then the topological information (in general) is not sufficient to determine those resolutions that have no basepoints.
Proof of Theorem (2.18.): Let $l', l'' \in \mathtt{m}_{X,0}$ be two generic linear functions. Then:

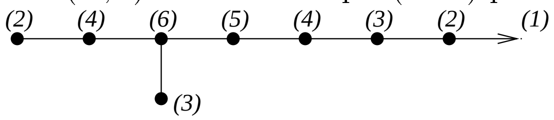
$$mult(X, 0) = (\pi \circ l') \cdot (\pi \circ l'') = [(l')_\Gamma + St(l')] \cdot (\pi \circ l'') \overset{(2.6)}{=}$$

$$= St(l') \cdot (\pi \circ l'') = St(l') \cdot [(l'')_\Gamma + St(l'')] = St(l') \cdot St(l'') + St(l') \cdot (l'')_\Gamma =$$

$$= St(l') \cdot St(l'') + [(\pi \circ l') - (l')_\Gamma] \cdot (l'')_\Gamma \overset{(2.6)}{=} St(l') \cdot St(l'') - (l')_\Gamma \cdot (l'')_\Gamma =$$
$$= St(l') \cdot St(l'') - Z_{max}^2$$

Since $St(l') \cdot St(l'') \geq 0$, we get $mult(X,0) \geq -Z_{max}^2$ and equality holds if and only if there are no basepoints (i.e. the generic linear functions l' and l'' can be chosen so that their strict transforms do not go through the same points of E and thus $St(l') \cdot St(l'') = 0$).

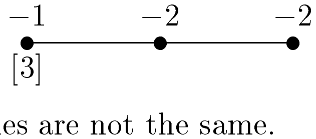
2.20. Examples.
(a) Take $(X,0) = (\{x^2 + y^3 + z^5 = 0\}, 0) \subset (\mathbf{C}^3, 0)$. Consider again $z : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ defined by $(x,y,z) \mapsto z$. By abuse of notation we write $z = z|_X$. The minimal resolution graph of $(X,0)$ is as in Example (1.20.) part (f). The multiplicities of z are:



Thus $Z_{min} = (z)_\Gamma$ and so $Z_{min} = Z_{max}$.
$-Z_{max}^2 = -Z_{min}^2 = -Z_{min} \cdot [(\pi \circ z) - St(z)] = Z_{min} \cdot St(z) = 2$. Since $mult(X,0) = 2$, this is an example of a singularity, whose minimal resolution has no basepoints and so $mult(X,0) = -Z_{max}^2$ as was just verified.
(b) Take $(X,0) = (\{x^2 + y^3 + z^{11} = 0\}, 0) \subset (\mathbf{C}^3, 0)$, and consider its minimal resolution. Then it can be verified that $Z_{min} = (z)_\Gamma$ again, so that $Z_{min} = Z_{max}$. In this case however $-Z_{max}^2 = -Z_{min}^2 = 1$, hence $mult(X,0) = 2 > -Z_{max}^2$. This shows that there is at least one basepoint in the minimal resolution and in this case $\mathfrak{m}_{X,0}\mathcal{O}_{\tilde{X}} = \mathcal{I}\mathcal{O}_{\tilde{X}}(-Z_{max})$ for some proper ideal $\mathcal{I} \neq \mathcal{O}_{\tilde{X}}$. (Actually, in this case there is exactly one base point p, and $\mathcal{I} = \mathfrak{m}_{\tilde{X},p}$.)

2.21. When is the multiplicity topological? The previous discussion shows that if $Z_{max} = Z_{min}$ and $\mathcal{O}_{\tilde{X}}(-Z_{max}) = \mathfrak{m}_{X,0}\mathcal{O}_{\tilde{X}}$ then we have $mult(X,0) = -Z_{min}^2$. Therefore, if these two assumptions are satisfied, then the multiplicity (which is a priori an analytic invariant) can be a posteriori determined from the topological data. However, in general, the multiplicity is not a topological invariant.

2.22. Example. Consider the normal hypersurface singularities $X_1 = \{x^2 + y^7 + z^{42} = 0\}$ and $X_2 = \{x^3 + y^4 + z^{36} = 0\}$. Then using the algorithm of Appendix 1 we can verify that the minimal resolution graph for both singularities is $\Gamma_{\tilde{X}_1} = \Gamma_{\tilde{X}_2}$:



But their multiplicities are not the same.

II. The canonical cycle.

2.23. The Canonical Divisor. Recall that $\Omega^2_{\tilde{X}}$ denotes the sheaf of holomorphic 2-forms on $\widetilde{X}$. Since it is a holomorphic line bundle on $\widetilde{X}$, it corresponds to a class of divisors (modulo the principal divisors). More precisely, if $\tilde{\omega}$ is a meromorphic section of $\Omega^2_{\tilde{X}}$ (i.e. a global meromorphic 2-form on $\widetilde{X}$), then its divisor $(\tilde{\omega})$ is independent of the choice of $\tilde{\omega}$ modulo a principal divisor. This divisor is called the canonical divisor, and it is denoted by K or $K_{\tilde{X}}$.

2.24. Definition. If we can find a section $\tilde{\omega}$ of $\Omega^2_{\tilde{X}}$ supported by E, then the singularity $(X,0)$ is called Gorenstein. This means that $(X,0)$ is Gorenstein, if there is a global section $\omega \in H^0(X-\{0\}, \Omega^2_{X-\{0\}})$ that is never vanishing on $X-\{0\}$; or equivalently, if $\Omega^2_{X-\{0\}}$ is a holomorphically trivial line bundle. (Cf. Exercise (2.43.); and (2.33.), (2.34.) and (2.37.).)

Since K is well-defined modulo a principal divisor, and for any principal divisor (g) one has $(g)\cdot E_i = 0$ for all i (cf. Exercise (2.6.)), the intersection multiplicities $\{K\cdot E_i\}_i$ do not depend on the choice of $\tilde{\omega}$. These integers represent K *numerically.*

2.25. The Adjunction Formula when each E_i is smooth. *Consider a resolution with the property that each irreducible exceptional curve is smooth, e.g. the case of a "good" resolutions. Then the integers $\{K\cdot E_i\}_i$ can be determined from the graph by the following relations:*

$$K\cdot E_i = -E_i^2 + 2g_i - 2 \quad \text{for all } i.$$

Proof of the Adjunction Formula: Let T denote the tangent bundle and N_{E_i} the normal bundle of E_i in $\widetilde{X}$, then:

$$0 \longrightarrow T_{E_i} \longrightarrow T_{\tilde{X}}|_{E_i} \longrightarrow N_{E_i} \longrightarrow 0 \qquad \text{or}$$

$$0 \longrightarrow N^*_{E_i} \longrightarrow \Omega^1_{\tilde{X}}|_{E_i} \longrightarrow \Omega^1_{E_i} \longrightarrow 0.$$

Hence $\Omega^2_{\tilde{X}}|_{E_i} = \det \Omega^1_{\tilde{X}}|_{E_i} = \Omega^1_{E_1}\otimes N^*_{E_i}$, and taking the degree of these line bundles (or the first Chern-numbers) one has:

$$K_{\tilde{X}}\cdot E_i = deg\,\Omega^2_{\tilde{X}}|_{E_i} = deg\,\Omega^1_{E_i} + deg\,N^*_{E_i} = -\chi_{top}(E_i) - E_i^2 = -E_i^2 + 2g_i - 2,$$

where $\chi_{top}(E_i)$ denotes the topological Euler characteristic of E_i. □

2.26. The case of singular irreducible exceptional curves. Now assume that we have a resolution with some singular irreducible exceptional curves. Then in the above adjunction formula we have to add a correction term: the delta–invariant of the singular curves.

2.27. The delta–invariant of singular curves. [27]

Let $\rho_i : \tilde{E}_i \to E_i$ be the normalization of the irreducible curve E_i. We define the *delta–invariant* δ_i of E_i by:

$$\delta_i = \dim_{\mathbf{C}} (\rho_i)_* \mathcal{O}_{\tilde{E}_i} / \mathcal{O}_{E_i}.$$

Actually, the sheaf $(\rho_i)_* \mathcal{O}_{\tilde{E}_i} / \mathcal{O}_{E_i}$ is supported by the set of singular points $Sing(E_i)$ of E_i. Moreover,

$$\delta_i = \sum_{Q \in Sing(E_i)} \delta_Q(E_i),$$

where $\delta_Q(E_i)$ is the local delta invariant ("the number of nodes and cusps") of the plane curve singularity (E_i, Q). For the convenience of the reader, we give also the definition of these local invariants.

Assume that the germ (E_i, Q) has r_i local irreducible components: $(E_i, Q) = \cup_{j=1}^{r_i} (E_{ij}, Q)$. Let $\rho_{ij} : (\tilde{E}_{ij}, \tilde{Q}) \to (E_{ij}, Q)$ be the normalization, then

$$\delta_Q(E_{ij}) := \dim (\rho_{ij})_* \mathcal{O}_{\tilde{E}_{ij}, \tilde{Q}} / \mathcal{O}_{E_{ij}, Q}.$$

Obviously, by the definition of the normalization, $(\rho_{ij})_* \mathcal{O}_{\tilde{E}_{ij}, \tilde{Q}}$ is exactly the *integral closure* of $\mathcal{O}_{E_{ij}, Q}$ in its field of fraction. Finally,

$$\delta_Q(E_i) := \sum_j \delta_Q(E_{ij}) + \sum_{i<k} \text{intersection multiplicity}(E_i, E_k)_Q.$$

2.28. Exercise. In (b) and (c) the singular germ is considered in $(\mathbf{C}^2, 0)$.

(a) $\delta_Q(E_i) = 0$ if and only if E_i is smooth at Q.

(b) $\delta_0(\{x^2 + y^3 = 0\}) = 1$.

(c) $\delta_0(\{xy = 0\}) = 1$.

(d) For any positive integer k, construct a plane curve singularity with delta–invariant $= k$.

2.29. The delta–invariant of a plane curve singularity is a topological invariant. In order to explain this statement we need some topological preparations. Consider the plane curve singularity $(E_i, Q) \subset (\tilde{X}, Q)$. If B is a sufficiently small ball with center at Q, then the intersection $L_{E_i} := \partial B \cap E_i$ is an oriented 1–manifold in $\partial B (= 3$–sphere $S^3)$. Consider now this link $L_{E_i} \subset S^3$. The number of link–components of L_{E_i} is exactly the number of local analytic irreducible components of E_i, i.e. by the above notations: r_i. Fix a minimal Seifert surface S for this link. Define the "Milnor number" $\mu_Q(E_i)$ by $rank\, H^1(S, \mathbf{Z})$. Then, by a formula of J. Milnor [20]:

$$2\delta_Q(E_i) = \mu_Q(E_i) + r_i - 1.$$

In particular, $\delta_Q(E_i)$ is a topological invariant of the embedded germ $(E_i, Q) \subset (\tilde{X}, Q)$. (For a different definition of the Milnor numbers see [20], or the author's work "Some topological invariants..." in this proceedings.)

2.30. The adjunction formula in the general case. Consider an arbitrary resolution. Recall, that if some of the irreducible exceptional divisors are not smooth, in the decoration of the graph we put all the embedded topological invariants of the the local germs $(E_i, Q) \subset (\tilde{X}, Q)$, for all singular points Q of E_i. In particular, the delta invariant $\delta_i = \delta(E_i) = \sum_Q \delta_Q(E_i)$ is completely determined from the resolution graph.

Actually, in some cases, what we really put on the graph is only the list of the singular points with their delta–invariants.

Now, the adjunction formula in this general case has the following form:

$$K \cdot E_i = -E_i^2 + 2\delta_i + 2g_i - 2 \quad \text{for all } i.$$

For the proof and other discussions, see the book of Serre [27].

2.31. The case of the minimal resolutions. Assume that $K \cdot E_i < 0$ for some i. Then, by the Adjunction Formula, $-E_i^2 + 2\delta_i + 2g_i - 2 < 0$. But $-E_i^2 > 0$, $\delta_i \geq 0$, and $g_i \geq 0$, hence this can only happen if $E_i^2 = -1$ and $\delta_i = g_i = 0$. This means that E_i is a smooth rational curve with self intersection -1. By the Castelnuovo Theorem (cf. (3.19.)) this curve can be blown down. In particular, in a minimal resolution such a curve cannot occur. This show that for a *minimal* resolution $K \cdot E_i \geq 0$ for any i.

2.32. The Canonical Cycle. Above we proved that the numerical information $\{K \cdot E_i\}_i$ of the canonical divisor K is completely topological, and can be read from the graph. This motivates the following definition: the rational cycle Z_K is called the canonical cycle if $Z_K \cdot E_i = -KE_i$ for all i, i.e.

$$Z_K \cdot E_i = E_i^2 - 2\delta_i - 2g_i + 2 \quad \text{for all } i. \tag{1}$$

Thus the canonical cycle Z_K is defined by a system of equations (whose matrix is exactly $(E_i \cdot E_j)_{i,j}$), which always has a rational solution $Z_K \in Div_{\tilde{X}} \otimes \mathbf{Q}$, since the intersection matrix $(E_i \cdot E_j)_{i,j}$ is non–degenerate.

Let us emphasize, that Z_K is a topological object, while K (or $\Omega^2_{\tilde{X}}$) is not! Moreover, the divisor K is defined only modulo the principal divisors, but Z_K is a well–defined cycle.

2.33. Definition If the coefficients of Z_K are integers, then the singularity $(X, 0)$ is called numerical Gorenstein.

Now we want to explain a little bit more the relation between K and Z_K. They are both defined in $Div(\tilde{X})$, but we also can work with their homology classes. Since K, in general, is not compact, its class $[K]$ is in $H_2(\tilde{X}, \partial\tilde{X}, \mathbf{Z})$; on the other hand Z_K is a rational cycle with compact support, hence $[Z_K] \in H_2(\tilde{X}, \mathbf{Q})$. (Since the homology class of any principal divisor is trivial, by a similar argument as in (2.6.), the homology class $[K]$ is well–defined.)

Now consider the exact sequence:

$$H_2(\tilde{X},\mathbf{Z}) \xrightarrow{j} H_2(\tilde{X},\partial\tilde{X},\mathbf{Z}) \xrightarrow{\partial} H_1(\partial\tilde{X},\mathbf{Z}) .$$

The intersection pairing $\langle\cdot,\cdot\rangle : H_2(\tilde{X},\partial\tilde{X},\mathbf{Z})\otimes H_2(\tilde{X},\mathbf{Z}) \to \mathbf{Z}$ is perfect, and one can connect with the intersection form $(\cdot,\cdot) : H_2(\tilde{X},\mathbf{Z})\otimes H_2(\tilde{X},\mathbf{Z}) \to \mathbf{Z}$ by $\langle j(x),y\rangle = (x,y)$. The form $(,)$ is non-degenerate (but in general it is not unimodular), hence j is injective, but in general is not onto. But its cokernel is finite, hence $j_{\mathbf{Q}} := j\otimes 1_{\mathbf{Q}}$ is an isomorphism. Moreover, $j_{\mathbf{Q}}(x) = z \Leftrightarrow (x,y) = \langle j(x),y\rangle = \langle z,y\rangle$ for any $y\in H_2(\tilde{X},\mathbf{Z})$. Since $H_2(\tilde{X},\mathbf{Z})$ is generated by the classes $\{E_i\}_i$, the relations $Z_K\cdot E_i = -K\cdot E_i$ (for all i) reads as $j_{\mathbf{Q}}([Z_K]) = -[K]$. The above exact sequence shows that Z_K has integer coefficient, if and only if $[K]$ is in the image of the integer operator j, or equivalently, if and only if $\partial[K]\in H_1(\partial\tilde{X},\mathbf{Z})$ is vanishing.

Now we recall the following relation between Chern numbers and divisors (see e.g. [5], page 22): If D is a divisor on a compact surface S, then $c_1(\mathcal{L}(D)) = PD([D])\in H^2(S,\mathbf{Z})$, where PD denotes the Pincaré Duality. In our case $\tilde{X}$ is not compact, but the Poincaré dual of $[K]\in H_2(\tilde{X},\partial\tilde{X},\mathbf{Z})$ is in $H^2(\tilde{X},\mathbf{Z})$ and still satisfies $PD([K]) = c_1(\mathcal{L}(K)) = c_1(\Omega^2_{\tilde{X}})$.

Then, it is easy to see that the Poincaré dual (in $\partial\tilde{X}$) of $\partial[K]$ is exactly the restriction of the above Chern class to $\partial\tilde{X}$. Since $\partial\tilde{X}$ admits a deformation retract in $\tilde{X}-E$, we deduce that $\partial[K]=0$ if and only if $c_1(\Omega^2_{\tilde{X}-E})=0$. Therefore, we proved:

2.34. Proposition. $c_1(\Omega^2_{\tilde{X}-E}) = 0 \;\Leftrightarrow\; Z_K$ *has integer coefficients.*

2.35. Exercise. ([7]) *Let ξ denote a two-dimensional complex vector bundle over a CW complex L with $H^i(L)=0$ for $i>3$. Then the following conditions are equivalent:*

(i) ξ is topologically trivial.

(ii) The first Chern class $c_1(\xi)\in H^2(L,\mathbf{Z})$ is trivial.

(iii) The second exterior power $\Lambda^2\xi$ is a topologically trivial line bundle.

2.36. Exercise. Fix a resolution $\tilde{X}\to X$ of a a normal surface singularity $(X,0)$. Prove that the above exercise can be applied for $L := \tilde{X}-E \approx X-\{0\}$. [Hint: L has the same homotopy type as the link L_X of $(X,0)$.]

2.37. Corollary. $(X,0)$ *is a numerical Gorenstein singularity* $\Leftrightarrow$

$$\Omega^2_{X-\{0\}} \textit{ is topologically trivial} \Leftrightarrow \Omega^1_{X-\{0\}} \textit{ is topologically trivial} \Leftrightarrow$$

$$\Omega^2_{\tilde{X}-E} \textit{ is topologically trivial} \Leftrightarrow \Omega^1_{\tilde{X}-E} \textit{ is topologically trivial} \Leftrightarrow c_1(\Omega^2_{\tilde{X}-E}) = 0.$$

2.38. Exercise. Prove that if $(X,0)$ is Gorenstein, then it is numerical Gorenstein.

2.39. Exercise. Give an example of a singularity which is not numerical Gorenstein.

2.40. Exercise. Show that if $Z_K = 0$ then $\delta_i = g_i = 0$ and $E_i^2 = -2$ for all i. Determine all the possible resolution graphs with this property (i.e. determine all the possible graphs with $E_i^2 = -2$ and $(E_i \cdot E_j)_{i,j}$ negative definite). (They are exactly the so called $A - D - E$ graphs, for their form see (1.20.).)

2.41. Exercise. Assume that π is a minimal resolution. Prove that if $Z_K \neq 0$ and $Z_K = \sum n_i E_i$ ($n_i \in \mathbf{Q}$), then $n_i > 0$ for all i.

2.42. Exercise. Assume that $(X, 0)$ is a numerical Gorenstein singularity and π is a minimal resolution. Then $Z_{min} \leq Z_K$ or $Z_K = 0$ (cf. (2.31.)).

2.43. Exercise. Consider the hypersurface singularity $(X, 0) = (\{g = 0\}, 0)$, where $g : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ defines an isolated singularity. Take

$$\omega = \frac{dx \wedge dy}{g_z}|_X = \frac{dy \wedge dz}{g_x}|_X = \frac{dz \wedge dx}{g_y}|_X.$$

(Above g_x, g_y, g_z are the partial derivatives of g.)
Prove that $\omega \in H^0(X - \{0\}, \Omega^2_{X-\{0\}})$ is a well–defined global section of Ω^2 over $X - \{0\}$, and it has no zeros on $X - \{0\}$. In particular, $(X, 0)$ is Gorenstein and $Z_K = -(\pi^*\omega)$ i.e. Z_K measures the poles of $\pi^*\omega$ along E.

Using ω, compute Z_K in the case $g = x^n + y^n + z^n$ and verify that Z_K satisfies the Adjunction Formula.

2.44. Exercise. Generalize Exercise (2.43.) for isolated complete intersections $(f_1, \ldots, f_k) : (\mathbf{C}^{k+2}, 0) \to (\mathbf{C}^k, 0)$. Prove that they are Gorenstein.
[Hint:

$$\text{Take } \omega = dx_1 \wedge dx_2/\text{Jacobian determinant of } \frac{\partial(f_1, \ldots, f_k)}{\partial(x_3, \ldots, x_{k+2})}.]$$

III. Appendix 2. Cohomology results.

The main technique in the study of singularities is the cohomology of sheaves. The rich geometry is described in the language of the cohomology groups of type:

$$H^i(\widetilde{X}, \mathcal{O}_{\tilde{X}}(-D)) \text{ and } H^i(Z, \mathcal{O}_Z(-D)),$$

where Z is an effective cycle.

Here we make a short list of properties which will be used in the sequel, and they are necessary in the understanding of the next sections. For the proofs of these statements and as a general reference see [11], [12].

