The Regularity Lemma and its applications in Graph Theory

János Komlós, Ali Shokoufandeh, Miklós Simonovits
and Endre Szemerédi
Rutgers University
Hungarian Academy of Sciences

Abstract

Szemerédi's Regularity Lemma is an important tool in discrete mathematics. It says that, in some sense, all graphs can be approximated by random-looking graphs. Therefore the lemma helps in proving theorems for arbitrary graphs whenever the corresponding result is easy for random graphs. In the last few years more and more new results were obtained by using the Regularity Lemma, and also some new variants and generalizations appeared. Komlós and Simonovits have written a survey on the topic [96]. The present survey is, in a sense, a continuation of the earlier survey. Here we describe some sample applications and generalizations. To keep the paper self-contained we decided to repeat (sometimes in a shortened form) parts of the first survey, but the emphasis is on new results.

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Preface

The Regularity Lemma [127] is one of the most powerful tools of (extremal) graph theory. It was invented as an auxiliary lemma in the proof of the famous conjecture of Erdős and Turán [53] that sequences of integers of positive upper density must always contain long arithmetic progressions. Its basic content could be described by saying that every graph can, in some sense, be well approximated by random graphs. Since random graphs of a given edge density are much easier to treat than all graphs of the same edge-density, the Regularity Lemma helps us to carry over results that are trivial for random graphs to the class of all graphs with a given number of edges. It is particularly helpful in "fuzzy" situations, i.e., when the conjectured extremal graphs have no transparent structure.

Remark. Sometimes the Regularity Lemma is called Uniformity Lemma, see e.g., [61] and [4].

Notation. In this paper we only consider simple graphs – undirected graphs without loops and multiple edges: $G = (V, E)$ where $V = V(G)$ is the vertex-set of G and $E = E(G) \subset \binom{V}{2}$ is the edge-set of G. $v(G) = |V(G)|$ is the number of vertices in G (order), $e(G) = |E(G)|$ is the number of edges in G (size). G_n will always denote a graph with n vertices. $deg(v)$ is the degree of vertex v and $deg(v, Y)$ is the number of neighbours of v in Y. $\delta(G), \Delta(G)$ and $t(G)$ are the minimum degree, maximum degree and average degree of G. $\chi(G)$ is the chromatic number of G. $N(x)$ is the set of neighbours of the vertex x, and $e(X, Y)$ is the number of edges between X and Y. A bipartite graph G with color-classes A and B and edge-set E will sometimes be written as $G = (A, B, E)$, $E \subset A \times B$. For disjoint X, Y, we define the **density**

$$d(X, Y) = \frac{e(X, Y)}{|X| \cdot |Y|}.$$

$G(U)$ is the restriction of G to U and $G - U$ is the restriction of G to $V(G) - U$. For two disjoint subsets A, B of $V(G)$, we write $G(A, B)$ for the subgraph with vertex set $A \cup B$ whose edges are those of G with one endpoint in A and the other in B.

For graphs G and H, $H \subset G$ means that H is a subgraph of G, but often we will use this in the looser sense that G has a subgraph isomorphic to H (H is embeddable into G), that is, there is a one-to-one map (injection) $\varphi : V(H) \to V(G)$ such that $\{x, y\} \in E(H)$ implies $\{\varphi(x), \varphi(y)\} \in E(G)$. $\|H \to G\|$ denotes the number of labelled copies of H in G. The cardinality of a set S will mostly be denoted by $|S|$, but sometimes we write $\#S$. We will be somewhat sloppy by often disregarding rounding.

1. Introduction

1.1. The structure of this survey

We will start with some historical remarks, then we state the Regularity Lemma. After that we introduce the basic notion of the Reduced Graph of a graph corresponding to a partition of the vertex-set, and state a simple but useful tool (Embedding Lemma). The much stronger version called Blow-up Lemma is mentioned later. The latter has found many applications since [96] was published. (For a short survey on the Blow-up Lemma, see [87].)

We will also touch upon some algorithmic aspects of the Regularity Lemma, its relation to quasi-random graphs and extremal subgraphs of a random graph. We also shortly mention a sparse version.

The results quoted here only serve as illustrations; we did not attempt to write a comprehensive survey. An extended version is planned in the near future.

1.2. Regular pairs

Regular pairs are highly uniform bipartite graphs, namely ones in which the density of any reasonably sized subgraph is about the same as the overall density of the graph.

Definition 1.1 (Regularity condition). Let $\varepsilon > 0$. Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is ε-regular if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

The next one is the most important property of regular pairs.

Fact 1.2 (Most degrees into a large set are large). *Let (A, B) be an ε-regular pair with density d. Then for any $Y \subset B$, $|Y| > \varepsilon|B|$ we have*

$$\#\{x \in A : deg(x, Y) \le (d - \varepsilon)|Y|\} \le \varepsilon|A|.$$

For other basic properties of regular pairs see [96].

We will also use another version of regularity:

Definition 1.3 (Super-regularity). Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is (ε, δ)-super-regular if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$e(X, Y) > \delta|X||Y|,$$

and furthermore,

$$deg(a) > \delta|B| \quad \text{for all} \quad a \in A, \quad \text{and} \quad deg(b) > \delta|A| \quad \text{for all} \quad b \in B.$$

1.3. The Regularity Lemma

The Regularity Lemma says that every dense graph can be partitioned into a small number of regular pairs and a few leftover edges. Since regular pairs behave as random bipartite graphs in many ways, the Regularity Lemma provides us with an approximation of a large dense graph with the union of a small number of random-looking bipartite graphs.

Theorem 1.4 (Regularity Lemma, Szemerédi 1978 [127]). *For every $\varepsilon > 0$ there exists an integer $M = M(\varepsilon)$ with the following property: for every graph G there is a partition of the vertex set into k classes $V = V_1 + V_2 + \ldots + V_k$ such that*

- $k \leq M$,
- $|V_i| \leq \lceil \varepsilon |V| \rceil$ *for every* i,
- $||V_i| - |V_j|| \leq 1$ *for all* i, j *(equipartition),*
- (V_i, V_j) *is* ε*-regular in* G *for all but at most* εk^2 *pairs* (i, j).

The classes V_i will be called **groups** or **clusters**.

If we delete the edges within clusters as well as edges that belong to irregular pairs of the partition, we get a subgraph $G' \subset G$ that is more uniform, more random-looking, and therefore more manageable. Since the number of edges deleted is small compared to $|V|^2$, the Regularity Lemma provides us with a good approximation of G by the random-looking graph G'. Of course, if we have a sequence (G_n) of graphs with $e(G_n) = o(n^2)$, the Regularity Lemma becomes trivial: G_n are approximated by empty graphs. Thus the Regularity Lemma is useful only for **large, dense graphs.**

Remark 1.5. A drawback of the result is that the bound obtained for $M(\varepsilon)$ is extremely large, namely a tower of 2's of height proportional to ε^{-5}. That this is not a weakness of Szemerédi's proof but rather an inherent feature of the Regularity Lemma was shown by Timothy Gowers [70] (see also [9]).

The Regularity Lemma asserts in a way that every graph can be approximated by generalized random graphs.

Definition 1.6 ([118]). Given an $r \times r$ symmetric matrix (p_{ij}) with $0 \leq p_{ij} \leq 1$, and positive integers $n_1, \ldots, n_r$, we define a **generalized random graph** R_n (for $n = n_1 + \ldots + n_r$) by partitioning n vertices into classes V_i of size n_i and then joining the vertices $x \in V_i$, $y \in V_j$ with probability p_{ij}, independently for all pairs $\{x, y\}$.

Remark 1.7. Often, the application of the Regularity Lemma makes things transparent but the same results can be achieved without it equally easily. One would like to know when one can replace the Regularity Lemma with "more elementary" tools and when the application of the Regularity Lemma is unavoidable. The basic experience is that when in the conjectured extremal graphs for a problem the densities in the regular partition are all near to 0 or 1, then the Regularity Lemma can probably be eliminated. On the other hand, if these densities are strictly bounded away from 0 and 1 then the application of the Regularity Lemma is often unavoidable.

1.4. The road to the Regularity Lemma

The following is a basic result in combinatorial number theory.

Theorem 1.8 (van der Waerden 1927 [131]). *Let k and t be arbitrary positive integers. If we color the integers with t colors, at least one color-class will contain an arithmetic progression of k terms.*

A standard compactness argument shows that the following is an equivalent form.

Theorem 1.9 (van der Waerden - finite version). *For any integers k and t there exists an n such that if we color the integers $\{1, \ldots, n\}$ with t colors, then at least one color-class will contain an arithmetic progression of k terms.*

This is a Ramsey type theorem in that it only claims the existence of a given configuration in one of the color classes without getting any control over which class it is. It turns out that the van der Waerden problem is *not* a true Ramsey type question but of a density type: the only thing that matters is that at least one of the color classes contains relatively many elements. Indeed, answering a very deep and difficult conjecture of P. Erdős and P. Turán from 1936 [53], Endre Szemerédi proved that positive upper density implies the existence of an arithmetic progression of k terms.

Theorem 1.10 (Szemerédi 1975 [126]). *For every integer $k > 2$ and $\varepsilon > 0$ there exists a threshold $n_0 = n_0(k, \varepsilon)$ such that if $n \geq n_0$, $A \subset \{1, \ldots, n\}$ and $|A| > \varepsilon n$, then A contains an arithmetic progression of k terms.*

Remark. For $k = 3$ this is a theorem of K.F. Roth [103] that dates back to 1954, and it was already an important breakthrough when Szemerédi succeeded in proving the theorem in 1969 for $k = 4$ [124]. One of the interesting questions in this field is the speed of convergence to 0 of $r_k(n)/n$, where $r_k(n)$ is the maximum size of a subset of $[n]$ not containing an arithmetic progression of length k. Szemerédi's proof used van der Waerden's theorem and therefore gave no reasonable bound on the convergence rate of $r_4(n)/n$. Roth found an analytical proof a little later [104, 105] not using van der Waerden's theorem and thus providing the first meaningful estimates on the convergence rate of $r_4(n)/n$ [104].