If $\mathcal{F}$ is a sheaf, its Euler-characteristic is

$$\chi(\mathcal{F}) = \sum_i (-1)^i h^i(\mathcal{F}), \quad \text{where } h^i(\mathcal{F}) = \dim H^i(\mathcal{F}).$$

1. <u>The formal neighbourhood theorem (or Theorem of formal functions):</u>

$$\begin{aligned} H^1(\widetilde{X}, \mathcal{O}_{\tilde{X}}(-D) &= \lim_{\substack{\longleftarrow \\ k\to\infty}} H^1(\mathcal{O}_{kE}, \mathcal{O}_{kE}(-D)) \\ &= \lim_{\substack{\longleftarrow \\ Z>0}} H^1(\mathcal{O}_Z, \mathcal{O}_Z(-D)) \end{aligned}$$

2. <u>The Riemann-Roch Theorem:</u> (for $Z > 0$)

$$\begin{aligned} \chi(\mathcal{O}_Z(-D)) &= -Z \cdot D \ + \ \chi(\mathcal{O}_Z) \\ \chi(\mathcal{O}_Z) &= -\frac{1}{2} Z \cdot (Z + K) = -\frac{1}{2} Z \cdot (Z - Z_K) \end{aligned}$$

   In particular, $\chi(\mathcal{O}_Z)$ and $\chi(\mathcal{O}_Z(-D))$ are topological, and can be determined from the resolution graph.

   As a consequence, for any two cycles A and B one has:

$$\chi(A + B) = \chi(A) + \chi(B) - AB.$$

3. <u>Serre duality:</u>

$$h^1(\mathcal{O}_Z(D)) = h^0(\mathcal{O}_Z(-D + Z + K))$$

4. <u>Dimension argument:</u>

$$h^i(\widetilde{X}, \mathcal{O}_{\tilde{X}}(-D)) = h^i(Z, \mathcal{O}_Z(-D)) = 0 \text{ for } i \geq 2.$$

5. <u>Kodaira type vanishing theorem:</u>
   If $\mathcal{L}$ is a line bundle with $\mathcal{L} \cdot E_i \geq K_{\tilde{X}} \cdot E_i$ for all i, then $h^1(\widetilde{X}, \mathcal{L}) = 0$, and $h^1(D, \mathcal{L}|_D) = 0$ for any effective cycle D. By duality: $H^0(\mathcal{O}_D(D) \otimes \mathcal{L}) = 0$ if $\mathcal{L} \cdot E_i \leq 0$ for all i.

6. The Long Exact Sequence: A short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0,$$

induces the long exact sequence

$$... \longrightarrow H^i(\mathcal{F}_1) \longrightarrow H^i(\mathcal{F}) \longrightarrow H^i(\mathcal{F}_2) \longrightarrow H^{i+1}(\mathcal{F}_1) \longrightarrow ...$$

7. The most frequent exact sequence: If $D_1 > 0$, $D_2 > 0$, then:

$$0 \longrightarrow \mathcal{O}_{D_2}(D - D_1) \longrightarrow \mathcal{O}_{D_1+D_2}(D) \longrightarrow \mathcal{O}_{D_1}(D) \longrightarrow 0$$

8. The case of $\mathbf{P}^1$:

   (a) The line bundles on $\mathbf{P}^1$ are classified by their first Chern number i.e. any $\mathcal{L} \in Pic(\mathbf{P}^1)$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(n)$, where $n = \deg_{\mathbf{P}^1} \mathcal{L} = c_1(\mathcal{L})$. $\Omega^1_{\mathbf{P}^1} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)$.

   (b) $\mathcal{O}_{\tilde{X}}(D)|_{E_i = \mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(E_i \cdot D)$.

   (c) $h^0(\mathcal{O}_{\mathbf{P}^1}(n) = \left\{ \begin{array}{ll} n+1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{array} \right\} = h^1(\mathcal{O}_{\mathbf{P}^1}(-n-2))$.

9. Actually, even for a non–rational E_i one has: $h^0(\mathcal{O}_{E_i}(D)) \neq 0$ implies $E_i \cdot D \geq 0$.

10. For any irreducible curve C one has: $\chi(\mathcal{O}_C) = 1 - g(C) - \delta(C)$.

2.45. Exercises. Recall the notation of Laufer's Algorithm (2.10.). Prove that:

1. $h^0(Z_{min}, \mathcal{O}_{Z_{min}}) = 1$. [Hint: show by induction that $H^0(Z_l, \mathcal{O}_{Z_l}) = \{\text{constants}\}$ for all l.]

2. $h^1(Z_{min}, \mathcal{O}_{Z_{min}}) = \sum_l h^1(\mathcal{O}_{E_{j_l}}(-Z_l))$, where $Z_{l+1} = Z_l + E_{j_l}$.
   In particular, if $E_i \approx \mathbf{P}^1$ for any i, then $h^1(Z_{min}, \mathcal{O}_{Z_{min}}) = \sum\limits_{E_{j_l} \cdot Z_l \geq 2} (E_{j_l} Z_l - 1)$.

2.46. Exercise. Prove that for a numerical Gorenstein singularity $p_g = h^1(Z_K)$. [Hint: For any $D > Z_K$, $H^1(\mathcal{O}(-Z_K)) \to H^1(\mathcal{O}_{D-Z_K}(-Z_K))$ is onto, and the vanishing of the first group implies the vanishing of the second one too. Now, from the exact sequence $0 \to \mathcal{O}_{D-Z_K}(-Z_K) \to \mathcal{O}_D \to \mathcal{O}_{Z_K} \to 0$ one gets that $H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{Z_K})$ is an isomorphism.]

2.47. Exercise. Prove that

$$h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \dim_{\mathbf{C}} \frac{H^0(X - \{0\}, \Omega^2)}{H^0_{L^2}(X - \{0\}, \Omega^2)},$$

i.e. the two definitions of the geometric genus are equivalent.
[Hint. Consider the exact sequence:

$$0 \to H^0_c(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^0(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^0(\tilde{X} - E, \Omega^2_{\tilde{X}}) \to H^1_c(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^1(\tilde{X}, \Omega^2_{\tilde{X}}).$$

Then $H^0_c(\tilde{X}, \Omega^2_{\tilde{X}}) = H^2(\mathcal{O}_{\tilde{X}})^* = 0$ by dimension argument, and $H^1(\tilde{X}, \Omega^2_{\tilde{X}}) = 0$ by Kodaira vanishing. Therefore, $H^1_c(\tilde{X}, \Omega^2_{\tilde{X}}) = H^1(\mathcal{O}_{\tilde{X}})^*$ satisfies:

$$0 \to H^0(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^0(\tilde{X} - E, \Omega^2_{\tilde{X}}) \to H^1(\mathcal{O}_{\tilde{X}})^* \to 0.$$

Now, notice that $H^0(\Omega^2_{\tilde{X}}) = H^0_{L^2}(X - \{0\}, \Omega^2_{X-\{0\}})$.]

2.48. A diagram. In the next diagram we want to emphasize the "parallelism" between analytic and topological invariants introduced above:

analytical invariants	topological invariants
the local ring $\mathcal{O}_{X,0}$	the link L_X
the (minimal) resolution $\widetilde{X} \xrightarrow{\pi} X$	the dual resolution graph $\Gamma_{\tilde{X}}$
the holomorphic bundle $\Omega^2_{X-\{0\}}$	the topological bundle $\Omega^2_{X-\{0\}}$
the canonical divisor K	the canonical cycle Z_K
$\mathcal{Z}_{an}$ and the maximal ideal cycle $Z_{max} \in \mathcal{Z}_{an}$	$\mathcal{Z}_{top}$ and the minimal cycle $Z_{min} \in \mathcal{Z}_{top}$
the geometric genus $p_g = \dim_{\mathbf{C}} H^1(\widetilde{X}, \mathcal{O}_{\tilde{X}})$	the arithmetic genus $p_a = 1 - \min_{Z>0} \chi(\mathcal{O}_Z)$ (as a topological lower bound of p_g.)

3§. Rational singularities.

3.1. Definition. Recall that the geometric genus of a normal surface singularity $(X,0)$ is defined by $p_g := \dim_{\mathbf{C}} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$, where $\pi : \tilde{X} \to X$ is any resolution of $(X,0)$. This definition does not depend on the choice of the resolution π.

3.2. Definition. A singularity $(X,0)$ is called <u>rational</u> if

$$p_g = \dim H^1(\widetilde{X}, \mathcal{O}_{\tilde{X}}) = 0.$$

I. Topological characterization of rational singularities.

For the sequel, we fix a resolution π. We start with the following general statement:

3.3. Lemma.
(a) If $Z_2 > Z_1 > 0$, then $H^1(\mathcal{O}_{Z_2}) \to H^1(\mathcal{O}_{Z_1})$ is surjective.
(b) $p_g = h^1(\widetilde{X}, \mathcal{O}_{\tilde{X}}) = \dim \varprojlim_{Z>0} H^1(\mathcal{O}_Z)$
Proof. Apply (7) and (1) of Appendix 2. □

3.4. Corollary. *$p_g = 0$ if and only if $h^1(\mathcal{O}_Z) = 0$ for all cycles $Z > 0$.*

3.5. Theorem. [Artin][1, 2]

$$p_g = 0 \iff \chi(\mathcal{O}_Z) \geq 1 \text{ for all positive cycles } Z > 0.$$

Note that this gives a <u>topological</u> characterization of rational singularities, since the right hand side depends only on the resolution graph $\Gamma_{\tilde{X}}$.
Proof of (3.5.) "$\Rightarrow$" $p_g = \varprojlim_{Z>0} h^1(\mathcal{O}_Z) \overset{(3.3)}{\geq} h^1(\mathcal{O}_Z) = h^0(\mathcal{O}_Z) - \chi(\mathcal{O}_Z) \geq 1 - \chi(\mathcal{O}_Z)$. (Here $h^0(\mathcal{O}_Z) \geq 1$, since the constant functions are in $H^0(\mathcal{O}_Z)$.)
"$\Leftarrow$" Assume that $p_g \neq 0$. Then by (3.4.) there exists an effective cycle $D > 0$ with $h^1(\mathcal{O}_D) \neq 0$. Let D be the minimal with this property. (Notice that if $D = E_i$ for some i, then $h^0(\mathcal{O}_D) = 1$ and $\chi(\mathcal{O}_D) \geq 1$ imply $h^1(\mathcal{O}_D) = 0$, hence such a minimal element exists.) Thus $h^1(\mathcal{O}_D) \neq 0$, but $h^1(\mathcal{O}_{D-E_i}) = 0$ for any $E_i \subset |D|$.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{E_i}(-D+E_i) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D-E_i} \longrightarrow 0.$$

Then $H^1(\mathcal{O}_{E_i}(-D+E_i)) \longrightarrow H^1(\mathcal{O}_D) \longrightarrow H^1(\mathcal{O}_{D-E_i}) \longrightarrow 0$ is exact (where $H^1(\mathcal{O}_D) \neq 0$ and $H^1(\mathcal{O}_{D-E_i}) = 0$), so $h^1(\mathcal{O}_{E_i}(-D+E_i)) \neq 0$ and by Serre duality $h^0(\mathcal{O}_{E_i}(D - E_i + E_i + K)) \neq 0$. Using (9) of Appendix 2, $E_i(D+K) \geq 0$ for all $E_i \subset |D|$. By summation $D(D+K) \geq 0$, or $\chi(D) \leq 0$. But this is a contradiction. □

3.6. Example. Consider a resolution graph that consists of a single vertex with self-intersection number $-e \leq -2$ and genus $g = 0$. Let the curve that corresponds to the vertex be denoted by E. Then the canonical cycle Z_K is $\frac{e-2}{e}E$. For a divisor $D = nE$ with $n \geq 1$, by (2) of Appendix 2, we have $\chi(D) = -\frac{1}{2}nE(nE - \frac{e-2}{e}E) = \frac{1}{2}en(en - e + 2) > 0$. This shows, via Theorem (3.5.), that a singularity $(X, 0)$ with such a resolution graph is rational.

3.7. Exercise. Show that if $(X, 0)$ is numerical Gorenstein and rational, then $Z_K = 0$. In particular, $(X, 0)$ is an $A - D - E$–singularity. (Cf. also Exercise (2.40.).)

In general it is difficult to verify that $\chi(\mathcal{O}_Z) > 0$ for *all* positive cycles $Z > 0$. The next theorem shows that it is enough to compute $\chi(\mathcal{O}_{Z_{min}}) = \chi(Z_{min})$.

3.8. Theorem. [Artin] [1, 2]

$$p_g = 0 \iff \chi(Z_{min}) = 1.$$

3.9. Some preliminaries. By Exercise (2.45.) $h^0(\mathcal{O}_{Z_{min}}) = 1$, hence

$$\chi(Z_{min}) = 1 \iff h^1(\mathcal{O}_{Z_{min}}) = 0.$$

Now, we verify that $h^1(\mathcal{O}_{Z_{min}}) = 0$ implies

$$\begin{cases} (i) & \text{each } E_i \approx \mathbf{P}^1 \text{ (i.e. } \delta_i = g_i = 0 \text{ for all } i); \\ (ii) & \Gamma_{\tilde{X}} \text{ is a tree;} \\ (iii) & E_{j_l} Z_l = 1 \text{ for all } l \text{ in Laufer's Algorithm (2.10.)} \end{cases}$$

Indeed, consider the computation sequence $\{Z_l\}_l$ in Laufer's Algorithm (2.10.) In the exact sequence $0 \longrightarrow \mathcal{O}_{E_{j_l}}(-Z_l) \longrightarrow \mathcal{O}_{Z_{l+1}} \longrightarrow \mathcal{O}_{Z_l} \longrightarrow 0$ the Chern number $-E_{j_l}Z_l < 0$, hence $H^0(\mathcal{O}_{E_{j_l}}(-Z_l)) = 0$.

By induction $H^0(\mathcal{O}_{Z_l}) = \{\text{constants}\} = \mathbf{C}$ and

$$(*) \qquad h^1(\mathcal{O}_{Z_m}) = \sum_l h^1(\mathcal{O}_{E_{j_l}}(-Z_l)).$$

If $\delta(E_{j_l}) + g(E_{j_l}) \geq 1$ for some i, then $\chi(\mathcal{O}_{E_{j_l}}(-Z_l)) = -E_{j_l}Z_l + 1 - \delta(E_{j_l}) - g(E_{j_l}) < 0$, hence $h^1(\mathcal{O}_{E_{j_l}}(-Z_l)) \neq 0$, which proves part (i).

Now, assume that the resolution graph $\Gamma_{\tilde{X}}$ has a cycle $E_1, E_2, ..., E_l, E_{l+1} = E_1$. Consider the following part of a computation sequence (suitable chosen) $E_1, E_1 + E_2, ..., E_1 + E_2 + ... + E_l$. Then $h^1(\mathcal{O}_{E_l}(-E_1 - ... - E_{l-1})) = h^1(\mathcal{O}_{\mathbf{P}^1}(-2)) = 1 \neq 0$, which proves part (ii). For the last part use again $(*)$. $\square$

Proof of Theorem (3.8.): "$\Rightarrow$" If $\chi(Z) \geq 1$ for all $Z > 0$, then $1 - h^1(\mathcal{O}_{Z_{min}}) \geq 1$. Thus $h^1(\mathcal{O}_{Z_{min}}) = 0$ and $\chi(Z_{min}) = 1$.

"$\Leftarrow$" Let $\{Z_l\}_l$ be a computation sequence in Laufer's Algorithm (2.10.). Consider the infinite sequence: $Z_1,\dots,Z_l,\dots,Z_{min},Z_{min}+Z_1,\dots,Z_{min}+Z_l,\dots,2Z_{min},\dots,tZ_{min},tZ_{min}+Z_1,\dots,tZ_{min}+Z_l,\dots,(t+1)Z_{min},\dots$.

In the long cohomology exact sequence associated with

$$0 \longrightarrow \mathcal{O}_{E_{j_l}}(-tZ_{min}-Z_l) \longrightarrow \mathcal{O}_{tZ_{min}+Z_{l+1}} \longrightarrow \mathcal{O}_{tZ_{min}+Z_l} \longrightarrow 0$$

the term

$$H^1(\mathcal{O}_{E_{j_l}}(-tZ_{min}-Z_l)) \overset{(3.9)}{=} H^1(\mathcal{O}_{\mathbf{P}^1}(-E_{j_l}(Z_l+tZ_{min}))) \overset{(3.9)}{=} H^1(\mathcal{O}_{\mathbf{P}^1}(-1-tE_{j_l}Z_{min}))=0.$$

Here we used $E_{j_l}\cdot Z_{min}\le 0$, hence the Chern number $-1-tE_{j_l}\cdot Z_{min}\ge -1$. Therefore, by induction, $h^1(\mathcal{O}_{tZ_{min}+Z_l})=0$ for all t and l. But $h^1(\mathcal{O}_{tZ_{min}})=0$ implies $p_g=0$ by Lemma (3.3.). $\square$

3.10. Exercises.

1. Show that the $A-D-E$–singularities are rational (cf. (1.21.) and (1.20.)).
2. Show that if the resolution graph of a singularity is a string of $\mathbf{P}^1$'s then $p_g=0$. This shows that the Hirzebruch–Jung singularities are rational (cf. (1.9.), (1.21.) and (1.20.)).
3. Show that $X=\{x^2+y^3+z^j=0\}$ for $j\ge 6$, defines a singularity $(X,0)$ for which $p_g\neq 0$. (In fact, $p_g=[\frac{j}{6}]$.) [Hint: use Exercises (3.7.) and (2.40.); cf. also with (4.6.) and (4.23.).]

II. The analytic invariants described topologically.

3.11. The Picard group $Pic(\widetilde{X})$. By definition, the Picard group $Pic(\widetilde{X})$ is the group of isomorphism classes of line bundles. Here the group operation is the tensor product of line bundles. It is well-known that $Pic(\widetilde{X})=H^1(\widetilde{X},\mathcal{O}^*_{\tilde{X}})$. Consider the exponential exact sequence $0\longrightarrow \mathbf{Z}_{\tilde{X}}\longrightarrow \mathcal{O}_{\tilde{X}}\xrightarrow{\exp}\mathcal{O}^*_{\tilde{X}}\longrightarrow 0$. This provides

$$H^1(\widetilde{X},\mathbf{Z})\longrightarrow H^1(\mathcal{O}_{\tilde{X}})\longrightarrow H^1(\mathcal{O}^*_{\tilde{X}})\xrightarrow{\delta} H^2(\widetilde{X},\mathbf{Z})\longrightarrow H^2(\mathcal{O}_{\tilde{X}}).$$

By dimension argument (see (4) of Appendix 2) $H^2(\mathcal{O}_{\tilde{X}})=0$. If $p_g=0$, then also $H^1(\mathcal{O}_{\tilde{X}})=0$. Therefore, for rational singularities, $H^1(\mathcal{O}^*_{\tilde{X}})\overset{\delta}{=}H^2(\widetilde{X},\mathbf{Z})=\mathbf{Z}^{\#\{E_i\}_i}$. The isomorphism is given by the Chern-numbers:

$$\mathcal{L}\in H^1(\mathcal{O}^*_{\tilde{X}})\longmapsto \{\text{Chern-number of } \mathcal{L}|_{E_i}\}_i$$

3.12. Corollary. *Assume that $(X,0)$ is rational. Then:*

$$\begin{aligned}\mathcal{O}_{\tilde{X}}(D)\text{ is holomorphically trivial} &\iff \mathcal{O}_{\tilde{X}}(D)\text{ is topologically trivial}\\ &\iff D\cdot E_i=0\text{ for all }E_i.\end{aligned}$$

3.13. Exercise. Prove that if δ is an isomorphism then $p_g = 0$, i.e. the statement of Corollary (3.12.) holds if and only if $(X,0)$ is a rational singularity.

3.14. Theorem. *Fix a resolution π. If $(X,0)$ is rational then:*

(a) $\mathcal{Z}_{an} = \mathcal{Z}_{top}$*;*

(b) $Z_{min} = Z_{max}$ *and* $\pi^*\mathtt{m}_{X,x} = \mathcal{O}(-Z_{max})$*;*

(c) $\pi^*\mathtt{m}^k_{X,x} = \mathcal{O}(-kZ_{max})$*.*

Proof: We have to prove that $\mathcal{Z}_{top} \subset \mathcal{Z}_{an}$. Fix $0 < Z \in \mathcal{Z}_{top}$, so that $Z = \sum n_i E_i$ with all $n_i > 0$ (cf. Exercise (2.8.)).

We need the following definition: a <u>cut</u> associated with a fixed E_i is a (non-compact) divisor C_i of $\widetilde{X}$ such that $C_iE_i = 1$ and $C_iE_j = 0$ for all $j \neq i$ (e.g. a cut is a transversal local analytic slice at a generic point of E_i).

If for the cycle Z chosen above $ZE_i = a_i \leq 0$, then consider the divisor $D \in Div(\widetilde{X})$ given by $D = Z + \sum_i(-a_i)C_i$, where C_i is a cut associated to E_i. Obviously, $DE_i = 0$ for all i (i.e. D is a topological candidate for a principal divisor(!) and $\sum_i(-a_i)C_i$ is the candidate for the strict transform of the corresponding function).

Now, consider the line bundle $\mathcal{L} = \mathcal{O}_{\tilde{X}}(D)$, $\mathcal{L} \in Pic(\widetilde{X})$. Since $DE_i = 0$ for all i, by Corollary (3.12.) $\mathcal{L}$ is trivial, hence D is a principal divisor indeed. Thus, there exists a *meromorphic* function g on $\widetilde{X}$ such that $(g) = D$. But D is effective $(-a_i, n_i \geq 0)$, therefore g has no poles. In other words, g is a *holomorphic* function on $\widetilde{X}$. Moreover, since $Z = \sum n_i E_i$ with all $n_i > 0$, one obtains that $g|_{E_i} = 0$ for all i. Since $X = \widetilde{X}/\cup E_i$, we have that $g = f \circ \pi$ for some *continuous* function $f : X \longrightarrow \mathbf{C}$. But π is an isomorphism above $X - \{0\}$, and this implies that f is holomorphic on $X - \{0\}$. Since f is continuous on X, it is bounded on $X - \{0\}$, therefore the normality of X guarantees that f is holomorphic (cf. Definition (1.3.)). This shows that the cycle $Z = (f)_\Gamma$ for some $f \in \mathtt{m}_{X,0}$, and thus $\mathcal{Z}_{top} \subset \mathcal{Z}_{an}$.

The fact that $\mathcal{Z}_{an} = \mathcal{Z}_{top}$ implies $Z_{min} = Z_{max}$ by the definition of these cycles. Since the choice of cuts C_i was arbitrary, $\pi^*\mathtt{m}_{X,0}$ has no basepoints, and thus $\pi^*\mathtt{m}_{X,0} = \mathcal{O}(-Z_{max})$. (c) follows similarly. $\square$

3.15. Corollary. (Cf. 2.21.) *For a rational singularity $(X,0)$:*

$$mult(X,0) = -Z^2_{min}.$$

In particular, $mult(X,0)$ is a topological invariant!

3.16. For any normal surface singularity $Z_{min} \leq Z_{max}$, therefore the mapping

$$f \in \mathtt{m}_{X,0} \longmapsto f \circ \pi \in H^0(\widetilde{X}, \mathcal{O}_{\tilde{X}}(-Z_{min}))$$

is well-defined and actually it is an isomorphism of $\mathbf{C}$-vector spaces.

Similarly, for any $k \geq 2$, one can define the injective map:

$$f \in \mathtt{m}^k_{X,0} \longmapsto f \circ \pi \in H^0(\widetilde{X}, \mathcal{O}_{\tilde{X}}(-kZ_{min})),$$

but this, in general, is not an isomorphism. However, for rational singularities the following results are valid:

3.17. Theorem. [Artin][1, 2] *For rational singularities the following facts hold:*

(a) $H^0(\widetilde{X}, \mathcal{O}_{\tilde{X}}(-kZ_{min})) \simeq \mathtt{m}^k_{X,0}\ \forall k \geq 1$.

(b) $\dim \mathtt{m}^k_{X,0} / \mathtt{m}^{k+1}_{X,0} = -kZ^2_{min} + 1$.

(c) $f_{HS}(k) = -k(k-1)/2 \cdot Z^2_{min} + k$ *for any* $k \geq 1$ *(which reproves* $mult(X,0) = -Z^2_{min}$*, cf. (1.12.)).*

Proof. We have to prove the inclusion $\mathtt{m}^k_{X,0} \supset H^0(\tilde{X}, \mathcal{O}(-kZ_{min}))$. In the proof we will write for Z_{min} simply Z. Set $\mathcal{L} := \mathcal{O}_Z(-Z)$ in $Pic(\mathcal{O}_Z)$, let $d := deg\,\mathcal{L} = -Z^2$.

Step. 1. Complete Z with some cuts A such that $(Z + A) \cdot E_i = 0$ and $Z + A$ is effective. Then, similarly as above, $(Z + A)$ is the divisor of an analytic function, in particular $\mathcal{O}_{\tilde{X}}(-Z - A) \approx \mathcal{O}_{\tilde{X}}$, which gives $\mathcal{O}_Z(-Z) \approx \mathcal{O}_Z(A)$. This shows also that there exists a section of $\mathcal{L}$ (or of $\mathcal{O}_Z(A)$) whose divisor of zeros is A. Fix such a section s_0. Multiplication by s_0 gives the following exact sequence:

$$0 \to \mathcal{O}_Z \overset{s_0}{\to} \mathcal{O}_Z(A) = \mathcal{L} \to \mathcal{O}_A(A) \to 0.$$

The last term can be simplified as follows. Take another section of $\mathcal{O}_Z(A)$ which is local basis of this line bundle near A. Then

$$0 \to \mathcal{O}_Z \overset{s_0}{\to} \mathcal{L} \overset{/s}{\to} \mathcal{O}_A \to 0,$$

where $/s$ means "devide by s". This provides the cohomology exact sequence:

$$0 \to H^0(\mathcal{O}_Z) \to H^0(\mathcal{L}) \overset{pr}{\to} H^0(\mathcal{O}_A) \to H^1(\mathcal{O}_Z) \to H^1(\mathcal{L}) \to 0.$$

In the above exact sequence $H^0(\mathcal{O}_Z) = \mathbf{C}$, $\mathcal{O}_A$ is an Artinian algebra of dimension d, hence $H^0(\mathcal{O}_A) = \mathcal{O}_A = \mathbf{C}^d$. Moreover, since $(X, 0)$ is rational, $H^1(\mathcal{O}_Z) = 0$, hence $H^1(\mathcal{L}) = 0$ as well. By dimension argument: $h^0(\mathcal{L}) = d + 1$.

Take $s_1, \ldots, s_d \in H^0(\mathcal{L})$ such that $pr(s_0), \ldots, pr(s_d)$ is a base of $H^0(\mathcal{O}_A)$. (Here, by the above notations, $pr(s_i) = [s_i/s]$.) Then $H^0(\mathcal{L})$, as a vector space is $\mathbf{C}\langle s_0, s_1, \ldots, s_d\rangle$.

Now, we can consider the above exact sequence of sheaves tensored by $\mathcal{L}$:

$$0 \to \mathcal{L} \overset{s_0}{\to} \mathcal{L}^2 \overset{/s^2}{\to} \mathcal{O}_A \to 0.$$

Its cohomology exact sequence is:

$$0 \to H^0(\mathcal{L}) \to H^0(\mathcal{L}^2) \to \mathcal{O}_A \to H^1(\mathcal{L}) \to H^1(\mathcal{L}^2) \to 0.$$

Using the vanishing of $H^1(\mathcal{L})$, we obtain $H^1(\mathcal{L}^2)=0$. Moreover, one can verify easily that the sections $s_0s_0, s_0s_1, \dots, s_0s_d, ss_1, \dots, ss_d$ form a base in $H^0(\mathcal{L}^2)$. Similarly, we can construct the exact sequences

$$0 \to \mathcal{L}^k \xrightarrow{s_0} \mathcal{L}^{k+1} \xrightarrow{/s^{k+1}} \mathcal{O}_A \to 0.$$

By induction we verify that $H^1(\mathcal{L}^k)=0$, and we obtain that in $H^0(\mathcal{L}^k)$ a possible base is given by:

$$s_0^k;\ \text{and}\ s_0^{k-l}s^{l-1}s_i\ \text{for}\ l=1,\dots,k\ \text{ and }\ i=1,\dots,d.$$

In particular, we obtain that

$$H^0(\mathcal{L})^{\otimes k} \to H^0(\mathcal{L}^k)\ \text{is onto for any}\ k.$$

Step. 2. Let $\{Z_l\}_l$ be a computation sequence for Z. Recall, that in the proof of Theorem (3.8.) we already verified that $H^0(\mathcal{O}_{kZ+Z_{l+1}}) \to H^0(\mathcal{O}_{kZ+Z_l})$ is onto, and $H^1(\mathcal{O}_{kZ})=0$ for any k. The first fact, by induction, gives that $H^0(\mathcal{O}_{kZ+rZ}) \to H^0(\mathcal{O}_{kZ})$ is onto. And using also the second fact, we obtain that

$$H^1(\mathcal{O}_{rZ}(-kZ))=0,\ \text{for any}\ r\geq 1\ \text{and}\ k\geq 1.$$

Now, by property 1 of Appendix 2, one has:

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-kZ_{min}))=0,\ \text{for any}\ k.$$

Step. 3. Consider the diagram:

$$\begin{array}{ccccc} & & m_{X,0}^{\otimes k} = [H^0(\mathcal{O}_{\tilde{X}}(-Z))]^{\otimes k} & \xrightarrow{(*)} & [H^0(\mathcal{O}_Z(-Z))]^{\otimes k} \\ & & \downarrow & & \downarrow (**) \\ H^0(\mathcal{O}_{\tilde{X}}(-kZ-Z)) & \to & H^0(\mathcal{O}_{\tilde{X}}(-kZ)) & \longrightarrow & H^0(\mathcal{O}_Z(-kZ)) \end{array}$$

The arrow $(*)$ is onto because of the vanishing result (5) of Appendix 2, $(**)$ is onto by Step 1. This diagram shows that $H^0(\mathcal{O}_{\tilde{X}}(-kZ)) \subset m_{X,0}^k + H^0(\mathcal{O}_{\tilde{X}}(-(k+1)Z))$. Now, by induction: $H^0(\mathcal{O}_{\tilde{X}}(-kZ)) \subset m_{X,0}^k + H^0(\mathcal{O}_{\tilde{X}}(-lZ))$ for any $l>k$. Since $\oplus_{k\geq 0} H^0(M, \mathcal{O}_{\tilde{X}}(-kZ))$ is finitely generated [25], (a) follows.

For (b), consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-kZ-Z) \longrightarrow \mathcal{O}_{\tilde{X}}(-kZ) \longrightarrow \mathcal{O}_Z(-kZ) \longrightarrow 0.$$

Step 2 and (a) imply that

$$0 \longrightarrow \mathtt{m}_{X,0}^{k+1} \longrightarrow \mathtt{m}_{X,0}^k \longrightarrow H^0(\mathcal{O}_Z(-kZ)) \longrightarrow 0.$$

But then $\dim \mathtt{m}_{X,0}^k / \mathtt{m}_{X,0}^{k+1} = h^0(\mathcal{O}_Z(-kZ)) \overset{\text{Step 2}}{=} \chi(\mathcal{O}_Z(-kZ)) = -kZ^2+1$.

For (c) notice that $\dim \mathtt{m}_{X,0}^k / \mathtt{m}_{X,0}^{k+1} = f_{\mathrm{HS}}(k+1) - f_{\mathrm{HS}}(k)$. $\square$

3.18. Particular case: the embedding dimension. The above theorem shows that for a rational singularity $(X,0)$ the embedding dimension is $\dim \mathfrak{m}_{X,0}/\mathfrak{m}^2_{X,0} = -Z^2_{min}+1$.

3.19. Examples. Consider a rational singularity $(X,0)$. Then:

1. $Z^2_{min} = -1$ implies, that $(X,0)$ is smooth.

   A nice application of this fact is *Castelnuovo's Contraction Criterion*: If a rational smooth curve E_0 on a smooth surface $\tilde{X}$ has self intersection $E_0^2 = -1$, then it can be contracted, and $\tilde{X}/E_0$ is smooth.

   Indeed, let $(X,0)$ be the singularity $\tilde{X}/E_0$, then one of the resolution graphs contains exactly one vertex corresponding to E_0. By an easy computation $\chi(nE_0) = n(n+1)/2 > 0$, hence $(X,0)$ is rational. But $Z_{min} = E_0$, hence $Z^2_{min} = -1$, hence by the above result $(X,0)$ is smooth.

   In this case even the contraction can be constructed rather explicitly. Fix the resolution with exceptional curve E_0, and construct two generic analytic slices A and B (cuts) with $A \cap B = \emptyset$. Then $E_0 + A = (f)$ and $E_0 + B = (g)$ for some holomorphic maps f and g defined in the neighbourhood U of E_0. Then $(f,g) : (U, E_0) \to (\mathbf{C}^2, 0)$ provides a contraction of E_0.

2. $Z^2_{min} = -2$ implies, that $mult(X,0) = 2$ and $\mathrm{emb}\dim = 3$, in particular $(X,0)$ is a hypersurface singularity. But the graphs of rational Gorenstein singularities are completely classified (cf. (2.40.) and (3.7.)), they are exactly the $A-D-E$ graphs. It is not very difficult to prove that any of these graphs carry a unique analytic structure (cf. (1.20.) and (1.21.)). These singularities are called also Du Val singularities or simple hypersurface singularities.

   Since complete intersections are Gorenstein, the only rational complete intersections are the Du Val singularities.

3. $Z^2_{min} = -3$ implies $mult(X,0) = 3$ and $\mathrm{emb}\dim(X,0) = 4$. This means that $(X,0)$ can be realized in $(\mathbf{C}^4,0)$ as $(X,0) = (\{f_1 = \ldots = f_r = 0\}, 0)$ with f_l having no linear terms. The above theorem gives that $\dim \mathfrak{m}^2_{X,0}/\mathfrak{m}^3_{X,0} = 7$. On the other hand, the vector space

$$\mathfrak{m}^2_{X,0}/\mathfrak{m}^3_{X,0} = \frac{\mathbf{C} < x_i x_j >_{1\leq i,j\leq 4}}{< \text{quadratic parts of } f_l >} = \frac{\mathbf{C}^{10}}{< \text{quadratic parts of } f_l >},$$

   which shows that *at least three* functions (with non-trivial quadratic parts) are needed to define X. This shows again that $(X,0)$ is <u>not</u> a complete intersection.

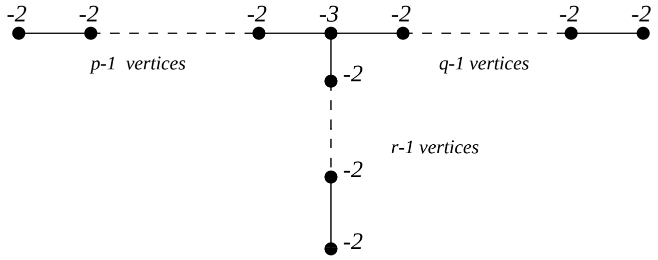
3.20. Exercise. Compute the multiplicity and the embedding dimension of the Hirzebruch–Jung singularities from their minimal resolution graphs (cf. (1.9.) and (1.20.)).

3.21. Example. For a graph Γ let s_v denote the number of edges, with endpoint v. Suppose that Γ is a tree with weights $g_v = 0$ for all vertices and $e_v = \begin{cases} -s_v & \text{if } s_v \neq 1 \\ -2 & \text{if } s_v = 1 \end{cases}$.

Then $Z_{min} = \sum_i E_i$ and $Z_{min}^2 = -\#\{v : \#_v = 1\} < 0$. Hence, the intersection matrix $(E_iE_j)_{i,j}$ is automatically negative definite (cf. Exercise (2.13.)). In particular, Γ is the minimal resolution graph of some normal surface singularity $(X, 0)$. Prove that $(X, 0)$ is rational. This exercise shows also that the multiplicity and the embedded dimension can be arbitrary big.

3.22. Exercise. Consider $(X, 0) \subset (\mathbf{C}^4, 0)$ given by: $\begin{cases} xy + xt^q + t^{p+q} &= 0 \\ yz + yt^r + t^{q+r} &= 0 \\ xz + zt^p + t^{p+r} &= 0 \end{cases}$

Prove that its minimal resolution graph Γ_X is:



3.23. Exercise. Let $(X, 0)$ be a rational singularity with link L_X. For any function $f : (X, 0) \longrightarrow (\mathbf{C}, 0)$, where $f \in \mathtt{m}_{X,0}$, define <u>the link</u> L_f <u>of f</u> by $L_f = \{f = 0\} \cap S_\epsilon^{2N-1}$, where ϵ is as in Definition (1.15.). Thus $L_f \hookrightarrow L_X = X \cap S_\epsilon^{2N-1}$ and L_f is a 1-dimensional, oriented real sub-manifold of (the real 3-dimensional) L_X.

Consider the set of all "algebraic links" of L_X:

$$\mathcal{L}_{an} = \{L_f \subset L_X\}_{f \in \mathtt{m}_{X,0}}.$$

Characterize $\mathcal{L}_{an}$ topologically, i.e. determine $\mathcal{L}_{an}$ only from the topology of L_X! Give an example of a "non-algebraic" link.

4§. "Weakly" Elliptic singularities.

4.1. Review of the definition of the rational singularities. If we fix a resolution graph of an arbitrary singularity, then $h^0(Z_{min}) = 1$, hence $\chi(Z_{min}) \leq 1$. By the results of section 3, $\chi(Z_{min}) = 1$ implies that $(X, 0)$ is rational, which is equivalent with $\chi(Z) \geq 1$ for any $Z > 0$. Therefore:

$$(X, 0) \text{ is rational (i.e. } p_g = 0) \iff \min_{Z>0} \chi(Z) \geq 1 \iff \chi(Z_{min}) = 1.$$

4.2. Now, we assume that $\min_{Z>0} \chi(Z) = 0$. Then $\chi(Z_{min}) \geq 0$; but $\chi(Z_{min}) = 1$ is impossible by (4.1.), hence $\chi(Z_{min}) = 0$. Conversely, if $\chi(Z_{min}) = 0$, then $\min_{Z>0} \chi(Z) = 0$ by a result of Laufer [16], page 1281. Laufer's argument uses the theory of analytic deformations. The next original new proof is purely combinatorial.

4.3. Proposition. *If* $\chi(Z_{min}) = 0$ *then* $\min_{Z>0} \chi(Z) = 0$.

Proof. Set $r := \min_{Z>0} \chi(Z)$, and $\mathcal{Z}_r := \{Z > 0 \mid \chi(Z) = r\}$. Then $\mathcal{Z}_r$ is finite (because the intersection form is negative definite). Hence it has a maximal element (in fact, by a similar proof as in (4.10.) one can verify that it has a unique maximal element, but this is not important here).

If Z_1 is a maximal element in $\mathcal{Z}_r$, then $Z_1 \in \mathcal{Z}_{top}$. Indeed, $\chi(Z_1) < \chi(Z_1 + E_i) = \chi(Z_1) + \chi(E_i) - Z_1 E_i$, hence $Z_1 E_i < \chi(E_i) \leq 1$.

Now notice that the statement is purely combinatorial, it depends only on the graph Γ. Assume that the statement is not true for some graph. Let Γ be a minimal graph with this property: (a) $\chi(Z_{min}(\Gamma)) = 0$ (b) $r(\Gamma) < 0$, and (c) for any proper subgaph $\Gamma' \subset \Gamma$ with $\chi(Z_{min}(\Gamma')) = 0$ the equality $r(\Gamma') = 0$ holds.

Take an arbitrary element L of $\mathcal{Z}_{top} \cap \mathcal{Z}_r$, where $r = r(\Gamma)$. By the above discussion this is possible. Since $L \in \mathcal{Z}_{top}$, by the definition of the fundamental cycle: $L \geq Z_{min}(\Gamma)$. Obviously (by (a) and (b)), $L \neq Z_{min}(\Gamma)$. Set $L' := Z_{min}(\Gamma) - L$. Then $r = \chi(L) = \chi(L') + \chi(Z_{min}(\Gamma)) - L' \cdot Z_{min}(\Gamma)$. Hence $\chi(L') = L' \cdot Z_{min}(\Gamma) + r \leq r$. On the other hand, $\chi(L') \geq r$ by the definition of r, hence we must have $\chi(L') = r$ and $L' \cdot Z_{min}(\Gamma) = 0$. Now, from the definition of the fundemantal cycle, and from the fact that the intersection matrix is non–degenerate, we deduce that the support Γ' of L' is strict smaller than Γ.

Now, we verify that Γ' satisfy the conditions (a) and (b), which leads to a contradiction. Indeed, since $Z_{min}(\Gamma') \leq Z_{min}(\Gamma)$ we have $h^1(Z_{min}(\Gamma')) \leq h^1(Z_{min}(\Gamma)) = 1$. But $h^1(Z_{min}(\Gamma'))$ cannot be zero, since this would imply that Γ' supports a rational singularities, which contradicts $\chi(L') = r \leq 0$. Therefore, $\chi(Z_{min}(\Gamma')) = 0$.

Moreover, $r(\Gamma') \geq r(\Gamma)$, in general; but $\chi(L') = r(\Gamma)$; therefore, in fact $r(\Gamma') = r(\Gamma) < 0$. But this is in contradiction with the minimality condition (c). □

So, it is natural to try to classify those normal surface singularities which are defined via the above properties. They are the elliptic singularities. Ph. Wagreich introduced them by $\min_{Z>0} \chi(Z) = 0$, H. Laufer by $\chi(Z_{min}) = 0$.

4.4. Definition. Fix a resolution of a normal surface singularity. The singularity is called *elliptic*

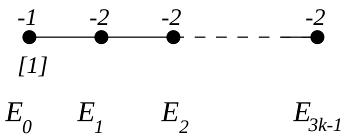
$$\text{if } \min_{Z>0} \chi(Z) = 0, \quad \text{or equivalently, if } \ \chi(Z_{min}) = 0.$$

The definition is independent of the choice of the reolution.

4.5. Exercise. Prove that for any surface singularity $p_g \geq 1 - \min_{Z>0} \chi(Z)$.

The above exercise and the results of section 3 show that elliptic singularities include all the singularities with $p_g = 1$. It is possible to prove (cf. Exercise 4.18.) that they include also all the Gorenstein singularities with $p_g = 2$. Moreover, an elliptic singularity can have arbitrary big geometric genus. (In the literature we find the following terminology: elliptic=“weakly” elliptic; $\{p_g = 1\}$=“strongly” elliptic. By this terminology we emphasize the difference between elliptic singularities and those with $p_g = 1$.)