Szemerédi's theorem (for general k) was also proved by Fürstenberg [66] in 1977 using ergodic theoretical methods. It was not quite clear first how different the Fürstenberg proof was from that of Szemerédi, but subsequent generalizations due to Fürstenberg and Katznelson [68] and later by Bergelson and Leibman [7] convinced the mathematical community that Ergodic Theory is a natural tool to attack combinatorial questions. The narrow scope of this survey does not allow us to explain these generalizations. We refer the reader to the book of R.L. Graham, B. Rothschild and J. Spencer, *Ramsey Theory* [71], which describes the Hales-Jewett theorem and how these theorems are related, and its chapter "Beyond Combinatorics" gives an introduction into related subfields of topology and ergodic theory. Another good source is the paper of Fürstenberg [67].

2. Early applications

Among the first graph theoretical applications, the Ramsey-Turán theorem for K_4 and the $(6,3)$-theorem of Ruzsa and Szemerédi were proved using (an earlier version of) the Regularity Lemma.

2.1. The $(6,3)$-problem

The $(6,3)$-problem is a special hypergraph extremal problem: Brown, Erdős and T. Sós asked for the determination of the maximum number of hyperedges an r-uniform hypergraph can have without containing ℓ hyperedges the union of which is at most k [17, 16]. One of the simplest cases they could not settle was this $(6,3)$-problem.

Theorem 2.1 (The $(6,3)$-theorem, Ruzsa-Szemerédi 1976 [111]). *If H_n is a 3-uniform hypergraph on n vertices not containing 6 points with 3 or more triples, then $e(H_n) = o(n^2)$.*

(Since the function $M(\varepsilon)$ grows incredibly fast, this would only give an upper bound $r_3(n) = O(n/\log^* n)$, much weaker than Roth's $r_3(n) = O(n/\log\log n)$, let alone the often conjectured $r_3(n) = O(n/\log n)$. The best known upper bound is due to Heath-Brown [80] and to Szemerédi [128] improving Heath-Brown's result, according to which $r_3(n) \le O(n/\log^{1/4-\varepsilon} n)$.)

The $(6,3)$ theorem was generalized by Erdős, Frankl and Rödl as follows. Let $g_r(n,v,e)$ denote the maximum number of r-edges an r-uniform hypergraph may have if the union of any e edges span more than v vertices.

Theorem 2.2 (Erdős-Frankl-Rödl [38]). *For all (fixed) r, $g_r(n, 3r-3, 3) = o(n^2)$.*

For another strengthening of the $(6,3)$ theorem, see [32].

2.2. Applications in Ramsey-Turán theory

Theorem 2.3 (Ramsey-Turán for K_4, Szemerédi 1972 [125]). *If G_n contains no K_4 and only contains $o(n)$ independent vertices, then $e(G_n) < \frac{1}{8}n^2 + o(n^2)$.*

Remark. Since most people believed that in Theorem 2.3 the upper bound $n^2/8$ can be improved to $o(n^2)$, it was quite a surprise when in 1976 Bollobás and Erdős [10] came up with an ingenious geometric construction which showed that the constant $1/8$ in the theorem is best possible. That is, they showed the existence of a graph sequence (H_n) for which

$$K_4 \not\subset H_n, \quad \alpha(H_n) = o(n) \quad \text{and} \quad e(H_n) > \frac{n^2}{8} - o(n^2).$$

Remark. A typical feature of the application of the regularity lemma can be seen above, namely that we do not distinguish between $o(n)$ and $o(m)$, since the number k of clusters is bounded (in terms of ε only) and $m \sim n/k$.

Remark. The problem of determining $\max e(G_n)$ under the condition

$$K_p \not\subset G_n \quad \text{and} \quad \alpha(G_n) = o(n)$$

is much easier for odd p than for even p. A theorem of Erdős and T. Sós [51] describing the odd case was a starting point of the theory of **Ramsey-Turán problems.** The next important contribution was the above-mentioned theorem of Szemerédi (and then the counterpart due to Bollobás and Erdős). Finally the paper of Erdős, Hajnal, T. Sós and Szemerédi [45] completely solved the problem for all even p by generalizing the above Szemerédi-Bollobás-Erdős theorems. It also used the Regularity Lemma.

One reason why the Regularity Lemma can be used here is that if we know that the reduced graph contains some graph L, (e.g., a K_3), then using the $o(n)$-condition we can guarantee a larger subgraph (e.g., a K_4) in the original graph. According to our philosophy, one reason why probably the use of the Regularity Lemma is unavoidable is that the edge-density in the conjectured extremal graph is 1/2; bounded away from 0 and 1.

There are many related Ramsey-Turán theorems; we refer the reader to [43] and [44], or to the survey [121]. The very first Ramsey-Turán type problem can be found in the paper [122] of Vera T. Sós.

2.3. Building small induced subgraphs

While the reduced graph R of G certainly reflects many aspects of G, when discussing induced subgraphs the definition should be changed in a natural way. Given a partition $V_1, \dots, V_k$ of the vertex-set V of G and positive parameters ε, d, we define the induced reduced graph as the graph whose vertices are the clusters $V_1, \dots, V_k$ and V_i and V_j are adjacent if the pair (V_i, V_j) is ε-regular in G with density *between* d and $1 - d$.

Below we will describe an application of the regularity lemma about the existence of small induced subgraphs of a graph, not by assuming that the graph has many edges but by putting some condition on the graph which makes its structure randomlike, fuzzy.

Definition 2.4. A graph $G = (V, E)$ has the property (γ, δ, σ) if for every subset $S \subset V$ with $|S| > \gamma|V|$ the induced graph $G(S)$ satisfies

$$(\sigma - \delta)\binom{|S|}{2} \le e(G(S)) \le (\sigma + \delta)\binom{|S|}{2}.$$

Theorem 2.5 (Rödl 1986 [107]). *For every positive integer k and every $\sigma > 0$ and $\delta > 0$ such that $\delta < \sigma < 1 - \delta$ there exists a γ and a positive integer n_0 such that every graph G_n with $n \ge n_0$ vertices satisfying the property (γ, δ, σ) contains all graphs with k vertices as induced subgraphs.*

Rödl also points out that this theorem yields an easy proof (see [101]) of the following generalization of a Ramsey theorem first proved in [28, 42] and [106]:

Theorem 2.6. *For every graph L there exists a graph H such that for any 2-coloring of the edges of H, H must contain an induced monochromatic L.*

The next theorem of Rödl answers a question of Erdős [8, 36].

Theorem 2.7. *For every positive integer k and positive σ and γ there exists a $\delta > 0$ and a positive integer n_0 such that every graph G_n with at least n_0 vertices having property (γ, δ, σ) contains all graphs with k vertices as induced subgraphs.*

(Erdős asked if the above theorem holds for $\frac{1}{2}, \delta, \frac{1}{2}$ and K_k.)

The reader later may notice the analogy and the connection between this theorem and some results of Chung, Graham and Wilson on quasi-random graphs (see Section 8).

2.4. Diameter-critical graphs

We shall need a notation: If H is an arbitrary graph with vertex set $\{x_1, \ldots, x_k\}$ and $a_1, \ldots, a_k$ are non-negative integers, then $H(a_1, \ldots, a_k)$ denotes the graph obtained from H_k by replacing x_j by a set X_i of a_i independent vertices, and joining each $x \in X_i$ to each $x' \in X_j$ for $1 \le i < j \le k$ exactly if $(x_i, x_j) \in E(H)$.

If $a_1 = a_2 = a_k = t$, then we use the shorter $H(t)$ *

Consider all graphs G_n of diameter 2. The minimum number of edges in such graphs is attained by the star $K(1, n-1)$. There are many results on graphs of diameter 2. An interesting subclass is the class of 2-diameter-critical graphs. These are minimal graphs of diameter 2: deleting any edge we get a graph of diameter greater than 2. The cycle C_5 is one of the simplest 2-diameter-critical graphs. If H is a 2-diameter-critical graph, then $H(a_1, \ldots, a_k)$ is also 2-diameter-critical. So $T_{n,2}$, and more generally of $K(a, b)$, are 2-diameter-critical. Independently, Murty and Simon (see in [21]) formulated the following conjecture:

Conjecture 2.8. *If G_n is a minimal graph of diameter 2, then $e(G) \le \lfloor n^2/4 \rfloor$. Equality holds if and only if G_n is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

Füredi used the Regularity Lemma to prove this.

Theorem 2.9 (Füredi 1992 [65]). *Conjecture 2.8 is true for $n \ge n_0$.*

Here is an interesting point: Füredi did not need the whole strength of the Regularity Lemma, only a consequence of it, the $(6, 3)$-theorem.

3. How to apply the Regularity Lemma

3.1. The Reduced Graph

Given an arbitrary graph $G = (V, E)$, a partition P of the vertex-set V into $V_1, \ldots, V_k$, and two parameters ε, d, we define the **Reduced Graph** (or Cluster graph) R as follows: its

**** Jancsi,tk. nem szeretem ezt a roviditest, mert nem eri meg, inkabb ki szoktam irni: $H(t, \ldots, t)$, itt most teljesen Rad hagyom.

vertices are the clusters $V_1, \dots, V_k$ and V_i is joined to V_j if (V_i, V_j) is ε-regular with density more than d. Most applications of the Regularity Lemma use Reduced Graphs, and they depend upon the fact that many properties of R are inherited by G.

The most important property of Reduced Graphs is mentioned in the following section.

3.2. A useful lemma

Many of the proofs using the Regularity Lemma struggle through similar technical details. These details are often variants of an essential feature of the Regularity Lemma: If G has a reduced graph R and if the parameter ε is small enough, then every small subgraph H of R is also a subgraph of G. In the first applications of the Regularity Lemma the graph H was fixed, but the greedy algorithm outlined in the section "Building up small subgraphs" works smoothly even when the order of H is proportional with that of G as long as H has bounded degrees. (Another standard class of applications - embedding trees into dense graphs - will be discussed later.)

The above mentioned greedy embedding method for bounded degree graphs is so frequently used that, just to avoid repetitions of technical details, it is worth while spelling it out in a quotable form.

For a graph R and positive integer t, let $R(t)$ be the graph obtained from R by replacing each vertex $x \in V(R)$ by a set V_x of t independent vertices, and joining $u \in V_x$ to $v \in V_y$ iff (x, y) is an edge of R. In other words, we replace the edges of R by copies of the complete bipartite graph $K_{t,t}$.