4.6. Example. Let $X_k = \{x^2 + y^3 + z^{18k} = 0\}$ with $n \geq 1$. The minimal resolution graph of the singularity $(X_k, 0)$ has $3k$ vertices



$Z_{min} = \sum_{i=0}^{3k-1} E_i$ and $Z_K = \sum_{i=0}^{3k-1} (3k-i)E_i$ so that $\chi(Z_{min}) = 0$ for all k. Thus $(X_k, 0)$ are elliptic for all k.

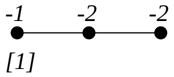
The computation of the geometric genus of this/any example, using our definition $p_g = h^1(\mathcal{O}_{\tilde{X}})$, can be very difficult. For hypersuface singularities we have a different, very convenient method. Let $\mu(f)$ be the Milnor number of the hypersurface isolated singularity $(X, 0) = (\{f(x, y, z) = 0\}, 0)$. This is by definition the dimension of the Milnor algebra $\mathbf{C}\{x, y, z\}/$(the ideal generated by the partial derivatives of f). Fix any resolution of $(X, 0)$, and in the resolution graph compute the “characteristic number” $Z_K^2 + \#\{E_i\}$ (where $\#\{E_i\}$ is the number of irreducible exceptional divisors $\{E_i\}_i$). Then Laufer proved the following relation between these invariants [17]:

$$\mu(f) = 12p_g - rank\, H_1(L_X, \mathbf{Z}) + Z_K^2 + \#\{E_i\}.$$

Recall that $rank\, H_1(L_X)$ can be easily computed from the graph, it is $c_\Gamma + 2\sum_i g_i$ (cf. 1.27.).

Now, we invite the reader to verify that for the above singularity the Milnor number is $\mu = 2(18k - 1)$ and $p_g = 3k$. (Cf. also with 4.23.)

4.7. Exercise/Example. (Cf. Exercise 2.15.) Consider $(X_1, 0) = \{x^2 + y^3 + z^{18} = 0\}$ and $(X_2, 0) = \{z^2 + y(x^4 + y^6) = 0\}$. Their minimal resolution graphs are the same. $\Gamma_{X_1} = \Gamma_{X_2}$:



Using the graph one can easily verify that both singularities are elliptic. On the other hand, using Laufer's formula presented in (4.6.), one can verify that $p_g(X_1) = 3$ and $p_g(X_2) = 2$. In particular, even for hypersurface singularities, we cannot expect that the topology determines the geometric genus.

4.8. Another warning. Actually, with the same graph as in (4.7.), there is another singularity with $p_g = 1$ (but this is not a hypersurface singularity). More generally, by another result of Laufer [16], if we start with an elliptic singularity, even with very high geometric genus, if we deform its analytic structure (but we keep its topological type), for the generic analytic structure $p_g = 1$. (This generic analytic structure, in general, is not Gorenstein.) This shows, that for singularities with $p_g \geq 1$, the geometric genus cannot be characterized topologically. The previous example shows, that even if we restrict ourselves to the case of hypersurface singularities, the geometric genus of singularities with $p_g \geq 2$ cannot be characterized topologically.

However, Laufer characterized a subclass of elliptic singularities topologically, and proved for them that their geometric genus and Hilbert–Samuel function can be determined from the minimal resolution graph. These singularities are the *minimally elliptic singularities*, and they will be discussed later.

I. The minimally elliptic cycle.

4.9. Convention. In the sequel we will assume that our resolution is minimal. Some of the results are true also for an arbitrary resolution, but the minimal resolution has a big advantage: in that case $E_i Z_K \leq 0$ for any i (cf. 2.31.). Therefore, in the numerical Gorenstein case Z_K is either zero, or it is an effective cycle because of (2.41.). This shows also that $Z_K \geq Z_{min}$.

Before we present the characterization of minimally elliptic singularities, we need some preparations.

4.10. Definition. [Laufer] [16] A nonzero effective cycle M is called minimally elliptic if $\chi(M) = 0$ and for any $0 < D < M$ one has $\chi(D) > 0$.

If $\chi(Z_{min}) = 0$ then obviously a minimally elliptic cycle exists: take any minimal element of the set $\{Z > 0 \ : \ \chi(Z) = 0\}$. The point is that if $\chi(Z_{min}) = 0$, then there is exactly one minimal element of this set!

To see this, consider first an arbitrary cycle $A > 0$ with $\chi(A) = 0$, and take $M = \min(A, Z_{min})$. Since $|Z_{min}| = E$ the cycle M is non–zero, and obviously $M \leq A$ and $M \leq Z_{min}$.

Now, (4.3.) applied for $A + Z_{min} - M$ gives: $0 \leq \chi(A + Z_{min} - M) = \chi(A) + \chi(Z_{min}) - \chi(M) - (Z_{min} - M)(A - M) \leq -\chi(M)$; but $\chi(M) \geq 0$ (again by 4.3.), hence $\chi(M) = 0$. Hence all the minimal elements C of $\{Z > 0 : \ \chi(Z) = 0\}$ satisfy $C \leq Z_{min}$.

Now, take two cycles $0 < A, B \leq Z_{min}$ with $\chi(A) = \chi(B) = 0$. Take $M := \min\{A, B\}$. Then we prove that $M \neq 0$ and $\chi(M) = 0$.

Indeed, assume that $M = 0$. Then $AB \geq 0$. This and (4.3.) give $0 \leq \chi(A+B) = \chi(A) + \chi(B) - AB \leq 0$. This shows that $AB = 0$, hence $|A| \cap |B| = \emptyset$ and $h^1(\mathcal{O}_{A+B}) = h^1(\mathcal{O}_A) + h^1(\mathcal{O}_B) \geq 2$. But $A + B \leq Z_{min}$, hence $h^1(A+B) \leq h^1(Z_{min}) = 1$, which leads to a contradiction.

The proof of $\chi(M) = 0$ is similar as above: $0 \leq \chi(A + B - M) = \chi(A) + \chi(B) - \chi(M) - (B - M)(A - M) \leq -\chi(M) \leq 0$, hence $\chi(M) = 0$. So, we proved:

4.11. Proposition. [Laufer] [16] *If $\chi(Z_{min}) = 0$ then there is a unique minimally elliptic cycle. In the sequel will be denoted by C. In the case of a minimal resolution: $C \leq Z_{min} \leq Z_K$.*

4.12. Exercise. (a) If $\chi(Z_{min}) = 0$, then there exist at most one irreducible exceptional divisor with $\delta_i + g_i > 0$. Actually, in this case $\delta_i + g_i = 1$.

(b) Consider an arbitrary singularity. If M is a minimally elliptic cycle, then there exist at most one irreducible exceptional divisor $E_i \subset |M|$ with $\delta_i + g_i > 0$. Actually, in this case $\delta_i + g_i = 1$, and $M = E_i$.

4.13. Lemma. [16] *Consider an arbitrary singularity. If M is a minimally elliptic cycle then M is the canonical cycle on its support $|M|$.*

Proof. If there is an $E_i \subset |M|$ with $\delta_i + g_i > 0$, then using (4.12.) the result is clear. Now assume that all the irreducible exceptional curves are smooth and rational. Take an arbitrary $E_i \subset |M|$. Then $\chi(M - E_i) > 0$, hence $0 = \chi(M) = \chi(M - E_i) + \chi(E_i) - (M - E_i)E_i$, therefore $E_i(M - E_i) \geq 2$, which is equivalent with $E_iM \geq -E_iK$, or $E_i(M + K) \geq 0$. By summation $-2\chi(M) \geq 0$, but we already know that $\chi(M) = 0$. Hence all the inequalities must be equalities: $E_i(M + K) = 0$. □

II. Minimally elliptic singularities.

Now we can formulate Laufer's characterization of minimally elliptic singularities:

4.14. Theorem. [Laufer] [16] *Consider the <u>minimal</u> resolution $\pi : \widetilde{X} \longrightarrow X$ of the singularity $(X, 0)$. Then the following five statements are equivalent. If a singularity satisfies them, it is called <u>minimally elliptic.</u>*

1. *$(X, 0)$ is numerical Gorenstein and $Z_K = Z_{min}$.*
2. *$\chi(Z_{min}) = 0$ and any proper subgraph of $\Gamma_{\tilde{X}}$ supports a rational singularity.*
3. *$(X, 0)$ is numerical Gorenstein and Z_K is a minimally elliptic cycle*
4. *Z_{min} is a minimally elliptic cycle.*

5. $p_g = 1$ *and* $(X, 0)$ *is Gorenstein.*

Therefore, for minimally elliptic singularities $Z_{min} = Z_K = C$.

Notice that this theorem gives a topological characterization of Gorenstein singularities with $p_g = 1$ (The first four conditions are topological. The second condition provides the name of the "family".)

Proof. Using Exercise (4.12.), or similar arguments, one can verify that if any of the above conditions is satisfied, and there is an irreducible exceptional divisor E_i with $\delta_i + g_i > 0$, then in fact $\delta_i + g_i = 1$, and $E = E_i = Z_{min} = Z_K$. In the sequel we will assume that all E_i's are smooth and rational, and we invite the reader to adopt the next proof for the singular case.
"$1 \Rightarrow 2$" Since $\chi(Z_K) = 0$ by Riemann–Roch, we obtain $\chi(Z_{min}) = 0$. Now, consider a proper subgraph $S \subset \Gamma$, fix a computation sequence of $Z_{min}(S)$, and complete to a computation sequence of Z_{min}. Since $1 = h^1(Z_{min}) = \sum(E_{i_l}Z_l - 1)$, there is exactly one index l for which $E_{i_l}Z_l = 2$, otherwise $E_{i_l}Z_l = 1$. We show that $E_{i_l}Z_l = 2$ exactly at the very last step. Indeed, $E_iZ_{min} = E_iZ_K = E_i^2 + 2$ implies that $(Z_{min} - E_i)E_i = 2$, hence at the last step this Chern number is 2. Since only one Chern number is 2, in the previous steps we have no contributions in $\sum(E_{i_l}Z_l - 1)$. Since in the computation of $\chi(Z_{min}(S))$ we need some of the previous steps only , the Euler characteristic $\chi(Z_{min}(S)) = 0$. Hence S supports a rational singularity by (4.1.).
"$2 \Rightarrow 3$" Since $\chi(Z_{min}) = 0$, the minimally elliptic cycle C exists (cf. 4.11.). If $|C| < \Gamma$ then C supports a rational singularity which is in contradiction with $\chi(C) = 0$. Therefore the support of C is the whole E. Then $C = Z_K$ by (4.13.).
"$3 \Rightarrow 4$" Since the resolution is minimal $Z_{min} \leq Z_K$ (cf. 4.9.). If $Z_{min} < Z_K$, by the definition of the minimally elliptic cycle, $\chi(Z_{min}) > 0$. This implies that the singularity is rational which contradicts $\chi(Z_K) = 0$. Therefore $Z_{min} = Z_K$.
"$4 \Rightarrow 1$" follows from (4.13.).

So, we proved $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$, and if any of these conditions is valid then $Z_{min} = Z_K = C$.
"$1, 2, 3, 4 \Rightarrow 5$"

First notice that $p_g = h^1(Z_K)$ by (2.46.). Therefore, $p_g = h^1(Z_K) = h^1(Z_{min}) = 1$.

In order to prove that $(X, 0)$ is Gorenstein we have to verify that $\mathcal{O}_{\tilde{X}}(K + Z_K) \approx \mathcal{O}_{\tilde{X}}$. In the proof of this fact we will follow the paper of M. Reid [25], from where we borrowed the next three lemmas.

From the definition of the cycle Z_K it is clear that $\mathcal{O}_{\tilde{X}}(K + Z_K)$ is numerically trivial, we have to prove that it is analytically trivial as well. First we will prove that $\mathcal{L} := \mathcal{O}_{Z_K}(K + Z_K)$ is trivial. This will be verified by the next lemma. Here we will use the following terminology: an effective cycle D is k-connected, if for any $0 < A < D$ one has $A(D - A) \geq k$.

4.15. Lemma. *Fix a numerically trivial line bundle* $\mathcal{L}$ *in* $Pic^0(D)$ *(i.e. all the Chern numbers* $\mathcal{L}E_i$ *are vanishing). Assume that* D *is 1-connected, and* $h^0(\mathcal{L}) > 0$. *Then* $\mathcal{L}$ *is* *analytically trivial.*

Proof. Fix a nonzero global section s of $\mathcal{L}$, and let A be the maximal cycle where s is vanishing. Write $B = D - A$. From the exact sequence $0 \to \mathcal{L} \otimes \mathcal{O}_B(-A) \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_A \to 0$, one gets that s is a global section of $\mathcal{L} \otimes \mathcal{O}_B(-A)$, and it generates the fibers. The section s restricted on each $E_i \subset |B|$ is non-zero, hence $E_i\mathcal{L} + E_i(-A) \geq 0$, which gives $AB \leq 0$. By the connectedness of D this can happen only if $A = 0$ ($A \neq D$ since $s \neq 0$). $A = 0$ means that s generically generates the fibers of $\mathcal{L}$, but the restriction of s to any E_i is a non-zero constant, hence s is never vanishing, and gives a trivialization of $\mathcal{L}$. $\square$

Now, we want to apply this lemma for our line bundle $\mathcal{L} := \mathcal{O}_{Z_K}(K + Z_K)$. For this we need:

4.16. Lemma. *If Z_K is a minimally elliptic cycle then it is 2-connected, (hence 1-connected too).*

Proof. Indeed, for any $0 < D < Z_K$, by the definition of the minimally elliptic cycle: $\chi(D) \geq 1$, but this via Riemann-Roch is $D(Z_K - D) \geq 2$. $\square$

By (4.15.) and (4.16.) we deduce that $\mathcal{L}$ is trivial. Now, by the Kodaira type vanishing theorem (cf. Appendix 2) $H^0(\mathcal{O}_{\tilde{X}}(K + Z_K)) \to H^0(\mathcal{L}) = H^0(\mathcal{O}_{Z_K})$ is onto, which shows that there is a section s in $H^0(\mathcal{O}_{\tilde{X}}(K + Z_K))$ which is a nonzero constant in $H^0(\mathcal{O}_{Z_K})$, therefore $K + Z_K$ is a principal divisor (of s).
"$5 \Rightarrow 3$" A Gorenstein singularity is obviously numerical Gorenstein, so we have to prove only that Z_K is a minimally elliptic cycle. By Riemann-Roch $\chi(Z_K) = 0$. So, the only remaining part is that for any $0 < D < Z_K$ the inequality $\chi(D) > 0$ folds. Obviously, it is enough to prove that $h^1(\mathcal{O}_D) = 0$. First notice that since $p_g = 1$, by (2.46.) $h^1(Z_K) = 1$. Now, the result follows from the next lemma:

4.17. Lemma. *In the minimal resolution of a Gorenstein singularity, for any $0 < D < Z_K$ the strict inequality $h^1(\mathcal{O}_D) < h^1(\mathcal{O}_{Z_K})$ holds.*

Proof. By Serre duality, it is enough to prove that $h^0(\mathcal{O}_D(-(Z_K - D))) < h^0(\mathcal{O}_{Z_K})$. Consider the exact sequence: $0 \to \mathcal{O}_D(-(Z_K - D)) \to \mathcal{O}_{Z_K} \to \mathcal{O}_{Z_K - D} \to 0$. Then in the cohomology exact sequence $H^0(\mathcal{O}_{Z_K}) \to H^0(\mathcal{O}_{Z_K - D})$ is non-trivial (because both contain the constant sections), hence the result follows. $\square$

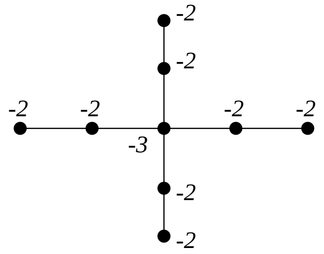
4.18. Exercise. Using Lemma 4.17. for $D = Z_{min}$, prove that any Gorenstein singularity with $p_g = 2$ is elliptic.

4.19. Example. Consider the hypersurface singularity $(\{x^3 + y^3 + z^3 = 0\}, 0)$.
Its minimal resolution graph $\Gamma_{\tilde{X}}$ is $\overset{[1]}{\underset{-3}{\bullet}}$. Thus its link L_X is an S^1-bundle over the torus with Euler-number -3.

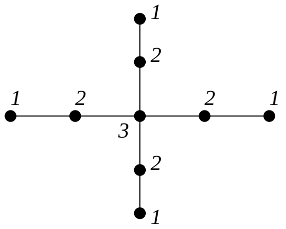
By Laufer's Algorithm $Z_{min} = E$, and by the adjunction formula $Z_K = E$ as well. Thus it is minimally elliptic.

4.20. Example. Consider the hypersurface singularity $(\{x^3 + y^3 + z^4 = 0\}, 0)$.

Its minimal resolution graph $\Gamma_{\tilde{X}}$ is:

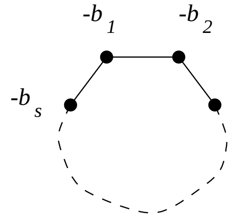


By Laufer's Algorithm and the adjunction formula $Z_K = Z_{min} = \sum n_i E_i$, where the coefficients n_i are shown as weights:



Since $Z_K = Z_{min}$, Theorem 4.2. implies that $p_g = 1$.

4.21. Cusp Singularities. Cusp singularities are by definition singularities whose minimal resolution graph is of the form

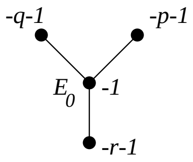


with $b_i \geq 2$ for all $i = 1, 2, ..., s$ $(s \geq 3)$ and negative definite intersection matrix $(E_i E_j)_{i,j}$ (or equivalently, $\sum_i (b_i - 2) > 0$).

In this case L_X is a torus bundle over the circle S^1 with monodromy (written multiplicatively) $\begin{pmatrix} b_s & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix}$. ([24])

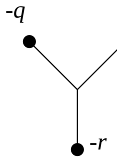
Again $Z_{min} = Z_K = \sum E_i$, which means that the cusp singularities are minimally elliptic. Notice that all proper subgraphs support Hirzebruch-Jung singularities, which are rational.

4.22. Triangle Singularities. (Cf. 1.22.) A good resolution of a triangle singularity is of the form:



, with genus $g_i = 0$ for all i.

Laufer's Algorithm yields $Z_{min} = E_1+E_2+E_3+3E_0$ and the adjunction formula gives $Z_K = E_1 + E_2 + E_3 + 2E_0$, so that $Z_{min} \neq Z_K$ for this resolution, although $(X,0)$ is a minimally elliptic singularity. To see this, consider the minimal resolution graph:



where the graph corresponds to the intersection matrix: $(E_iE_j)_{i,j} = \begin{pmatrix} -p & 1 & 1 \\ 1 & -q & 1 \\ 1 & 1 & -r \end{pmatrix}$.
Notice that for the minimal resolution graph $Z_K = Z_{min}(=\sum E_i)$ and $\chi(Z_{min}) = 0$.

4.23. Remark. In some cases we can compute the geometric genus by some different methods as well. For example, the geometric genus of a Brieskorn singularity $(\{x^a + y^b + z^c = 0\}, 0) \subset (\mathbf{C}^3, 0)$ is given by $p_g = \#\{(i,j,k) | i,j,k > 0,\ \frac{i}{a}+\frac{j}{b}+\frac{k}{c} \leq 1\}$. In particular, a Brieskorn singularity is rational, if and only if, $\frac{i}{a}+\frac{j}{b}+\frac{k}{c} > 1$. The only possible values (a,b,c) are: $(2,2,n)$ (A_{n-1}), $(2,3,3)$ (D_4), $(2,3,4)$ (E_6) and $(2,3,5)$ (E_8).

Now, in some of the above examples we can compute p_g also by this formula. It is somehow mysterious, that this formula gives $p_g = 1$ if and only if $Z_{min} = Z_K$ in the minimal resolution graph!

4.24. Remark. Laufer in [16] computed the Hilbert–Samuel function of a minimally elliptic singularity in terms of Z_{min} (defined in the minimal resolution graph). This result will be presented in the next section in a more general context.

The facts proved in the following exercises will be helpful in those discussions. All the facts condensed in these exercises are contained in Laufer's paper [16].

In all the following exercises we assume that $(X,0)$ is a minimally elliptic singularity, $\pi : \tilde{X} \to X$ is a minimal resolution, and all the irreducible exceptional divisors are smooth and rational.

4.25. Exercise. Let $\{Z_l\}_{l=1}^k$ be a computation sequence for Z_{min} (i.e. $Z_k = Z_{min}$). Then $\chi(Z_l) = 1$ for all $l < k$; $Z_l \cdot E_{j_l} = 1$ for all $l < k-1$; and in the last step: $Z_{k-1} \cdot E_{j_{k-1}} = 2$. (Cf. the proof of 4.14.)

4.26. Exercise. Fix any pair E_0 and E_1 $(E_0 \neq E_1)$ of irreducible exceptional divisors. Show that there exists a computation sequence for Z_{min} which starts with E_0 (i.e. $Z_1 = E_0$) and ends with E_1 (i.e. $E_{j_{k-1}} = E_1$).
[Hint: Start a computation sequence with $Z_1 = E_0$. Suppose that E_1 may not be chosen to be the last. Then the computation sequence reaches a cycle D when we are forced to put the last E_1, but still $D + E_1 \neq Z_{min}$. This means that D has the following properties: $D + E_1 < Z_{min}$, $DE_i \leq 0$ for all $i \neq 1$, and $E_1 \not\subset |Z_{min} - D - E_1|$.

By (4.25.), $\chi(D+E_1)=1$ and $DE_1=1$ which imply that $\chi(D)=1$.

Since $Z_{min}-D<Z_{min}$ and Z_{min} is a minimally elliptic cycle, one has $\chi(Z_{min}-D)\geq 1$. Therefore $\chi(Z_{min})=\chi(Z_{min}-D+D)=\chi(Z_{min}-D)+\chi(D)-(Z_{min}-D)D\geq 2-(Z_{min}-D)D=1-(Z_{min}-D-E_1)D$. Since $E_iD\leq 0$ for $i\neq 1$, and $E_1\not\subset|Z_{min}-D-E_1|$, one has $(Z_{min}-D-E_1)D\leq 0$. This together with the above inequality gives that $\chi(Z_{min})\geq 1$ which is certainly a contradiction.]

4.27. Exercise. Assume that $E_0Z_{min}=0$ then $h^1(\tilde{X},\mathcal{O}_{\tilde{X}}(-Z_{min}-E_0))=0$.
[Hint: Fix E_1 with $E_1Z_{min}<0$. Then consider a computation sequence $\{Z_l\}_l$ which starts with E_0 and ends with E_1. Construct the sequnece $E_0,\ldots,Z_l,\ldots,Z_{min},Z_{min}+E_0,Z_{min}+Z_l,\ldots,2Z_{min},\ldots$ as in the proof of (3.8.). Then using the exact sequneces (for $t\geq 1$):

$$0\to\mathcal{O}(-tZ_{min}-Z_{l+1})\to\mathcal{O}(-tZ_{min}-Z_l)\to\mathcal{O}_{E_{j_l}}(-tZ_{min}-Z_l)\to 0$$

one can verify that the Chern number $E_{j_l}(-tZ_{min}-Z_l)\geq -1$, hence by induction, the natural map $\alpha: H^1(\mathcal{O}(-tZ_{min}-E_0))\to H^1(\mathcal{O}(-Z_{min}-E_0))$ is onto. But for t sufficiently large α is the trivial map (by (1) of Appendix 2).]

4.28. Exercise.

(a) Assume that $E_0Z_{min}=0$. Then $H^0(\mathcal{O}_{\tilde{X}}(-Z_{min}))\to H^0(\mathcal{O}_{E_0}(-Z_{min}))=\mathbf{C}$ is onto.

(b) There exists $f\in H^0(\tilde{X},\mathcal{O}_{\tilde{X}}(-Z_{min}))$ such that the vanishing order of f along E_0 is exactly the multiplicity of E_0 in Z_{min} for any E_0 with $E_0Z_{min}=0$.

5§. "Weakly" Elliptic Gorenstein singularities.

In section 4, we noticed that singularities with $p_g = 1$ (without Gorenstein assumption), or even Gorenstein singularities with $p_g = 2$ (or $p_g \geq 2$) cannot be characterized topologically. In this second case, one can easily construct pairs of hypersurface singularities with the same resolution graph but different p_g (cf. 4.7.). But in all these examples (known by the author) either there is a non-rational exceptional divisor in the resolution, or the graph is not a tree, i.e. $H^1(E, \mathbf{Z}) \neq 0$. (Notice that $H^1(E, \mathbf{Z}) = 0$ if and only if the link of $(X, 0)$ is a rational homology sphere.)

In this section we show that for elliptic Gorenstein singularities with $H^1(E, \mathbf{Z}) = 0$ *the Artin–Laufer program can be continued.*

The definition of the elliptic sequence and some of its properties (namely (5.5.a,b,c), (5.10.), (5.11. a,b,c,d)) are taken from the papers of S. S.-T. Yau. All the other results of this section are original, obtained by the author; a part of them will be published in [23]. In the subsections IV and V the author was substantially influenced by the Park City notes of M. Reid [25].

I. The elliptic sequence.

5.1. Recall that the elliptic singularities were introduced by

$$(X, 0) \text{ is elliptic} \Leftrightarrow \chi(Z_{min}) = 0 \Leftrightarrow \min_{D>0} \chi(D) = 0.$$

This class contains all the singularities with $p_g = 1$, and all the Gorenstein singularities with $p_g = 2$; but an elliptic singularity can have arbitrary high geometric genus.

5.2. We start our discussion with a numerical Gorenstein elliptic singularity. We consider its *minimal* resolution, Z_{min} denotes its minimal cycle, Z_K its canonical cycle, and C is its minimally elliptic cycle. Then, as we already know (4.11.):

$$C \leq Z_{min} \leq Z_K.$$

5.3. If $(X, 0)$ is elliptic, then the topology of the irreducible exceptional divisors E_i cannot be very complicated (cf. 4.12.). Actually, precisely one of the followings hold: (a) Precisely one component E_{i_0} satisfies $\chi(E_{i_0}) = 0$ (which is either a smooth elliptic curve or a rational curve with $\delta_{i_0} = 1$) and the other irreducible exceptional divisors are smooth rational curves. (In this case $C = E_{i_0}$). Or (b): all the exceptional divisors are smooth rational curves.

Actually, from the uniqueness of the minimally elliptic cycle it follows that all the connected components of $E \setminus |C|$ support the exceptional set of rational singularities, hence the dual graph of $(X, 0)$ can be obtained from the dual graph of $|C|$ by gluing trees whose vertices correspond to smooth rational curves. In particular, the restriction map

$$H^1(E, \mathbf{Z}) \to H^1(|C|, \mathbf{Z}) \text{ is an isomorphism.}$$

5.4. The construction of the elliptic sequence. [S. S.-T. Yau] Consider the minimal resolution of an elliptic numerical Gorenstein singularity. The elliptic sequence consists of the sequence $\{Z_{B_j}\}_{j=0}^m$, where Z_{B_j} is the minimal cycle of $B_j \subset E$. We define $\{B_j\}_j$ inductively as follows. For $j=0$ take $B_0 = E$ hence $Z_{B_0} = Z_{min}$. By (5.2.): $C \leq Z_{min} \leq Z_K$.

If $Z_{B_0} < Z_K$, we define $B_1 := |Z_K - Z_{min}|$. Then B_1 is connected. Indeed, assume that B_1 has more connected components $\{B_{1,t}\}_t$; then write $Z_K - Z_{min} = \sum_t D_t$ with $|D_t| = B_{1,t}$. Then the vanishing $\chi(Z_K - Z_{min}) = 0$ and (5.1.) imply that $\chi(D_t) = 0$ for all t. Then each $B_{1,t}$ supports a minimally elliptic cycle. This contradicts the uniqueness of the minimally elliptic cycle (4.11.). Now, using again $\chi(Z_{min}) = 0$, one gets $Z_{min}(Z_K - Z_{min}) = 0$. Since $Z_{min}E_i \leq 0$ for all E_i, it follows that for any $E_i \subset B_1$ the equality $Z_{min}E_i = 0$ holds. In particular, by the non-degeneracy of the intersection form, $B_1 \neq B_0$. Moreover, for any $E_i \subset B_1$ one has: $E_i(Z_K - Z_{B_0}) = E_iZ_K$, hence B_1 supports a numerical Gorenstein singularity with canonical cycle $Z_K - Z_{B_0}$. This singularity is again elliptic. Indeed, $\min_{D\subset B_1}\chi(D) \geq \min_{D\subset B_0}\chi(D) = 0$, but $\chi(Z_K - Z_{min}) = 0$, hence $\min_{D\subset B_1}\chi(D) = 0$. In particular, (cf. 5.2.), $C \leq Z_{B_1} \leq Z_K - Z_{B_0}$.

Now we repeat the above arguments. If $Z_{B_1} < Z_K - Z_{B_0}$, we define $B_2 := |Z_K - Z_{B_0} - Z_{B_1}|$ and we verify that it supports an elliptic singularity with canonical cycle $Z_K - Z_{B_0} - Z_{B_1}$ and for any $E_i \subset B_2$ the vanishing $E_iZ_{B_1} = 0$ holds. If we repeat this procedure, after a finite step we will obtain $Z_{B_m} = Z_K - Z_{B_0} - \cdots - Z_{B_{m-1}}$, i.e. the numerical cycle and the canonical cycle of B_m coincides, hence by (4.14.) B_m supports a minimally elliptic singularity with $C = Z_{B_m}$. In particular $B_m = |C|$.

The length of the elliptic sequence $\{Z_{B_j}\}_{j=0}^m$ is $m+1$.

If we contract the connected exceptional divisor $B_j \subset \tilde{X}$, we obtain a unique singular point; this will be denoted by $(\tilde{X}/B_j, p_j)$ $(0 \leq j \leq m)$. It is convenient to introduce the notations $C_t = \sum_{i=0}^t Z_{B_i}$ and $C'_t = \sum_{i=t}^m Z_{B_i}$ $(0 \leq t \leq m)$ too. By these notations, $C_0 = Z_{min}$, $C'_m = C$, and $C_m = C'_0 = Z_K$.

5.5. Definition/first properties of the elliptic sequence. *(a)* $B_0 = E$, $B_1 = |Z_K - Z_{B_0}|$, $B_2 = |Z_K - Z_{B_0} - Z_{B_1}|$, $\ldots$, $B_m = |C|$; *each* B_j *is connected and the inclusions* $B_{j+1} \subset B_j$ *are strict. Moreover,* $Z_{min} = Z_{B_0} \supset Z_{B_1} \supset \cdots \supset Z_{B_m} = C$.

(b) If $E_i \subset B_{j+1}$ *then* $E_iZ_{B_j} = 0$ *for all* i *and* j. *In particular,* $Z_{B_i}Z_{B_j} = 0$ *for all* $0 \leq i < j \leq m$.

(c) $Z_K = \sum_{i=0}^m Z_{B_i}$.

(d) $E_iC'_t = E_iZ_K$ *for any* $E_i \subset |C'_t|$. *In other words,* C'_t *is the canonical cycle of* $|C'_t| = B_t$ *(i.e. of* $(\tilde{X}/B_t, p_t)$*).*

(e) For any $E_i \subset E$ *the inequality* $E_i \cdot C_t \leq 0$ *holds.*

Proof. a, b, c, d follows from the above construction. The proof of (e) is as follows. If $E_i \subset B_t$ then $E_iZ_{B_j} \leq 0$ for any $j \leq t$ by the definition of the minimal cycle, hence $E_iC_t \leq 0$. If $E_i \not\subset B_t$ then $E_iC_t = E_i(Z_K - C'_{t+1})$. Now, $E_iZ_K \leq 0$ (by the minimality of the resolution) and $E_iC'_{t+1} \geq 0$ (because $|C'_{t+1}| \subset B_t$). □

The next result shows that the properties (d) and (e) characterize the elliptic sequence.

5.6. Lemma. *Assume that $(X,0)$ is a numerical Gorenstein elliptic singularity.*

(a) If a cycle $Z\geq 0$ satisfies $E_i(Z-Z_K)\geq 0$ for all $E_i\subset |Z|$, then in fact $E_i(Z-Z_K)=0$ for all $E_i\subset |Z|$, and $Z\in\{C_0',C_1',\dots,C_m',0\}$.

(b) If a cycle $0\leq Z\leq Z_K$ satisfies $E_iZ\leq 0$ for all $E_i\subset E$, then $Z\in\{0,C_0,C_1,\dots,C_m\}$.

Proof. If $E_i(Z-Z_K)\geq 0$, then $Z(Z-Z_K)\geq 0$, hence $\chi(Z)\leq 0$. This together with (5.1.) guarantees that $\chi(Z)=0$ and $E_i(Z-Z_K)=0$ for all $E_i\subset |Z|$. Now, if $|Z|=B_0$, then by the non–degeneracy of the intersection form $Z=Z_K$. If $|Z|\neq B_0$, then for $E_i\subset|Z|$ obviously $E_i(Z_K-Z)=0$; for $E_i\not\subset |Z|$ one has $E_i(Z_K-Z)\leq 0$ (because $E_iZ_K\leq 0$ and $E_iZ\geq 0$). Hence $E_i(Z_K-Z)\leq 0$ for any E_i, therefore $Z_K-Z\geq Z_{B_0}$, hence $Z\leq Z_K-Z_{B_0}$. In particular, $|Z|\subset B_1$. If $|Z|=B_1$, then Z is the canonical cycle C_1' of B_1, otherwise, by the same argument as above $Z\leq Z_K-Z_{B_0}-Z_{B_1}$, hence $|Z|\subset B_2$. Continuing the precess, (a) follows. For the second part, apply (a) for Z_K-Z. □

5.7. Since $C\leq Z_{B_j}\leq Z_{min}$,

$$h^1(\mathcal{O}_{Z_{B_j}})=1 \text{ hence } \chi(Z_{B_j})=\chi(C_j)=\chi(C_j')=0 \text{ for all } 0\leq j\leq m.$$

5.8. Since any Z_{B_i} is a minimal cycle, there is a "computation sequence" which can start with any of the irreducible exceptional divisors $E_j\subset B_i$ and ends with Z_{B_i}. Moreover, for any two integers $0\leq i<j\leq m$, since Z_{B_i} is a minimal cycle with $Z_{B_j}<Z_{B_i}$, there is a computation sequence which starts with Z_{B_j} and ends with Z_{B_i}. More precisely, there is a sequence $Z_0,\dots,Z_k$ with $Z_0=Z_{B_j}, Z_k=Z_{B_i}$, $Z_{l+1}=Z_l+E_{i_l}$ and $E_{i_l}Z_l>0$ for $0\leq l\leq k-1$. Actually, in this second case, since $h^1(\mathcal{O}_{Z_{B_i}})=h^1(\mathcal{O}_{Z_{B_j}})=1$ (cf. 5.7.), for any $0\leq l\leq k-1$ the curve E_{i_l} is smooth rational and $E_{i_l}Z_l=1$. We will call a sequence like this a "computation sequence which connects Z_{B_j} and Z_{B_i}".

5.9. Let D be a cycle with $DE_k=0$ for any $E_k\subset B_i$, and consider a computation sequence $\{Z_l\}_l$ which connects Z_{B_j} and Z_{B_i} (where $j>i$). In the exact sequence $0\to\mathcal{O}_{E_{i_l}}(-D-Z_l)\to\mathcal{O}_{D+Z_{l+1}}\to\mathcal{O}_{D+Z_l}\to 0$ $(0\leq l\leq k-1)$ the Chern number is $E_{i_l}(-D-Z_l)=-1$, hence the natural map

$$H^k(\mathcal{O}_{D+Z_{B_i}})\to H^k(\mathcal{O}_{D+Z_{B_j}})\ (k=1,2) \text{ is an isomorphism.}$$

For example, if $D=C_{i-1}$ and $j=m$, then

$$H^k(\mathcal{O}_{C_i})\to H^k(\mathcal{O}_{C_{i-1}+C})\ (1\leq i\leq m;\ k=1,2) \text{ is an isomorphism.}$$

Now consider the exact sequence

$$0\to\mathcal{O}_C(-C_{i-1})\to\mathcal{O}_{C_{i-1}+C}\to\mathcal{O}_{C_{i-1}}\to 0.$$

5.10. Since C is 2–connected (4.16.) $\mathcal{O}_C(-C_{i-1})$ is trivial if and only if it has a non-zero section (4.15.); and this is happening if and only if $h^k(\mathcal{O}_C(-C_{i-1}))=1$ for $k=1,2$. Therefore (5.9.) imply $h^1(\mathcal{O}_{C_{i-1}})\leq h^1(\mathcal{O}_{C_i})\leq h^1(\mathcal{O}_{C_{i-1}})+1$ for $1\leq i\leq m$. Since $h^1(\mathcal{O}_{C_0})=1$, one obtains S. S.-T. Yau 's result ([37, (3.9)]):

$$h^1(\mathcal{O}_{C_i})\leq i+1 \text{ (for } 0\leq i\leq m), \text{ hence } p_g=h^1(\mathcal{O}_{C_m})\leq m+1.$$

The next theorem gives some characterizations of the "extremal property" $p_g=m+1$.

5.11. First characterization of $p_g=m+1$. *Assume that $(X,0)$ is a numerical Gorenstein elliptic singularity. Then the following facts are equivalent:*

(a) $p_g=m+1$;

(b) $h^1(\mathcal{O}_{C_i})=i+1$ for all $0\leq i\leq m$;

(c) The line bundles $\mathcal{O}_C(-C_j)$ are trivial (in $Pic(C)$), and $H^0(\mathcal{O}_{C_j+C})\to H^0(\mathcal{O}_{C_j})$ is surjective for any $0\leq j\leq m-1$;

(d) $h^1(\mathcal{O}(-C_j))=m-j$ for all $0\leq j\leq m$;

(e) $h^1(\mathcal{O}_{C'_j})=m-j+1$ for all $0\leq j\leq m$;

(f) For any $0\leq j\leq m-1$, there exists $f_j\in H^0(\tilde{X},\mathcal{O}(-C_j))$, such that for any $E_l\subset B_{j+1}$ the vanishing order of f_j on E_l is exactly $m_{E_l}(C_j)$;

(g) The line bundles $\mathcal{O}_{C'_{j+1}}(-C_j)$ are trivial for $0\leq j\leq m-1$;

(h) The line bundles $\mathcal{O}_{C'_{j+1}}(-Z_{B_j})$ are trivial for $0\leq j\leq m-1$.

Proof. $a\Leftrightarrow b\Leftrightarrow c$ is just a reformulation of the discussion in (5.9.) and (5.10.).

$d\Rightarrow a$ is easy: for $j=0$ one has $m=h^1(\mathcal{O}(-Z_{min}))\overset{2.45.}{=}h^1(\mathcal{O})-h^1(\mathcal{O}_{Z_{min}})\overset{2.45.}{=}p_g-1$. Conversely, for $(a,b,c)\Rightarrow d$, notice that $H^0(\mathcal{O})\to H^0(\mathcal{O}_{C_m})$ is onto by a Kodaira type vanishing, and $H^0(\mathcal{O}_{C_m})\to H^0(\mathcal{O}_{C_j})$ is onto by part (c) and (5.9.) applied several times. Therefore $H^0(\mathcal{O})\to H^0(\mathcal{O}_{C_j})$ is onto, hence $h^1(\mathcal{O}(-C_j))=h^1(\mathcal{O})-h^1(\mathcal{O}_{C_j})=m+1-j-1$.

$e\Rightarrow a$ again is trivial (take $j=0$). In order to prove $a\Rightarrow e$, we construct a sequence $\{Z_l\}_l$. We start with $Z_0=Z_K$. If $Z_l>0$ is already constructed, then we define $Z_{l+1}=Z_l-E_{i_l}$ as follows. If there is at least one $E_i\subset|Z_l|$ with $E_i(Z_l-Z_K)<0$ then take for E_{i_l} one of these E_i's. Otherwise E_{i_l} is an arbitrary $E_i\subset|Z_l|$. In this second case, by (5.6.), $E_{i_l}(Z_l-Z_K)=0$ and $Z_l\in\{C'_0,\ldots,C'_m\}$. If $Z_l=0$ then we stop.

Now, from the exact sequence $0\to\mathcal{O}_{E_{i_l}}(-Z_{l+1})\to\mathcal{O}_{Z_l}\to\mathcal{O}_{Z_{l+1}}\to 0$ and Serre duality one gets that $h^1(\mathcal{O}_{Z_l})=h^1(\mathcal{O}_{Z_{l+1}})$ in the first case, and in second case $h^1(\mathcal{O}_{Z_{l+1}})=h^1(\mathcal{O}_{Z_l})-\epsilon_l$, where $\epsilon_l\in\{0,1\}$. Since h^1 must drop exactly $m+1$ times, we obtain that in the above sequence we reach all the cycles C'_j and at every time $\epsilon_l=1$.

$a\Rightarrow f$. Fix an arbitrary $E_k\subset B_{j+1}$. Consider a computation sequence $\{Z_l\}_l$ of $Z_{B_{j+1}}$ which starts with E_k (cf. 5.8.). Since the Chern numbers $E_{i_l}(-Z_l-C_j)<0$, using the sequences $0\to\mathcal{O}_{E_{i_l}}(-Z_l-C_j)\to\mathcal{O}_{Z_{l+1}}(-C_j)\to\mathcal{O}_{Z_l}(-C_j)\to 0$, we obtain that $\beta: H^0(\mathcal{O}_{Z_{B_{j+1}}}(-C_j))\to H^0(\mathcal{O}_{E_k}(-C_j))$ is injective. Now, $H^0(\mathcal{O}_{E_k}(-C_j))=\mathbf{C}$. Indeed, if $E_k\approx\mathbf{P}^1$ then $H^0(\mathcal{O}_{E_k}(-C_j))=H^0(\mathcal{O}_{\mathbf{P}^1})$. If $E_k\not\approx\mathbf{P}^1$ then $E_k=C$ by (5.3.), and $\mathcal{O}_{E_k}(-C_j)$ is trivial by (c. Now, considers the diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}(-C_j)) & \xrightarrow{\alpha_k} & H^0(\mathcal{O}_{E_k}(-C_j)) \\ \Big\uparrow = & & \Big\uparrow \beta \\ H^0(\mathcal{O}(-C_j)) & \xrightarrow{\alpha} & H^0(\mathcal{O}_{Z_{B_{j+1}}}(-C_j)) \to H^1(\mathcal{O}(-C_{j+1})) \to \end{array}$$

$$H^1(\mathcal{O}(-C_j)) \to H^1(\mathcal{O}_{Z_{B_{j+1}}}(-C_j)) \to 0$$

By part (d): $h^1(\mathcal{O}(-C_t)) = m-t$, hence by dimension argument $h^1(\mathcal{O}_{Z_{B_{j+1}}}(-C_j)) \neq 0$. Since $\chi(\mathcal{O}_{Z_{B_{j+1}}}(-C_j)) = 0$ and β is injective, we obtain that, in fact, β is an isomorphism, and α is onto; hence α_k is onto as well. Finally, the surjectivity of α_k for all $E_k \subset B_{j+1}$ implies (f).