Theorem 3.1 (Embedding Lemma). *Given $d > \varepsilon > 0$, a graph R, and a positive integer m, let us construct a graph G by replacing every vertex of R by m vertices, and replacing the edges of R with ε-regular pairs of density at least d. Let H be a subgraph of $R(t)$ with h vertices and maximum degree $\Delta > 0$, and let $\delta = d - \varepsilon$ and $\varepsilon_0 = \delta^\Delta/(2 + \Delta)$. If $\varepsilon \le \varepsilon_0$ and $t - 1 \le \varepsilon_0 m$, then $H \subset G$. In fact,*

$$\|H \to G\| > (\varepsilon_0 m)^h.$$

Remark. Note that $v(R)$ didn't play any role here.

Remark. Often we use this for R itself (that is, for $t = 1$): If $\varepsilon \le \delta^{\Delta(R)}/(2 + \Delta(R))$ then $R \subset G$, in fact, $\|R \to G\| \ge (\varepsilon m)^{v(R)}$.

Remark. Using the fact that large subgraphs of regular pairs are still regular (with a different value of ε), it is easy to replace the condition $H \subset R(\varepsilon_0 m)$ with the assumptions

(*) every component of H is smaller than $\varepsilon_0 m$,

(**) $H \subset R((1 - \varepsilon_0)m)$.

Most of the classical proofs using the Regularity Lemma can be simplified by the application of the Embedding Lemma. However, this only helps presentability; the original proof ideas –

basically building up subgraphs vertex-by-vertex – are simply summarized in the Embedding Lemma.

One can strengthen the lemma tremendously by proving a similar statement for all bounded degree subgraphs H of the full $R(m)$. This provides a very powerful tool (Blow-up Lemma), and it is described in Section 4.6.

Proof of the Embedding Lemma. We prove the following more general estimate.

$$\text{If} \quad t-1 \le (\delta^{\Delta} - \Delta\varepsilon)m \quad \text{then} \quad \|H \to G\| > \left[(\delta^{\Delta} - \Delta\varepsilon)m - (t-1)\right]^{h}.$$

We embed the vertices $v_1, \dots, v_h$ of H into G by picking them one-by-one. For each v_j not picked yet we keep track of an ever shrinking set C_{ij} that v_j is confined to, and we only make a final choice for the location of v_j at time j. At time 0, C_{0j} is the full m-set v_j is *a priori* restricted to in the natural way. Hence $|C_{0j}| = m$ for all j. The algorithm at time $i \ge 1$ consists of two steps.

Step 1 - *Picking* v_i. We pick a vertex $v_i \in C_{i-1,i}$ such that

$$deg_G(v_i, C_{i-1,j}) > \delta |C_{i-1,j}| \quad \text{for all} \quad j > i \quad \text{such that} \quad \{v_i, v_j\} \in E(H). \tag{1}$$

Step 2. - *Updating the* C_j*'s.* We set, for each $j > i$,

$$C_{ij} = \begin{cases} C_{i-1,j} \cap N(v_i) & \text{if } \{v_i, v_j\} \in E(H) \\ C_{i-1,j} & \text{otherwise.} \end{cases}$$

For $i < j$, let $d_{ij} = \#\{\ell \in [i] : \{v_\ell, v_j\} \in E(H)\}$.

Fact. *If* $d_{ij} > 0$ *then* $|C_{ij}| > \delta^{d_{ij}} m$. (If $d_{ij} = 0$ then $|C_{ij}| = m$.)

Thus, for all $i < j$, $|C_{ij}| > \delta^{\Delta} m \ge \varepsilon m$, and hence, when choosing the exact location of v_i, all but at most $\Delta\varepsilon m$ vertices of $C_{i-1,i}$ satisfy (1). Consequently, we have at least

$$|C_{i-1,i}| - \Delta\varepsilon m - (t-1) > (\delta^{\Delta} - \Delta\varepsilon)m - (t-1)$$

free choices for v_i, proving the claim. ∎

Remark. We did not use the full strength of ε-regularity for the pairs (A, B) of m-sets replacing the edges of H, only the following one-sided property:

$$X \subset A, |X| > \varepsilon |A|, Y \subset B, |Y| > \varepsilon |B| \quad \text{imply} \quad e(X, Y) > \delta |X||Y|.$$

We already mentioned that in a sense the Regularity Lemma says that all graphs can be approximated by generalized random graphs. The following observation was used in the paper of Simonovits and T. Sós [118] to characterize quasi-random graphs.

Theorem 3.2. *Let $\delta > 0$ be arbitrary, and let $V_0, V_1, \dots, V_k$ be a regular partition of an arbitrary graph G_n with $\varepsilon = \delta^2$ and each cluster size less than δn. Let Q_n be the random graph obtained by replacing the edges joining the classes V_i and V_j (for all $i \neq j$) by independently chosen random edges of probability $p_{i,j} := d(V_i, V_j)$, and let H be any graph with ℓ vertices. If $n \geq n_0$, then*

$$\|H \to Q_n\| - C_\ell \delta n^\ell \leq \|H \to G_n\| \leq \|H \to Q_n\| + C_\ell \delta n^\ell.$$

almost surely, where C_ℓ is a constant depending only on ℓ.

Most applications start with applying the Regularity Lemma for a graph G and finding the corresponding Reduced Graph R. Then usually a classical extremal graph theorem (like the König-Hall theorem, Dirac's theorem, Turán's theorem or the Hajnal-Szemerédi theorem) is applied to the graph R. Then an argument similar to the Embedding Lemma (or its strengthened version, the Blow-up Lemma) is used to lift the theorem back to the graph G.

3.3. Some classical extremal graph theorems

This is only a brief overview of the standard results from extremal graph theory most often used in applications of the Regularity Lemma. For a detailed description of the field we refer the reader to [8, 117, 64].

The field of extremal graph theory started with the historical paper of Pál Turán in 1941, in which he determined the minimal number of edges that guarantees the existence of a p-clique in a graph. The following form is somewhat weaker than the original theorem of Turán, but it is perhaps the most usable form.

Theorem 3.3 (Turán 1941 [130]). *If G_n is a graph with n vertices and*

$$e(G) > \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2},$$

then $K_p \subset G_n$.

In general, given a family $\mathcal{L}$ of **excluded** graphs, one would like to find the maximum number of edges a graph G_n can have without containing any subgraph $L \in \mathcal{L}$. This maximum is denoted by $\mathrm{ex}(n, \mathcal{L})$ and the graphs attaining the maximum are called **extremal graphs**. (We will use the notation $\mathrm{ex}(n, \mathcal{L})$ for hypergraphs, too.) These problems are often called Turán type problems, and are mostly considered for **simple graphs or hypergraphs**, but there are also many results for **multigraphs and digraphs** of bounded edge- or arc-multiplicity (see e.g. [13, 14, 15, 18, 114]).

Using this notation, the above form of Turán's theorem says that

$$\mathrm{ex}(n, K_p) \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$

The following theorem of Erdős and Stone determines $\mathrm{ex}(n, K_p(t, \dots, t))$ asymptotically.

Theorem 3.4 (Erdős-Stone 1946 [52] - Weak Form). *For any integers $p \geq 2$ and $t \geq 1$,*

$$\mathrm{ex}(n, K_p(t, \ldots, t)) = \left(1 - \frac{1}{p-1}\right)\binom{n}{2} + o(n^2).$$

(For strengthened versions, see [25, 26].) This is, however, much more than just another Turán type extremal result. As Erdős and Simonovits pointed out in [46], it implies the general asymptotic description of $\mathrm{ex}(n, \mathcal{L})$.

Theorem 3.5. *If $\mathcal{L}$ is finite and $\min_{L \in \mathcal{L}} \chi(L) = p > 1$, then*

$$\mathrm{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p-1}\right)\binom{n}{2} + o(n^2).$$

So this theorem plays a crucial role in extremal graph theory. (For structural generalizations for arbitrary $\mathcal{L}$ see [33, 34, 115].)

The proof of the Embedding Lemma gives the following quantitative form (see also Frankl-Pach [60], and [118]).

Theorem 3.6 (Number of copies of H). *Let H be a graph with h vertices and chromatic number p. Let $\beta > 0$ be given and write $\varepsilon = (\beta/6)^h$. If a graph G_n has*

$$e(G_n) > \left(1 - \frac{1}{p-1} + \beta\right)\frac{n^2}{2}$$

then

$$\|H \to G_n\| > \left(\frac{\varepsilon n}{M(\varepsilon)}\right)^h.$$

It is interesting to contrast this with the following peculiar fact observed by Füredi. If a graph has few copies of a sample graph (e.g., few triangles), then they can all be covered by a few edges:

Theorem 3.7 (Covering copies of H). *For every $\beta > 0$ and sample graph H there is a $\gamma = \gamma(\beta, H) > 0$ such that if G_n is a graph with at most $\gamma n^{v(H)}$ copies of H, then by deleting at most βn^2 edges one can make G_n H-free.*

The above mentioned theorems can be proved directly without the Regularity Lemma, e.g., using sieve-type formulas, see [97, 98, 48, 18].

4. Building subgraphs

4.1. Building small subgraphs

It is well-known that a random graph G_n with fixed edge-density $p > 0$ contains any fixed graph H almost surely (as $n \to \infty$). In some sense this is trivial: we can build up this H

vertex by vertex. If we have already fixed ℓ vertices of H then it is easy to find an appropriate ($\ell + 1$-th vertex with the desired connections. The Regularity Lemma (and an application of the Embedding Lemma) achieves the same effect for dense graphs.

4.2. Packing with small graphs

The Alon-Yuster conjecture

The conjecture of Noga Alon and Raphael Yuster [4] generalizes the Hajnal-Szemerédi theorem [73] from covering with cliques to covering with copies of an arbitrary graph H:

Conjecture 4.1 (Alon-Yuster). *For every graph H there is a constant K such that*

$$\delta(G_n) \geq \left(1 - \frac{1}{\chi(H)}\right) n$$

implies that G_n contains a union of vertex-disjoint copies of H covering all but at most K vertices of G_n.

A simple example in [4] shows that $K = 0$ cannot always be achieved even when $v(H)$ divides $v(G)$. After approximate results of Alon and Yuster [4, 5], an exact solution for large n has been given in [95].