For $f \Rightarrow b$ notice that the image of the elements $1, f_0, \ldots, f_{i-1}$ are linearly independent in $H^0(\mathcal{O}_{C_i})$, this together with (5.10.) proves (b).

$f \Leftrightarrow g$. For any $E_l \subset B_{j+1}$ consider the composed map

$$H^0(\tilde{X}, \mathcal{O}(-C_j)) \xrightarrow{u} H^0(\mathcal{O}_{C'_{j+1}}(-C_j)) \xrightarrow{v} H^0(\mathcal{O}_{E_l}(-C_j)).$$

By vanishing theorem (Appendix 2) u is onto. Hence $v \circ u$ is onto if and only if v is onto. Notice that $h^0(\mathcal{O}_{E_l}(-C_j)) = 1$ always (if (f) is true then see the proof of $a \Rightarrow f$; if (g) is true, then $\mathcal{O}_{E_l}(-C_j)$ is a trivial line bundle).

$g \Leftrightarrow h$ is easy: $\mathcal{O}_{C'_{j+1}}(-C_j) = \otimes_{i \le j} \mathcal{O}_{C'_{i+1}}(-Z_{B_i})|_{C'_{j+1}}$, and $\mathcal{O}_{C'_{j+1}}(-Z_{B_j}) = \mathcal{O}_{C'_{j+1}}(-C_j) \otimes \mathcal{O}_{C'_j}(-C_{j-1})|^{-1}_{C'_{j+1}}$. $\square$

5.12. Corollary. *Assume that $(X,0)$ is an elliptic Gorenstein singularity.*

a) Then $p_g = m+1$ if and only if the line bundles $\mathcal{O}_{C'_{j+1}}(-Z_{B_j})$ are trivial for all $1 \le j \le m-1$.

b) If $p_g = m+1$, then $h^1(\tilde{X}, \mathcal{O}(-lZ_{min})) = h^1(Z_K, \mathcal{O}_{Z_K}(-lZ_{min})) = p_g - 1$ for any $l \ge 1$.

Proof of (a). We have to prove that in the Gorenstein case the triviality of $\mathcal{O}_{C'_1}(-Z_{B_0})$ follows from the triviality of the other line bundles. Indeed, if all the other line bundles are trivial, then 5.11.a$\Leftarrow h$), applied for $(\tilde{X}/B_1, p_1)$, gives $h^1(\mathcal{O}_{C'_1}) = m$. But the Gorenstein property implies that for any positive cycle $D < Z_K$ the strict inequality $h^1(\mathcal{O}_D) < h^1(\mathcal{O}_{Z_K}) = p_g$ holds (4.17.). Hence $h^1(\mathcal{O}_{Z_K}) > h^1(\mathcal{O}_{C'_1}) = m$. Therefore $p_g = m+1$. Now, we prove (b). Using the vanishing theorem from Appendix 2, $h^1(\mathcal{O}(-lZ_{min})) = h^1(\mathcal{O}_{Z_K}(-lZ_{min})) = h^1(\mathcal{O}_{C'_1}(-lZ_{min}))$ for $l \ge 1$. But the line bundle $\mathcal{O}_{C'_1}(-lZ_{min})$ is trivial by (5.11.g), hence its first Betti–number is $p_g - 1$ by (5.11.e). $\square$

5.13. Definition. *In the sequel, we call the line bundles $\mathcal{O}_{C'_{j+1}}(-Z_{B_j})$ $(0 \le j \le m-1)$ "obstruction line bundles".*

The next example shows that the property $p_g = m+1$ is not always true (even if we deal with Gorenstein singularities).

5.14. Example. Consider the elliptic singularities $(X_1, 0) = \{x^2 + y^3 + z^{18} = 0\} \subset (\mathbf{C}^3, 0)$ and $(X_2, 0) = \{z^2 = y(x^4 + y^6)\} \subset (\mathbf{C}^3, 0)$ discussed in (2.15.) and (4.7.).

Denote by E_0, E_1, E_2 the irreducible exceptional divisors (starting from left). Then $E_0 = C = Z_{B_2}$, $E_0 + E_1 = Z_{B_1}$, $E_0 + E_1 + E_2 = Z_{min}$, and $Z_K = 3E_0 + 2E_1 + E_2$. Therefore $m = 2$.

On the other hand, the geometric genus in the first case is $p_g = 3$, and in the second case is $p_g = 2$.

II. Characterization of the trivial line bundles in $Pic(C'_j)$.

5.15. As usual, for any positive cycle D, we denote the isomorphism classes of invertible sheaves over $\mathcal{O}_D$ by $Pic(D)$ $(= H^1(\mathcal{O}^*_D))$. The kernel of the degree map $deg : Pic(D) \to \mathbf{Z}^{\#\{E_i | E_i \subset |D|\}}$ defined by $\mathcal{L} \to \{deg_{E_i} \mathcal{L}\}_i$ is denoted by $Pic^0(D)$.

If $D = C$ is a minimally elliptic cycle, then

$$\begin{cases} \mathcal{L} \in Pic^0(C) \text{ is trivial} \Leftrightarrow H^0(C, \mathcal{L}) \neq 0; \\ \text{moreover, if } \mathcal{L} \text{ is trivial then } h^0(C, \mathcal{L}) = 1. \end{cases}$$

This basically follows from the 2–connectivity of C: apply (4.16.) and (4.15.) for the singularity supported by $|C|$). But, in general, it is not really easy to provide similar result for $Pic^0(D)$.

5.16. Fortunately, in the case of an elliptic singularity, for any $0 \leq j \leq m$, $Pic^0(Z_{B_j})$ is as simple as $Pic^0(C)$. Indeed, consider the "exponential cohomology sequences" of the cycles Z_{B_j} and C. Then the natural maps $H^1(|Z_{B_j}|, \mathbf{Z}) \to H^1(|C|, \mathbf{Z})$ and $H^1(\mathcal{O}_{Z_{B_j}}) \to H^1(\mathcal{O}_C)$ are isomorphism (cf. 5.3. and 5.9. for $D = 0$). Hence $Pic^0(Z_{B_j}) \to Pic^0(C)$ is an isomorphism as well. Moreover, if we consider a computation sequence $\{Z_l\}_l$ which connects $C = Z_{B_m}$ and Z_{B_j}, then from the exact sequences $0 \to \mathcal{O}_{E_{i_l}}(-Z_l) \otimes \mathcal{L} \to \mathcal{L}|Z_{l+1} \to \mathcal{L}|Z_l \to 0$ follows that $H^0(Z_{B_j}, \mathcal{L}) = H^0(E, \mathcal{L}|_C)$. Therefore, for any line bundle $\mathcal{L} \in Pic^0(Z_{B_j})$:

$$\begin{cases} h^0(Z_{B_j}, \mathcal{L}) \leq 1, \text{ and} \\ h^0(Z_{B_j}, \mathcal{L}) = 1 \text{ if and only if } \mathcal{L} \text{ is trivial.} \end{cases}$$

5.17. Theorem. *Assume that the elliptic numerical Gorenstein singularity $(X, 0)$ satisfies $p_g = m + 1$; and fix an integer $0 \leq j \leq m$. Then the followings are equivalent.*

a) $\mathcal{L} \in Pic^0(C'_j)$ is trivial;

b) $h^0(C'_j, \mathcal{L}) \geq m - j + 1$.

Actually, if $\mathcal{L}$ is trivial, then $h^0(C'_j, \mathcal{L})$ is exactly $m - j + 1$.

Proof. $a \Rightarrow b$ and the equality follow from (5.11.$a \Rightarrow e$) and (5.7.). We prove $b \Rightarrow a$ by descending induction over j. If $j = m$, then $C'_j = C$, hence (a) follows from (5.15.).

Assume that $b \Rightarrow a$ is true for $j+1$, and $h^0(C'_j, \mathcal{L}) \geq m-j+1$. Consider the exact sequence

$$0 \to \mathcal{O}_{C'_{j+1}}(-Z_{B_j}) \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}|Z_{B_j} \to 0.$$

Then $h^0(\mathcal{L}|Z_{B_j}) \leq 1$ by (5.16.), hence $h^0 := h^0(\mathcal{O}_{C'_{j+1}}(-Z_{B_j}) \otimes \mathcal{L}) \geq m-j$. Hence, by the inductive step $\mathcal{O}_{C'_{j+1}}(-Z_{B_j}) \otimes \mathcal{L}$ is trivial in $Pic^0(C'_{j+1})$, and $h^0 = m-j$. Therefore $h^0(\mathcal{L}|Z_{B_j})$ must be 1, $\mathcal{L}|Z_{B_j}$ must be trivial, and $r : H^0(\mathcal{L}) \to H^0(\mathcal{L}|Z_{B_j})$ onto. In particular, there exists $s \in H^0(\mathcal{L})$ such that $r(s)$ generates $\mathcal{O}(\mathcal{L}|Z_{B_j})$. Since $|Z_{B_j}| = |C'_j|$, it follows that s generates the sections of $\mathcal{L}$ as well. $\square$

5.18. Corollary. *Assume that $(X,0)$ is an elliptic Gorenstein singularity, and we fix an integer $0 \leq j \leq m$. Assume that $h^1(\mathcal{O}_{C'_j}) = m-j+1$. Then $\mathcal{O}_{C'_j}(-C_{j-1})$ is trivial in $Pic^0(C'_j)$.*

For example, for $j=m$ we obtain that $\mathcal{O}_C(-C_{m-1})$ is trivial in $Pic^0(C)$ (the assumption $h^1(\mathcal{O}_C)=1$ is automatically satisfied). For $j=m-1$, if contracting B_{m-1} one has $p_g((\tilde{X}/B_{m-1}, p_{m-1}))=2$, then $\mathcal{O}_{C'_{m-1}}(-C_{m-2})$ is trivial in $Pic^0(C'_{m-1})$.

Proof. First notice that contracting B_j we obtain a numerical Gorenstein singularity with canonical cycle C'_j, and with the "extremal property" $p_g(\tilde{X}/B_j, p_j) =$ length of the elliptic sequence of $B_j = m-j+1$. Therefore, (5.17.) for this singularity gives: $\mathcal{L} \in Pic^0(C'_j)$ is trivial $\Leftrightarrow h^0(C'_j, \mathcal{L}) = m-j+1$. But by Serre duality $h^0(\mathcal{O}_{C'_j}(-C_{j-1}) = h^1(\mathcal{O}_{C'_j}) = m-j+1$. $\square$

5.19. Second characterization of $p_g = m+1$. *Assume that $(X,0)$ is a numerical Gorenstein elliptic singularity. Consider its minimal resolution $\tilde{X} \to X$ and the singularity $(\tilde{X}/B_j, p_j)$ obtained by the contraction of B_j $(0 \leq j \leq m)$. Then $(X,0)$ satisfies the extremal property $p_g = m+1$ if and only if the singularities $(\tilde{X}/B_j, p_j)$ are Gorenstein for all $0 \leq j \leq m-1$. (Notice that $(\tilde{X}/B_m, p_m)$ is automatically Gorenstein by 4.14.)*

Proof. "$\Rightarrow$" First we prove that a numerical Gorenstein singularity $(X,0)$ with $p_g = m+1$ is Gorenstein. (This fact was proved by S. S.-T. Yau in [37, (3.11)]; we present here a very short proof in the spirit of [25, (4.21)]; cf. also with the papers of L. Bădescu [3, 4]; and also with the proof of (4.14.)).

Consider the line bundle $\mathcal{L} := \mathcal{O}_{Z_K}(Z_K + K) \in Pic^0(Z_K)$. By Serre duality $h^0(\mathcal{L}) = h^1(\mathcal{O}_{Z_K})$, hence $h^0(\mathcal{L}) = p_g = m+1$. Therefore, by (5.17.), $\mathcal{L}$ is trivial. Moreover, by Kodaira type vanishing $h^1(\mathcal{O}(K)) = 0$, hence $H^0(\tilde{X}, \mathcal{O}(Z_K+K)) \to H^0(\mathcal{L}) \approx H^0(\mathcal{O}_{Z_K})$ is onto. Therefore, there is a global section of $\mathcal{O}(Z_K+K)$ which has no zeros in the neighbourhood of E, hence Z_K+K is linearly equivalent to the zero cycle. Hence $(X,0)$ is Gorenstein.

Now, $\tilde{X}/B_j$ is an elliptic numarical Gorenstein singularity with numerical cycle C'_j and the length of its elliptic sequence$= h^1(\mathcal{O}_{C'_j}) = m-j+1$. Therefore, the above fact applied for the singularities $\tilde{X}/B_j$ ends the proof.

"$\Leftarrow$" By (5.11.h), it is enough to prove that the obstruction line bundles $\mathcal{L}_j := \mathcal{O}_{C'_j}(-Z_{B_{j-1}})$ are trivial for $1 \leq j \leq m$. We will prove this by induction. Assume that for a fixed $1 \leq j_0 \leq m$ the line bundles $\mathcal{L}_j$ are trivial for $j > j_0$ (if $j_0 = m$ then this assumption is vacuous). Since the singularity $\tilde{X}/B_{j_0-1}$ is Gorenstein, by (5.11.h) and (5.12.a) (applied for $\tilde{X}/B_{j_0-1}$), the triviality of $\{\mathcal{L}_j\}_{j>j_0}$ implies the triviality of $\mathcal{L}_{j_0}$. $\square$

5.20. Examples. Assume that $(X,0)$ is a numerical Gorenstein elliptic singularity.

(a) If $m = 1$ then $p_g \leq 2$, and:

$$p_g = 2 \Leftrightarrow \mathcal{O}_C(-Z_{min}) \text{ is trivial } \Leftrightarrow (X,0) \text{ is Gorenstein.}$$

Examples with $p_g = 1$ exists (actually this is the generic case!). E.g. take the minimal resolution of any Gorenstein singularity with $p_g = 2$ and $m = 1$. Then deform its analytic structure. Generically we obtain a non–Gorenstein singularity with $p_g = 1$ (cf. [16], page 1279).

(b) Assume that $m = 2$. Then in the numerical Gorenstein case we have two obstruction line bundles. For the generic analytic structure $p_g = 1$ as above. If $(X,0)$ is Gorenstein, then $p_g = h^1(\mathcal{O}_{Z_K}) > h^1(\mathcal{O}_C) = 1$, hence $p_g \geq 2$. From (5.12.a), $p_g = 3$ if and only if $\mathcal{O}_C(-Z_{B_1})$ is trivial. From (5.18.) $\mathcal{O}_C(-Z_{min} - Z_{B_1})$ is trivial, hence if $(X,0)$ is Gorenstein, then:

$$p_g = 3 \Leftrightarrow \mathcal{O}_C(-Z_{min}) \text{ is trivial} \Leftrightarrow (\tilde{X}/B_1, p_1) \text{ is Gorenstein.}$$

III. Torsion properties of the "obstruction line bundles".

The main result of this subsection is the following:

5.21. Theorem. *Consider the minimal resolution of an elliptic Gorenstein singularity $(X,0)$ with elliptic sequence $\{Z_{B_j}\}_{j=0}^m$ $(m \geq 1)$, and fix an integer $1 \leq k \leq m$.*

Assume that for $k+1 \leq j \leq m$ the "obstruction line bundles" $\mathcal{O}_{C'_j}(-Z_{B_{j-1}}) \in Pic^0(\mathcal{O}_{C'_j})$ are trivial. (If $k = m$ then this assumption is vacuous.) Then $\mathcal{O}_{C'_k}(-Z_{B_{k-1}}) \in Pic^0(\mathcal{O}_{C'_k})$ has finite order.

Actually, there exist integers $1 \leq l_r \leq k - r$ $(0 \leq r \leq k-1)$ such that for any $0 \leq r \leq k-1$ the line bundle $\mathcal{O}_{C'_k}(l_0 \cdots l_r\ Z_{B_r})$ is trivial. In particular, the order of $\mathcal{O}_{C'_k}(-Z_{B_{k-1}})$ in $Pic^0(\mathcal{O}_{C'_k})$ is not greater than $k!$.

Proof. First assume that $k = 1$. Then $\mathcal{O}_{C'_1}(-Z_{B_0})$ is trivial by (5.12.a) and (5.11.$a \Leftrightarrow h$). So, in the sequel we will assume that $k \geq 2$.

Since $(\tilde{X}/B_k, p_k)$ is a numerical Gorenstein singularity with canonical class C'_k (cf. 5.5. d), the assumption together with (5.11.) (applied for this singularity) provide that:

$$h^1(\mathcal{O}_{C'_k}) = m - k + 1 \qquad (5.22.)$$

5.23. Lemma. *For any $0 \le r \le t \le k-2$ one has $h^1(\mathcal{O}(-C_t - Z_{B_r})) \ge m-k+1$.*
Proof. Notice that $h^1(\mathcal{O}(-C_{t+1})) \overset{App2}{=} h^1(\mathcal{O}_{C'_{t+2}}(-C_{t+1})) = h^0(\mathcal{O}_{C'_{t+2}}) \overset{5.7.}{=} h^1(\mathcal{O}_{C'_{t+2}}) \ge h^1(\mathcal{O}_{C'_k})$ (because $C'_{t+2} \ge C'_k$). Hence, by (5.22.)

$$h^1(\mathcal{O}(-C_{t+1}) \ge m-k+1. \tag{5.24.}$$

Now, consider a computation sequence $\{Z_l\}_l$ which connects $Z_{B_{t+1}}$ with Z_{B_r} (cf. 5.8.) (i.e. $Z_{l+1} = Z_l + E_{i_l}$ where E_{i_l} is smooth rational curve with $E_{i_l}Z_l = 1$). In the exact sequence $0 \to \mathcal{O}(-C_t - Z_{l+1}) \to \mathcal{O}(-C_t - Z_l) \to \mathcal{O}_{E_{i_l}}(-C_t - Z_l) \to 0$ the Chern number $E_{i_l}(-C_t - Z_l)$ is $-E_{i_l}C_t - 1 \ge -1$ (cf. 2.11.e). Hence $H^1(\mathcal{O}(-C_t - Z_{l+1})) \to H^1(\mathcal{O}(-C_t - Z_l))$ is onto for any l, in particular, $h^1(\mathcal{O}(-C_t - Z_{B_r})) \ge h^1(\mathcal{O}(-C_t - Z_{B_{t+1}}))$. The lemma follows from this and (5.24.). $\square$

Now consider the exact sequence:

$$0 \to \mathcal{O}(-Z_K - Z_{B_r}) \to \mathcal{O}(-C_t - Z_{B_r}) \to \mathcal{O}_{C'_{t+1}}(-C_t - Z_{B_r}) \to 0. \tag{5.25.}$$

5.26. Lemma. $H^1(\mathcal{O}(-Z_K - Z_{B_r}) = 0$.
Proof. Consider a computation sequence $\{Z_l\}_l$ which connects Z_{B_r} and $Z_{B_0} = Z_{min}$ (cf. 5.8.). Consider the exact sequences:

$$0 \to \mathcal{O}(-Z_K - Z_{l+1}) \to \mathcal{O}(-Z_K - Z_l) \to \mathcal{O}_{E_{i_l}}(-Z_K - Z_l) \to 0$$

and notice that $E_{i_l}(-Z_K - Z_l) = -E_{i_l}Z_K - 1 \ge -1$ (cf. 2.30.). Hence $H^1(\mathcal{O}(-Z_K - Z_{min})) \to H^1(\mathcal{O}(-Z_K - Z_{B_r}))$ is onto. But $H^1(\mathcal{O}(-Z_K - Z_{min})) = 0$ (by Appendix 2). $\square$

Now, from (5.25.) and duality:

$$H^1(\mathcal{O}(-C_t - Z_{B_r})) = H^1(\mathcal{O}_{C'_{t+1}}(-C_t - Z_{B_r})) = H^0(\mathcal{O}_{C'_{t+1}}(C'_{t+1} + C_t + Z_{B_r} - Z_K)).$$

Hence (5.23.) reads as follows: for $0 \le r \le t \le k-2$:

$$h^0(\mathcal{O}_{C'_{t+1}}(C'_{t+1} + C_t + Z_{B_r} - Z_K)) \ge m-k+1.$$

Now, for any positive cycle $0 < D \le C'_{t+1}$ write $o(D) := \mathcal{O}_D(D + C_t + Z_{B_r} - Z_K)$. For any irreducible exceptional divisor $E_i \subset |D|$, one has the exact sequence:

$$0 \to o(D - E_i) \to o(D) \to \mathcal{O}_{E_i}(D + C_t + Z_{B_r} - Z_K) \to 0.$$

The Chern number $E_i(D + C_t + Z_{B_r} - Z_K) = E_i(D - Z_K)$, for $E_iZ_{B_r} = E_iC_t = 0$. Therefore, if $E_i(D - Z_K) < 0$, then $H^0(o(D - E_i)) \to H^0(o(D))$ is an isomorphism. In the sequel, we will construct a decreasing sequence of cycles as follows. The starting cycle is $D = C'_{t+1}$. If D is already constructed, and there exists $E_i \subset |D|$ with $E_i(D - Z_K) < 0$, then the next term is $D - E_i$. During this step $h^0(o(D))$ will stay constant.