Komlós [88] has fine-tuned these covering questions by finding a different degree condition that is (asymptotically) necessary and sufficient. It uses the following quantity:

Definition 4.2. For an r-chromatic graph H on h vertices we write $\sigma = \sigma(H)$ for the smallest possible color-class size in any r-coloring of H. The **critical chromatic number** of H is the number

$$\chi_{cr}(H) = (r-1)h/(h-\sigma).$$

Theorem 4.3 (Tiling Turán Theorem [88]). *For every graph H and $\varepsilon > 0$ there is a threshold $n_0 = n_0(H, \varepsilon)$ such that, if $n \geq n_0$ and a graph G_n satisfies the degree condition*

$$\delta(G_n) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n,$$

then G_n contains an H-matching that covers all but at most εn vertices.

4.3. Embedding trees

So far all embedding questions we discussed dealt with embedding bounded degree graphs H into dense graphs G_n. General Ramsey theory tells us that this cannot be relaxed substantially without putting strong restrictions on the structure of the graph H. (Even for bipartite H, the largest complete bipartite graph $K_{\ell,\ell}$ that a dense graph G_n can be expected to have is for $\ell = O(\log n)$.) A frequently used structural restriction on H is that it is a tree (or a forest). Under this strong restriction even very large graphs H can be embedded into dense graphs G_n.

The two extremal cases are when H is a large star, and when H is a long path. Both cases are precisely and easily handled by classical extremal graph theory (Turán theory or Ramsey theory). The use of the Regularity Lemma makes it possible, in a sense, to reduce the case of general trees H to these two special cases by splitting the tree into "long" and "wide" pieces. After an application of the Regularity Lemma one applies, as always, some classical graph theorem, which in most cases is the König-Hall matching theorem, or the more sophisticated Tutte's theorem (more precisely, the Gallai-Edmonds decomposition).

The Erdős-Sós conjecture for trees

Conjecture 4.4 (Erdős-Sós 1963 [50]). *Every graph on n vertices and more than $(k-1)n/2$ edges contains, as subgraphs, all trees with k edges.*

In other words, if the number of edges in a graph G forces the existence of a k-star, then it also guarantees the existence of every other subtree with k edges. The theorem is known for k-paths (Erdős-Gallai 1959 [40]).

This famous conjecture spurred much activity in graph theory in the last 30 years.

Remark. The assertion is trivial if we are willing to put up with loosing a factor of 2: If G has average degree at least $2k-2>0$, then it has a subgraph G' with $\delta(G') \ge k$, and hence the greedy algorithm guarantees that G' contains all k-trees.

Using an *ad hoc* sparse version of the Regularity Lemma, Ajtai, Komlós, Simonovits and Szemerédi solved the Erdős-Sós conjecture for large n. ([1], in preparation.)

The Loebl conjecture

In their paper about graph discrepancies P. Erdős, Z. Füredi, M. Loebl and V. T. Sós [39] reduced some questions to the following conjecture of Martin Loebl:

Conjecture 4.5 (Loebl Conjecture). *If G is a graph on n vertices, and at least $n/2$ vertices have degrees at least $n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

J. Komlós and V. T. Sós generalized Loebl's conjecture for trees of any size. It says that any graph G contains all trees with size not exceeding the medium degree of G.

Conjecture 4.6 (Loebl-Komlós-Sós Conjecture). *If G is a graph on n vertices, and at least $n/2$ vertices have degrees greater than or equal to k, then G contains, as subgraphs, all trees with k edges.*

In other words, the condition in the Erdős-Sós conjecture that the *average* degree be greater than $k-1$, would be replaced here with a similar condition on the *median* degree.

This general conjecture is not easier than the Erdős-Sós conjecture. Large instances of both problems can be attacked with similar methods.

4.4. Embedding large bipartite subgraphs

The following theorem is implicit in Chvátal-Rödl-Szemerédi-Trotter 1983 [24] (according to [2]).

Theorem 4.7. *For any $\Delta, \beta > 0$ there is a $c > 0$ such that if $e(G_n) > \beta n^2$, then G_n contains as subgraphs all bipartite graphs H with $|V(H)| \leq cn$ and $\Delta(H) \leq \Delta$.*

4.5. Embedding bounded degree spanning subgraphs

This is probably the most interesting class of embedding problems. Here the proofs (when they exist) are too complicated to quote here, but they follow a general pattern. When embedding H to G (they have the same order now!), we first prepare H by chopping it into (a constant number of) small pieces, then prepare the host graph G by finding a regular partition of G, throw away the usual atypical edges, and define the reduced graph R. Then typically we apply to R the matching theorem (for bipartite H) or the Hajnal-Szemerédi theorem (for r-partite H). At this point, we make an assignment between the small pieces of H and the "regular r-cliques" of the partitioned R. There are two completely different problems left. Make the connections between the r-cliques, and embed a piece of H into an r-clique. The first one is sometimes easy, sometimes very hard, but there is no general recipe to apply here. The second part, however, can typically be handled by referring to the so-called Blow-up Lemma - a new general purpose embedding tool discussed below.

The Pósa-Seymour conjecture

Paul Seymour conjectured in 1973 that any graph G of order n and minimum degree at least $\frac{k}{k+1}n$ contains the k-th power of a Hamiltonian cycle. For $k = 1$, this is just Dirac's theorem. For $k = 2$, the conjecture was made by Pósa in 1962. Note that the validity of the general conjecture would imply the notoriously hard Hajnal-Szemerédi theorem.