If at a moment we reach a cycle $D>0$ with the property $E_i(D-Z_K)\geq 0$ for all $E_i\subset |D|$, then by (5.6.), in fact $E_i(D-Z_K)=0$ for all $E_i\subset|D|$ and D must be one of the cycles C'_s. If there is a $E_i\subset|D|$ such that

$$r_i: H^0(\mathcal{O}_D(D+C_t+Z_{B_r}-Z_K))\to H^0(\mathcal{O}_{E_i}(D+C_t+Z_{B_r}-Z_K))$$

is the trivial map, then the term following D is $D-E_i$ for this E_i. Hence, in this case again, $h^0(o(D))=h^0(o(D-E_i))$. If the above map r_i is nontrivial for all $E_i\subset|D|$, then the term following D is $D-E_i$, where E_i arbitrary with $E_i\subset|D|$. In this later case r_i has rank one, hence $h^0(o(D))$ will decrease by one. If $D=0$ we stop.

Since finally $h^0(o(D))$ must decrease to zero, and $h^0(o(C'_{t+1}))\geq m-k+1$, $h^0(o(D))$ must decrease at least $m-k+1$ times. Therefore, the first time when we decrease h^0 must happen for a cycle $D=C'_s$ with $s\leq k$ (i.e. the length of the sequence $\{C'_s,\dots,C'_m\}$ must be at least $m-k+1$). Obviously $C'_s\leq C'_{t+1}$ (because we deal with cycles $D\leq C'_{t+1}$), hence: $s\geq t+1$.

This shows that there exists $t+1\leq s\leq k$ such that

$$H^0(\mathcal{O}_{C'_s}(C'_s+C_t+Z_{B_r}-Z_K))\to H^0(\mathcal{O}_{E_i}(C'_s+C_t+Z_{B_r}-Z_K))=\mathbf{C}$$

is onto for all $E_i\subset|C'_s|$. In other words, $\mathcal{O}_{C'_s}(C'_s+C_t+Z_{B_r}-Z_K)$ is a trivial line bundle. Taking its restriction to C'_k, we obtain:

$$\text{(5.27.)}\qquad \begin{cases} \text{for any pair } (r,t) \text{ with } 0\leq r\leq t\leq k-2,\\ \text{there exists } s \text{ with } t+1\leq s\leq k, \text{ such that}\\ \mathcal{O}_{C'_k}(C'_s+C_t+Z_{B_r}-Z_K) \text{ is a trivial line bundle in } Pic^0(C'_k).\end{cases}$$

5.28. Lemma. *For any $0\leq r\leq k-1$, there exists l_r with $1\leq l_r\leq k-r$, such that*

$$\mathcal{O}_{C'_k}(l_0\cdots l_r\ Z_{B_r}) \text{ is trivial in } Pic^0(C'_k).$$

Proof. First we prove for $r=0$. If there is at least one s in (5.27.) with $s=t+1$, then $C'_s+C_t-Z_K=0$, hence $\mathcal{O}_{C'_k}(Z_{B_0})$ is trivial. Now, assume that $s\geq t+2$ for any t. This means that for any $0\leq t\leq k-2$, there exists $t+2\leq s(t)\leq k$ such that:

$$(*_t)\qquad\qquad \mathcal{O}_{C'_k}(Z_{B_0})\approx\mathcal{O}(Z_{B_{t+1}}+\cdots+Z_{B_{s(t)-1}}).$$

Now, we set $t_1:=0$. If $s(t_1)=k$ we stop; otherwise consider $t_2:=s(t_1)-1$. If $s(t_2)=k$ we stop, otherwise we continue until we obtain $s(t_u)=k$ for some $u\geq 1$. If we multiply the identities $(*_t)$ for $t_1,\dots,t_u$, we obtain:

$$\mathcal{O}_{C'_k}(uZ_{B_0})\approx\mathcal{O}_{C'_k}(Z_{B_1}+\cdots+Z_{B_{k-1}}).$$

Here u satisfies $1\leq u\leq k-1$ ($u=k-1$ exactly when $s(t)=t+2$ for every t, i.e. when we have to consider all the isomorphisms $(*_t)$). But from (5.18.) and (5.22.):

(5.29.) $$\mathcal{O}_{C'_k}(Z_{B_0}+\cdots+Z_{B_{k-1}}) \text{ is trivial,}$$

hence with the notation $l_0 := u+1$ one has $\mathcal{O}_{C'_t}(l_0 Z_{B_0})$ is trivial.

Now, take $r=1$ and consider again (5.27.). If $s=t+1$ for at least one t, then $\mathcal{O}_{C'_k}(Z_{B_1})$ is trivial. If $s\geq t+2$ for all $1\leq t\leq k-2$, by a similar argument as above one obtains that $\mathcal{O}_{C'_k}(uZ_{B_1})\approx \mathcal{O}_{C'_t}(Z_{B_2}+\cdots+Z_{B_{k-1}})$ for some $1\leq u\leq k-2$ ($u=k-2$ occurs exactly when we have to consider all the identities corresponding to $1\leq t\leq k-2$). This together with (5.29.) gives that $\mathcal{O}_{C'_k}(Z_{B_0}+(u+1)Z_{B_1})$ is trivial. Now consider its l_0^{th} power and take $l_1 := u+1$.

Similar argument can be used for all $0\leq r\leq k-2$. For $r=k-1$, just take (5.29.) again and take its $l_0\cdots l_{k-2}$-th power. $\square$

5.30. Example. (cf. 5.20. b) Assume that $(X,0)$ is elliptic Gorenstein, and $m=2$. Recall that $\mathcal{O}_C(-Z_{B_1}-Z_{min})\approx\mathcal{O}_C$. The above theorem (for $m=k=2$) says that the order of $\mathcal{O}_C(-Z_{min})$ divides 2, hence:

$$\mathcal{O}_C(-Z_{min}) \text{ is trivial} \Leftrightarrow p_g=3; \qquad \mathcal{O}_C(-Z_{min}) \text{ has order } 2 \Leftrightarrow p_g=2.$$

The main results of this subsection is the following:

5.31. Theorem. *Assume that $(X,0)$ is an elliptic Gorenstein singularity with $H^1(E,\mathbf{Z})=0$. Then $p_g=m+1=$ the length of the elliptic sequence in the minimal resolution of $(X,0)$. In particular p_g is a topological invariant.*

Proof. If $H^1(E,\mathbf{Z})=0$, then by the exponential exact sequence ([5], page 49) $Pic^0(D)\approx H^1(\mathcal{O}_D)$ for any positive cycle D, in particular, it is torsion free. Therefore, (5.21.) (as an inductive step) proves that all the obstruction line bundles are trivial. Hence $p_g=m+1$ by (5.11.). $\square$

5.32. Remark. If $m\geq 1$, then in (5.31.) both assumptions are necessary. Without Gorenstein condition p_g generically is one ([16, page 1279]). Moreover, Example (4.7.) shows that $p_g<m+1$ can occur if $H^1(E,\mathbf{Z})\neq 0$.

5.33. Corollary. *For a Gorenstein singularity $(X,0)$ with $H^1(E,\mathbf{Z})=0$ the following facts are equivalent:*

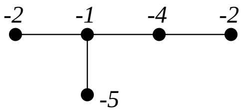
$$p_g=2 \;\Leftrightarrow\; \chi(Z_{min})=0 \text{ and } Z_K=Z_{min}+C.$$

(Notice that the right hand side is completely topological.)

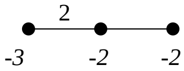
Proof. Indeed, if $(X,0)$ is Gorenstein with $p_g=2$, then $h^1(Z_{min})$ cannot be zero (because of (4.1.)); cannot be greater than or equal to two (that would imply via (2.46.) and (4.17.) that $Z_{min}=Z_K$, which characterize the minimally elliptic singularities); hence it is one. Therefore $(X,0)$ is elliptic. The rest follows from the above results. $\square$

5.34. Remark. S. S.-T. Yau in [37], Theorem B, proved that a Gorenstein singularity with $p_g = 2$ is elliptic. Moreover, in [35], he proved also that a *hypersurface* singularity with $H^1(E, \mathbf{Z}) = 0$ and $p_g = 2$ satisfies $m = 1$. His proof (of this second fact) is based on the classification of all possible dual resolution graphs of hypersurface singularities with $p_g = 2$ (250 cases).

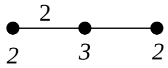
5.35. Example. (Cf. 1.22.) Consider the hypersurface singularity $\{x^2 + y^5 + z^7 = 0\}$. A non-minimal resolution graph obtained using the algorithm of Appendix 1 is

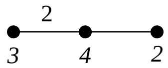


, with genus $g_i = 0$ for all i.

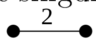
Thus the minimal resolution graph $\Gamma_{\tilde{X}}$ is 

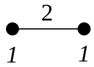
, where the graph corresponds to the following intersection matrix: $(E_iE_j)_{i,j} = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

By Laufer's Algorithm the minimal cycle is $Z_{min} = \sum_{i=1}^{3} n_i E_i$, where the coefficients n_i are shown as weights in the next figure: 

Similarly, the figure 

shows the coefficients of the canonical cycle $Z_K = \sum_{i=1}^{3} m_i E_i$. By a computation $\chi(Z_{min}) = 0$, so that $(X, 0)$ is an elliptic singularity.

The elliptic sequence is the following. $B_0 = \Gamma_{\tilde{X}}$, $B_1 = |Z_K - Z_{min}| =$ 

, for which $Z_{B_1} = Z_{min}(B_1)$ is determined by 

Thus $Z_K = Z_{min}(B_1) + Z_{min}(B_0)$, hence $m = 1$ and $p_g = 2$.

5.36. Exercises. Set $(X_i, 0) \subset (\mathbf{C}^3, 0)$ given by $(X_1, 0) = \{z^2 + y^3 + x^{9+6m} = 0\}$, $(X_2, 0) = \{z^2 + y^3 + x^{11+6m} = 0\}$, $(X_3, 0) = \{z^3 + y^3 + x^{4+3m} = 0\}$, $(X_4, 0) = \{z^3 + y^3 + x^{5+3m} = 0\}$ (where $m \geq 0$). Then, in all these cases, $H^1(E, \mathbf{Z}) = 0$, $(X_i, 0)$ is elliptic, and $p_g = m + 1 =$ the length of the elliptic sequence ($1 \leq i \leq 4$).

IV. The multiplicity.

5.37. Assume that $(X, 0)$ is an elliptic Gorenstein singularity with $p_g = m + 1$. In this subsection we will prove that its multiplicity $mult(X, 0)$ is a topological invariant, in fact depends only on Z^2_{min}.

The next theorem generalizes Laufer's result about the multiplicity of minimally elliptic singularities [16].

5.38. Theorem. *Assume that* $(X,0)$ *is an elliptic Gorenstein singularity with* $p_g = m+1$. *(If* $m=0$ *or* $H^1(E,\mathbf{Z})=0$ *then the last assumption is satisfied.) Then:*

If $Z^2_{min} \leq -2$, *then* $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}} = \mathcal{O}(-Z_{min})$, *hence* $mult(X,0) = -Z^2_{min}$*;*

If $Z^2_{min} = -1$, *then* $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}} = m_Q\mathcal{O}(-Z_{min})$ *for some smooth point* Q *of* E, *and* $mult(X,0) = 2$.

For the simplicity of the proof we will assume that $m>0$. The case $m=0$ is similar. We invite the reader to adopt the next arguments to the case $m=0$. In our proofs we will use B_1 (as a byproduct of the elliptic sequence), and many times (5.11.f). In the case $m=0$ take for B_1 the set $\cup E_i$, where the union is over $\{i : E_iZ_{min}=0\}$. Moreover, replace (5.11.f) by (4.28.). The details are left to the reader (see also the original paper of Laufer [16]).

We prove the above theorem in several steps.

First notice that by (5.11.f) there exists $f_0 \in H^0(\tilde{X}, \mathcal{O}(-Z_{min}))$ such that for any $E_l \subset B_1$, the order of vanishing of f_0 on E_l is exactly $m_{E_l}(Z_{min})$. In particular:

$$(5.39.) \qquad H^0(\tilde{X}, \mathcal{O}(-Z_{min})) \to H^0(\mathcal{O}_{E_l}(-Z_{min})) = \mathbf{C} \ \text{ is onto for all } E_l \subset B_1.$$

By Kodaira vanishing, if $|\mathcal{O}_{Z_K}(-Z_{min})|$ is free, then $|\mathcal{O}(-Z_{min})|$ is free too. By [29, page 426], if $|\mathcal{O}(-Z_{min})|$ is free, then $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}} = \mathcal{O}(-Z_{min})$, in which case $mult(X,0) = -Z^2_{min}$. In particular:

$$(5.40.) \qquad Z^2_{min} = -1 \ \Rightarrow -Z^2_{min} \neq mult(X,0) \Rightarrow \ |\mathcal{O}_{Z_K}(-Z_{min})| \text{ is not free.}$$

If $|\mathcal{O}(-Z_{min})|$ is not free, then $mult(X,0) > -Z^2_{min}$.

5.41. Exercise. If $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}} = m_Q\mathcal{O}(-Z_{min})$, where m_Q is the maximal ideal of a smooth point Q of E, then $mult(X,0) = -Z^2_{min}+1$.

Now we prove the converse of (5.40.) . For simplicity, in the sequel we will write $\mathcal{L} := \mathcal{O}_{Z_K}(-Z_{min})$.

5.42. Proposition. *Assume that* $|\mathcal{L}|$ *is not free. Then* $Z^2_{min} = -1$.

Proof. By (5.39.) , all the basepoints must lie outside of B_1. Since all the connected components of $E \setminus B_1$ are trees whose vertices correspond to smooth rational curves (cf. 5.3.), we can fix a basepoint Q such that Q is "as close to B_1 as possible". More precisely, we will fix a basepoint Q such that there exist irreducible exceptional divisors $\{E'_i\}_{i=0}^l$ with

$$(5.43.) \qquad \begin{cases} E'_i \not\subset B_1 \text{ and } E'_i \text{ is smooth rational curve } (0 \leq i \leq l); \\ Q \in E'_0, \ E'_iE'_{i+1} = 1 \text{ for } 0 \leq i \leq l-1, \text{ and } E'_lB_1 = 1; \\ \text{there is no basepoint of } |\mathcal{L}| \text{ on } \cup_{i>0}E'_i. \end{cases}$$

With this choice of Q, $E_0'Z_{min}$ cannot be zero. Indeed, since Q is a basepoint, the evaluation map $H^0(\mathcal{L}) \stackrel{\alpha}{\longrightarrow} H^0(\mathcal{O}_{E_0'}(-Z_{min})) \stackrel{e}{\longrightarrow} \mathbf{C}_Q$ is trivial. If, by assumption, $E_0'Z_{min}=0$ then e must be an isomorphism, hence α must be trivial as well. This shows that $E_0'\cap E_1'$ is also a basepoint, which contradicts the above choice of Q in (5.43.). For later reference:

$$\text{(5.44.)} \qquad E_0'Z_{min} < 0.$$

Since the evaluation map $H^0(Z_K,\mathcal{L}) \to \mathbf{C}_Q$ is zero, from the exact sequence $0\to m_Q\mathcal{L}\to \mathcal{L}\to \mathbf{C}_Q\to 0$ and (5.12.b), one gets $h^1(m_Q\mathcal{L}) = h^1(\mathcal{L})+1 = p_g$.

In the next paragraph, we follow the beginning of the proof of (4.23) of [25]. Take $0<D\leq Z_K$ minimal with the property $h^1(D, m_Q\mathcal{L}|_D) = p_g$. Minimality means:

$$h^1(D-E_i, m_Q\mathcal{L}|_{D-E_i}) \leq p_g - 1 \text{ for any } E_i \subset |D|.$$

Let K_i be the kernel of $m_Q\mathcal{L}|_D \to m_Q\mathcal{L}|_{D-E_i}$. Outside Q, clearly $K_i = \mathcal{L}\otimes\mathcal{O}_{E_i}(-D+E_i)$. Actually, if $Q\in|D-E_i|$, then even at Q one has $K_i = \mathcal{L}\otimes\mathcal{O}_{E_i}(-D+E_i)$. If $Q\in|D|$ but $Q\notin|D-E_i|$ (i.e. if $Q\in E_i\setminus\{\text{other components of } D\}$ and $m_{E_i}(D)=1$), then $K_i=\mathcal{L}\otimes m_Q\mathcal{O}_{E_i}(-D+E_i)$. We define $b_i:=0$ in the first two cases, and $b_i:=1$ in the third case.

Now, the minimality of D implies that for all i: $h^1(D,K_i)\neq 0$, in other words:

$$\text{(5.45.)} \qquad E_i\mathcal{L} + E_i(Z_K-D) \leq b_i \text{ for all } E_i\subset |D|.$$

5.46. Lemma. *The situation when $E_i\mathcal{L}+E_i(Z_K-D)=0$ for all $E_i\subset|D|$ cannot occur.*

Proof. Assume the contrary. Since $E_i\mathcal{L}\geq 0$ for all E_i, and $\chi(D)\geq 0$, the relation

$$0\leq D(Z_K-D) = \sum m_{E_i}(D)E_i(Z_K-D) = -\sum m_{E_i}(D)E_i\mathcal{L}\leq 0$$

says that $E_i\mathcal{L}=E_i(Z_K-D)=0$ for all $E_i\subset|D|$. In particular, $D=C_s'$ for some $0\leq s\leq m$ (by 5.6.), and $D\mathcal{L}=C_s'(-Z_{min})=0$. Since $C_0'(-Z_{min}) = -Z_{min}^2\neq 0$, we must have $s\geq 1$, i.e. $|D|\subset B_1$. On the other hand, (5.39.) guarantees that $Q\notin B_1$, hence $Q\notin|D|$. Therefore, $m_Q\mathcal{L}|_D = \mathcal{O}_{C_s'}(-Z_{min})$ for some $s\geq 1$. Using (5.17.) we obtain $h^1(\mathcal{O}_{C_s'}(-Z_{min})) = h^0(\mathcal{O}_{C_s'}(-Z_{min}))\leq m-s+1\leq p_g-1$, which contradicts $h^1(D,m_Q\mathcal{L}|_D)=p_g$. $\square$

The above lemma implies that there exists $E_{i_0}\subset|D|$ with $E_{i_0}\mathcal{L}+E_{i_0}(Z_K-D)=b_{i_0}=1$. (In particular, either $Q=E_{i_0}\cap E_0'$ and $E_0'\not\subset|D|$, or $E_{i_0}=E_0'$, cf. (5.43.). The first case will be eliminated later.) Summing (5.45.) over all components of D (recall $m_{E_{i_0}}(D)=1$) we obtain $D\mathcal{L}+D(Z_K-D)=1$. Since $D\mathcal{L}\geq 0$ (by the definition of Z_{min}) and $D(Z_K-D)$ is an *even* non–negative integer (by Riemann–Roch), we get:

(5.47.) $$D\mathcal{L} = 1 \text{ and } D(Z_K - D) = 0.$$

If we write $D = \sum D_i$, where $\{|D_i|\}_i$ are the the connected components of $|D|$, since $\chi(D)$=0 and $\chi(D_i) \geq 0$ for all i (cf. 4.4.), we must have $\chi(D_i)$ for all i. This shows that no D_i lies in $E \setminus B_1$ (for all the components of $E \setminus B_1$ support rational singularities whose positive cycles have $\chi > 0$, cf. (5.3.) and (4.1.)). Therefore, all the exceptional divisors $\{E_i'\}_{i=0}^l$ defined in (5.43.) must be components of $|D|$ and $E_0' = E_{i_0}$. Then (5.44.) reads as

(5.48.) $$E_{i_0} Z_{min} < 0.$$

Now, $E_j\mathcal{L} \geq 0$ for all $E_j \subset |D|$ and $D\mathcal{L} = 1$ (cf. (5.47.)), hence there is exactly one index j_0 with $E_{j_0} Z_{min} \leq -1$ (5.48.) guarantees that this j_0 is exactly i_0. Therefore, we get:

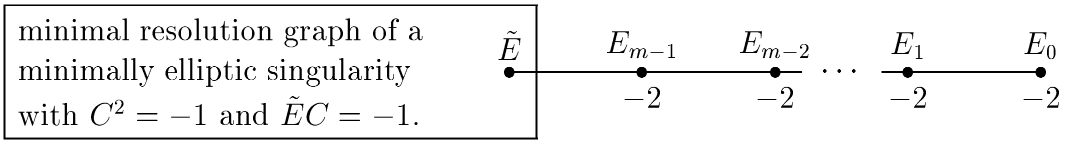
$$\begin{array}{lll} E_{i_0} Z_{min} = -1, & E_{i_0}(Z_K - D) = 0; & \\ E_j Z_{min} = 0, & E_j(Z_K - D) = 0 & \text{for } j \neq i_0;\ E_j \subset |D|. \end{array}$$

Again by (5.6.): $D = C_s'$ for some $0 \leq s \leq m$. Since $Q \in |D|$ but $Q \notin B_1$, the only possibility is $D = Z_K$. This means that $1 = D\mathcal{L} = Z_K(-Z_{min}) = -Z_{min}^2$, which ends the proof of (5.42.).

5.49. The special case $Z_{min}^2 = -1$. All the possible dual (minimal) resolution graphs of elliptic numerical Gorenstein singularities with $Z_{min}^2 = -1$ are classified. This is done by S. S.-T. Yau (in [34], (2.3) and Table 1) based on Laufer's classification of the minimal resolution graphs of the minimally elliptic singularities with $C^2 = -1$ ([16], Table 1.)

5.50. Proposition. [34, 16] *The minimal resolution graph of a numerical Gorenstein elliptic singularity with $m > 0$ and $Z_{min}^2 = -1$ has the form $E = B_1 \cup E_0$, where E_0 is a smooth rational curve with self–intersection -2, and it is unique with $E_0 Z_{min} = -1$. Moreover, $m_{E_0}(Z_{min}) = m_{E_0}(Z_K) = 1$. B_1 supports a singularity with $Z_{B_1}^2 = -1$. The curve E_0 intersects only one irreducible exceptional divisor $\tilde{E}$ of B_1, and it intersects this exactly in one point. The curve $\tilde{E}$ is the unique component of B_1 with the property $\tilde{E} Z_{B_1} = -1$.*

Therefore, by induction, $E = |C| \cup E_{m-1} \cup \cdots \cup E_0$, where all curves $\{E_i\}_{i=0}^{m-1}$ are smooth rational curves with self–intersection -2, $C^2 = -1$, $Z_{B_j} = C + \sum_{i \geq j} E_i$, and the dual graph has the following form:



We present the proof in the following steps. The details are left to the reader.
Step 1. If $Z_{min}^2 = -1$ then $Z_{B_1}^2 = -1$, hence by induction $Z_{B_i}^2 = -1$ for all i.
(Use $(Z_{B_0} - Z_{B_1})(Z_{B_0} + Z_{B_1}) = (Z_{B_0} - Z_{B_1})C_1 \leq 0$ by 5.5.e.)

Step 2. If $Z^2_{B_0} = Z^2_{B_1}$, then for any $E_i \subset |Z_{B_0} - Z_{B_1}|$ one has $E_i Z_K = 0$. In particular, any $E_i \not\subset B_1$ is a smooth rational curve with self–intersection -2 by (2.30.).
(Use $(Z_{B_0} - Z_{B_1})Z_K = (Z_{B_0} - Z_{B_1})(Z_{B_0} + Z_{B_1} + \cdots + Z_{B_m}) = 0$ by 5.5.)
Step 3. If $Z^2_{min} = -1$, there is a unique E_{i_0} with $E_{i_0} Z_{min} < 0$. For this curve $E_{i_0} Z_{min} = -1$ and $m_{E_{i_0}}(Z_{min}) = 1$. (Easy)
Step 4. If $Z^2_{min} = -1$, then there is a unique $E_i \not\subset B_1$ with $E_i \cap B_1 \neq \emptyset$. This is in fact E_{i_0}. (Use $E_i Z_{min} = E_i(Z_K - C'_1) \leq -E_i C'_1 < 0$ and Step 3.)
Step 5. Assume that $Z^2_{min} = -1$. Then the set $J = \{j | j \neq i_0,\ E_j \not\subset B_1$ and E_j intersects $E_{i_0}\}$ is empty.
(Set $\tilde{E} \subset B_1$ with $\tilde{E} \cap E_{i_0} \neq \emptyset$. Write $Z_{min} = \tilde{m}\tilde{E} + E_{i_0} + \sum_{j \in J} m_j E_j$+other terms. Then $-1 = Z_{min} E_{i_0} = \tilde{m} - 2 + \sum_j m_j$. But $\tilde{m} > 0$.)
Step 6. The proof of *Step 6* gives also $m_{\tilde{E}}(Z_{min}) = 1$. This together with $m_{E_{i_0}}(Z_{min}) = 1$ (cf. step 3) imply that $Z_{min} = Z_{B_1} + E_{i_0}$. (Use the definition of the numerical cycle.)
Step 7. $\tilde{E}$ is the unique irreducible divisor of B_1 with $\tilde{E} Z_{B_1} < 0$.
(Use $0 \geq \tilde{E} Z_{min} = \tilde{E}(Z_{B_1} + E_{i_0}) \Rightarrow \ \tilde{E} Z_{B_1} \leq -\tilde{E} E_{i_0} = -1$. But $Z^2_{B_1} = -1$ which gives the unicity.) $\square$

Now, we continue the discussion of the case $Z^2_{min} = -1$.

5.51. Proposition. *Assume that $(X, 0)$ is numerical Gorenstein elliptic singularity with $p_g = m + 1$ ($m > 0$) and $Z^2_{min} = -1$. Consider its minimal resolution graph and write $E = B_1 \cup E_0$ (cf. 5.50.). Then $|\mathcal{O}(-Z_{min})|$ has a unique basepoint $Q \in E_0 \setminus B_1$, and $\mathtt{m}_{X,0}\mathcal{O}_{\tilde{X}} = m_Q \mathcal{O}_{\tilde{X}}(-Z_{min})$.*
Proof. Recall that $|\mathcal{O}(-Z_{min})|$ must have a basepoint (5.40.), and by (5.39.) all the basepoints must lie in $E_0 \setminus B_1$. Consider the map $\alpha_1 : H^0(\mathcal{O}(-Z_{min})) \to H^0(\mathcal{O}_{E_0}(-Z_{min})) = \mathbf{C}^2$ (because $E_0 Z_{min} = -1$, and $\mathcal{O}_{E_0}(-Z_{min}) \approx \mathcal{O}_{\mathbf{P}^1}(1)$, cf. 5.50.). α_1 cannot be onto (because a basepoint exists), and cannot be trivial because there is a section which does not vanish at $E_0 \cap B_1$ (e.g. f_0 given by 5.11.f, cf. also 5.37.). This shows that $im(\alpha_1) = \mathbf{C}$, and there is a unique basepoint (the unique zero in E_0 of f_0). Recall that $m_{E_0}(Z_{min}) = 1$, hence we can consider a small neighbourhood U of Q with local coordinates (x, y), such that $U \cap E_0 = \{x = 0\}$ and f_0 in has local equation xy. Now, if we show that $|\mathcal{O}_{\tilde{X}}(-2Z_{min})|$ is free, then we are done. Indeed, then there is another section with local equation x^2, hence $\mathtt{m}_{X,0}\mathcal{O}_U = (x, y)\mathcal{O}_U(-Z_{min})$.

In the sequel we will show that $|\mathcal{O}_{\tilde{X}}(-2Z_{min})|$ is free. If $E_i \neq E_0$ then obviously $H^0(\mathcal{O}(-2Z_{min})) \to H^0(\mathcal{O}_{E_i}(-2Z_{min})) = \mathbf{C}$ is onto (the image is generated by f_0^2). Consider $H^0(\mathcal{O}_{E_0}(-2Z_{min})) \approx H^0(\mathcal{O}_{\mathbf{P}^1}(2)) = \mathbf{C}^3$ and the diagram:

$$\begin{array}{ccccc}
 & & H^0(\mathcal{O}(-2Z_{min})) & \xrightarrow{\alpha_2} & H^0(\mathcal{O}_{E_0}(-2Z_{min})) \\
 & & \downarrow p & \nearrow \gamma & \\
H^0(\mathcal{O}_{E_0}(-2Z_{min} - Z_K + E_0)) & \xhookrightarrow{\beta} & H^0(\mathcal{O}_{Z_K}(-2Z_{min})) & \longrightarrow & H^0(\mathcal{O}_{Z_K - E_0}(-2Z_{min}))
\end{array}$$

Since p is onto (by vanishing theorem) $im(\alpha_2) = im(\gamma)$. The Chern number $E_0(-2Z_{min} - Z_K + E_0) = 2 - 2\chi(E_0) = 0$, hence $H^0(\mathcal{O}_{E_0}(-2Z_{min} - Z_K + E_0))$ is one–dimensional. Since $m_{E_0}(Z_K) = 1$ (cf. 5.13) $\mathcal{O}_{E_0}(-2Z_{min} - Z_K + E_0) \hookrightarrow \mathcal{O}_{E_0}(-2Z_{min})$, hence $\gamma \circ \beta$ is injective. Let $\tilde{E}$ be in B_1 with $\tilde{E} \cap E_0 \neq \emptyset$. Notice that $m_{\tilde{E}}(Z_K) > m_{\tilde{E}}(Z_{min}) = 1$ (cf. 5.50. step 6; in fact $m_{\tilde{E}}(Z_K) = 2$ from $Z_K E_0 = 0$, cf. 5.50. step 2). Therefore the generator g of $im(\gamma \circ \beta)$ is vanishing at $E_0 \cap \tilde{E}$ with order 2 (in particular, as an element of $\mathcal{O}_{\mathbf{P^1}}(2)$, it has no other zeros). Hence g and $\alpha_2(f_0^2)$ have no common zeros, which proves that $|\mathcal{O}(-2Z_{min})|$ is free. This ends the proof of (5.51.) and (5.38.). $\square$

V. The Hilbert–Samuel function and the embedding dimension.

Assume that $(X, 0)$ is an elliptic Gorenstein singularity with $p_g = m + 1$. Consider its minimal resolution and set $\mathcal{L} := \mathcal{O}_{Z_K}(-Z_{min})$ and $d := -Z_{min}^2$.

5.52. Theorem. *If $d \geq 3$ then $\oplus_{k \geq 0} H^0(Z_K, \mathcal{L}^{\otimes k})$ is generated by elements of degree $k = 1$.*

Proof. The proof is again based on "Castelnuovo's free pencil trick" (cf. the proof of 3.17.).

By (5.38.) there exists $s_0 \in H^0(\mathcal{L})$ with $m_{E_i}(s_0) = m_{E_i}(Z_{min})$ for all E_i. Its divisor has form $Z_{min} + A$, where A is an effective Cartier divisor with $|A| \cap E$ finite, and $E_i A = E_i \mathcal{L}$. Obviously $\mathcal{O}_{Z_K}(A) \to \mathcal{L}$ (given by $f \mapsto f s_0$) is an isomorphism. We use another generic section $s \in H^0(\mathcal{L})$ to identify $\mathcal{L}$ with $\mathcal{O}_{Z_K}$ near A, hence we will write $\mathcal{O}_A$ instead of $\mathcal{L}|_A = \mathcal{O}_A(A)$. By construction, $\mathcal{O}_A$ is an Artinian scheme of degree d. Consider the exact sequence $0 \to \mathcal{O}_{Z_K} \to \mathcal{L} \to \mathcal{O}_A \to 0$, where the second arrow is the multiplication by s_0, (and the third is the "division by s near A"). Its long cohomology exact sequence is:

$$(*_1) \qquad 0 \to H^0(\mathcal{O}_{Z_K}) \xrightarrow{s_0} H^0(\mathcal{L}) \xrightarrow{r_1} \mathcal{O}_A \to H^1(\mathcal{O}_{Z_K}) \to H^1(\mathcal{L}) \to 0.$$

Since $h^1(\mathcal{O}_{Z_K}) = p_g$ and $h^1(\mathcal{L}) = p_g - 1$ (cf. 2.21.b), the rank of r_1 is $d - 1$.

In the following lemmas $A' \subset A$ is an arbitrary subscheme of length $d - 1$. Let $\mathcal{I}_{A'} \subset \mathcal{O}_{Z_K}$ be its ideal sheaf. If $\mathcal{I}_A \subset \mathcal{O}_{Z_K}$ is the ideal sheaf of A, then $\mathcal{I}_A \subset \mathcal{I}_{A'}$ and $\mathcal{I}_{A'}/\mathcal{I}_A = \mathbf{C}_Q$ for some point $Q \in E$.

5.53. Lemma. $h^1(\mathcal{I}_{A'}\mathcal{L}) = p_g - 1$.

Proof. The inclusion $\mathcal{I}_{A'}\mathcal{O}_{Z_K}(A) \subset \mathcal{O}_{Z_K}(A) = \mathcal{L}$ guarantee that $\mathcal{I}_{A'}\mathcal{L}$ is torsion free. Hence the exact sequence $(*)$ $0 \to \mathcal{O}_{Z_K} \xrightarrow{s_0} \mathcal{I}_{A'}\mathcal{L} \to \mathbf{C}_Q \to 0$ does not split. Hence in the exact sequence $0 \to Hom(\mathcal{I}_{A'}\mathcal{L}, \mathcal{O}_{Z_K}) \to Hom(\mathcal{O}_{Z_K}, \mathcal{O}_{Z_K}) \xrightarrow{\alpha} Ext^1(\mathbf{C}_Q, \mathcal{O}_{Z_K}) = \mathbf{C}$ the map α is not trivial. Hence by Serre duality $h^1(\mathcal{I}_{A'}\mathcal{L}) = h^1(\mathcal{O}_{Z_K}) - 1$. $\square$

5.54. Lemma. $H^0(\mathcal{L}) \to \mathcal{O}_{A'}$ *is surjective for any subscheme $A' \subset A$ of length $d - 1$.*

Proof. Use the cohomology exact sequence of $0 \to \mathcal{I}_{A'}\mathcal{L} \to \mathcal{L} \to \mathcal{O}_{A'} \to 0$ and $h^1(\mathcal{I}_{A'}\mathcal{L}) = h^1(\mathcal{L})$ by (5.53.) and 5.12.b). $\square$

Now consider again the sequence $(*_1)$. Recall that $h^0(\mathcal{L}) = p_g + d - 1$. Set a basis $s_0, s_1, \ldots, s_l$ $(l = p_g + d - 2)$ of $H^0(\mathcal{L})$ which extends the basis $s_0, \ldots, s_{p_g-1}$ of $H^0(\mathcal{O}_{Z_K})$. Lemma (6.3) says that $[s_{p_g} : \ldots : s_l]$ embeds A in $\mathbf{P}^{d-2}$ as a subscheme A_d of degree d, with the property that any subscheme of degree $d-1$ spans $\mathbf{P}^{d-2}$. For such a scheme A_d, the restriction map $H^0(\mathbf{P}^{d-2}, \mathcal{O}(2)) \to \mathcal{O}_{A_d}$ is surjective (by Claim 2, [25], page 115). This means that we can choose a set $\{t_1, \ldots, t_d\}$ of quadratic monomials in $\{s_{p_g}, \ldots, s_l\}$, that viewed as sections in $H^0(Z_K, \mathcal{L}^{\otimes 2})$, map to a basis of $\mathcal{O}_A$.

Now, consider the cohomology sequence of $0 \to \mathcal{L} \stackrel{s_0}{\to} \mathcal{L}^{\otimes 2} \stackrel{r_2}{\to} \mathcal{O}_A \to 0$ (where r_2 is the "division by s^2 near A"):

$$(*_2) \qquad 0 \to H^0(\mathcal{L}) \xrightarrow{s_0} H^0(\mathcal{L}^{\otimes 2}) \xrightarrow{r_2} \mathcal{O}_A \to H^1(\mathcal{L}) \to H^1(\mathcal{L}^{\otimes 2}) \to 0.$$

Then $H^0(\mathcal{L}^2) = s_0 \cdot H^0(\mathcal{L}) \oplus \mathbf{C}\langle t_1, \ldots, t_d\rangle$. In other words, $\{s_0 s_i\}_{i=0}^l$ and $\{t_i\}_{i=1}^d$ form a basis of $H^0(\mathcal{L}^2)$. More generally, for $k \geq 1$, in

$$(*_{k+1}) \qquad 0 \to H^0(\mathcal{L}^k) \xrightarrow{s_0} H^0(\mathcal{L}^{k+1}) \xrightarrow{r_{k+1}} \mathcal{O}_A \to H^1(\mathcal{L}^k) \to H^1(\mathcal{L}^{k+1}) \to 0$$

r_{k+1} is onto, $\{s^{k-1}t_i\}_{i=1}^d$ projects to a basis of $\mathcal{O}_A$, and $\{s_0^k s_i\}_{1\leq i\leq l}$, $\{s_0^p s^q t_i\}_{p+q=k-1, 1\leq i\leq d}$ is a basis in $H^0(\mathcal{L}^{k+1})$. □

5.55. Corollary. *For any elliptic Gorenstein singularity with $p_g = m+1$ the following facts hold:*

(a) $H^0(\tilde{X}, \mathcal{O}(-kZ_{min})) = \mathfrak{m}_{X,0}^k$ for any $k \geq 0$ provided that $Z_{min}^2 \leq -3$;

(b) $\dim \mathcal{O}_{X,0}/m_{X,0}^k = \chi(\mathcal{O}_{kZ_{min}}) + 1$, and $\dim \mathfrak{m}_{X,0}^k/\mathfrak{m}_{X,0}^{k+1} = -kZ_{min}^2$ for any $k \geq 1$ provided that $Z_{min}^2 \leq -3$;

(c) emb dim$(X,0) = \dim \mathfrak{m}_{X,0}/\mathfrak{m}_{X,0}^2 = \max(3, -Z_{min}^2)$.

Proof. Consider the diagram:

$$\begin{array}{ccc}
\mathfrak{m}_{X,0}^{\otimes k} = [H^0(\mathcal{O}(-Z_{min}))]^{\otimes k} & \xrightarrow{App2} & [H^0(\mathcal{O}_{Z_K}(-Z_{min}))]^{\otimes k} \\
\downarrow & & \downarrow (5.52.) \\
H^0(\mathcal{O}(-kZ_{min} - Z_K)) \to H^0(\mathcal{O}(-kZ_{min})) & \longrightarrow & H^0(\mathcal{O}_{Z_K}(-kZ_{min}))
\end{array}$$

This diagram and the inclusion $H^0(\mathcal{O}(-kZ_{min} - Z_K)) \subset H^0(\mathcal{O}(-(k+1)Z_{min}))$ shows that $H^0(\mathcal{O}(-kZ_{min})) \subset \mathfrak{m}_{X,0}^k + H^0(\mathcal{O}(-(k+1)Z_{min}))$. Now, by induction: $H^0(\mathcal{O}(-kZ_{min})) \subset \mathfrak{m}_{X,0}^k + H^0(\mathcal{O}(-lZ_{min}))$ for any $l > k$. Since $\oplus_{k\geq 0} H^0(\tilde{X}, \mathcal{O}(-kZ_{min}))$ is finitely generated, (a) follows (cf. also [25], page 115). For (b), consider the exact sequence $0 \to \mathcal{O}(-(k+1)Z_{min}) \to \mathcal{O}(-kZ_{min}) \to \mathcal{O}_{Z_{min}}(-kZ_{min}) \to 0$. Now, by part (a), (5.12.b) and an Euler-characteristic argument $\dim \mathfrak{m}_{X,0}^k/\mathfrak{m}_{X,0}^{k+1} = \chi(\mathcal{O}_{Z_{min}}(-kZ_{min})) = -kZ_{min}^2$. For the Hilbert-Samuel function, consider the exact sequence $0 \to \mathcal{O}(-kZ_{min}) \to \mathcal{O} \to \mathcal{O}_{kZ_{min}} \to 0$ and similar arguments. In order to prove (c) notice that if $Z_{min}^2 \geq -2$, then the multiplicity of $(X,0)$ is 2 (cf. 5.38.). It is well-known, that these germs are hypersurface singularities of type $\{z^2 + f(x,y) = 0\}$. □

5.56. The cases $d = -Z_{min}^2 = 1, 2$. (cf. with [25].) By a careful study of the exact sequences $(*_k)$ in the proof of (5.52.), we can write a generator set of the $\mathcal{O}_{X,0}$–algebra $R := \oplus_{k\geq 0} H^0(\tilde{X}, \mathcal{O}(-kZ_{min}))$ in the case $d \leq 2$ as well:

If $d = 2$, then R is generated by $\mathtt{m}_{X,0}$ in degree 1 and an element $y \in \mathtt{m}_{X,0} \setminus \mathtt{m}_{X,0}^2$ in degree two.

If $d = 1$, then R is generated by $\mathtt{m}_{X,0}$ in degree 1, by an element $y \in \mathtt{m}_{X,0} \setminus \mathtt{m}_{X,0}^2$ in degree 2, and $z \in \mathtt{m}_{X,0} \setminus (\mathtt{m}_{X,0}^2, y)$ in degree 3.

5.57. Remark. (Cf. with [16].) By (5.55.), if $-Z_{min}^2 = 1, 2$ or 3 then $(X, 0)$ is a hypersurface singularity. Moreover, it is possible to verify that if $-Z_{min}^2 = 4$ then $(X, 0)$ is a complete intersection; and for $-Z_{min}^2 > 4$ the singularity $(X, 0)$ is not a complete intersection.

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