For partial results, see the papers [58, 54, 55, 57, 56]. (Fan and Kierstead also announced a proof of the Pósa conjecture if the Hamilton cycle is replaced by Hamilton path.) We do not detail the statements in these papers, since they do not employ the Regularity Lemma.

The Seymour conjecture was proved in [94] for every fixed k and large n.

4.6. The Blow-up Lemma

Several recent results exist about embedding spanning graphs into dense graphs. Some of the proofs use the following new powerful tool. It basically says that regular pairs behave as complete bipartite graphs from the point of view of embedding bounded degree subgraphs. Note that for embedding spanning subgraphs, one needs all degrees of the host graph to be large. That's why using regular pairs is not sufficient any more, we need super-regular pairs. The Blow-up Lemma plays the same role in embedding spanning graphs H into G as the Embedding Lemma played in embedding smaller graphs H (up to $v(H) < (1 - \varepsilon)v(G)$).

Theorem 4.8 (Blow-up Lemma - Komlós-Sárközy-Szemerédi 1994 [91]). *Given a graph R of order r and positive parameters δ, Δ, there exists an $\varepsilon > 0$ such that the following holds. Let $n_1, n_2, \ldots, n_r$ be arbitrary positive integers and let us replace the vertices of R with pairwise disjoint sets $V_1, V_2, \ldots, V_r$ of sizes $n_1, n_2, \ldots, n_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup V_i$. The first graph $\mathbf{R}$ is obtained by replacing each edge $\{v_i, v_j\}$ of R with the complete bipartite graph between the corresponding vertex-sets V_i and V_j. A sparser graph G is constructed by replacing each edge $\{v_i, v_j\}$ with an (ε, δ)-super-regular pair between V_i and V_j. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $\mathbf{R}$ then it is already embeddable into G.*

The proof of the Blow-up Lemma starts with a probabilistic greedy algorithm, and then uses a König-Hall argument to finish the embedding. The proof of correctness is quite involved, and we will not present it here.

5. Applications in Ramsey Theory

5.1. The milestone

The following theorem is central in Ramsey theory. It says that the Ramsey number of a bounded degree graph is linear in the order of the graph. In other words, there is a function f such that the graph-Ramsey number $r(H)$ of *any* graph H satisfies $r(H) \leq f(\Delta(H))v(H)$. This was probably the first deep application of the Regularity Lemma, and certainly a milestone in its becoming a standard tool.

Theorem 5.1 (Chvátal-Rödl-Szemerédi-Trotter 1983 [24]). *For any $\Delta > 0$ there is a $c > 0$ such that if G_n is any n-graph, and H is any graph with $|V(H)| \leq cn$ and $\Delta(H) \leq \Delta$, then either $H \subset G_n$ or $H \subset \overline{G_n}$.*

5.2. Graph-Ramsey

The following more recent theorems also apply the Regularity Lemma.

Theorem 5.2 (Haxell-Luczak-Tingley 1999 [79]). *Let T_n be a sequence of trees with color-class sizes $a_n \geq b_n$, and let $M_n = \max\{2a_n, a_n + 2b_n\} - 1$ (the trivial lower bound for the Ramsey number $r(T_n)$). If $\Delta(T_n) = o(a_n)$ then $r(T_n) = (1 + o(1))M_n$.*

Theorem 5.3 (Łuczak 1999 [99]). *$R(C_n, C_n, C_n) \leq (3 + o(1))n$ for all even n, and $R(C_n, C_n, C_n) \leq (4 + o(1))n$ for all odd n.*

5.3. Random Ramsey

Given graphs $H_1, \ldots, H_r$ and G, we write $G \to (H_1, \ldots, H_r)$ if for every r-coloring of the edges of G there is an i such that G has a subgraph of color i isomorphic to H_i ('arrow notation'). The typical Ramsey question for random graphs is then the following. What is the threshold edge probability $p = p(n)$ for which $G(n,p) \to (H_1, \dots, H_r)$ has a probability close to 1.

Rödl and Ruciński [110] answered this in the symmetric case $G_1 = \dots = G_r$. A first step toward a general solution was taken by Kohayakawa and Kreuter [83] who used the Regularity Lemma to find the threshold when each G_i is a cycle.

6. New versions of the Regularity Lemma

6.1. The Frieze-Kannan version

Alan Frieze and Ravi Kannan [62] use a matrix decomposition that can replace the Regularity Lemma in many instances, and creates a much smaller number of parts. The authors describe their approximation algorithm as follows:

Given an $m \times n$ matrix A with entries between -1 and 1, say, and an error parameter ε between 0 and 1, a matrix D is found (by a probabilistic algorithm) which is the sum of $O(1/\varepsilon^2)$ simple rank 1 matrices so that the sum of entries of any submatrix (among the 2^{m+n}) of $(A - D)$ is at most εmn in absolute value. The algorithm takes time dependent only on ε and the allowed probability of failure (but not on m, n).

The rank one matrices in the Frieze-Kannan decomposition correspond to regular pairs in the Regularity Lemma, but the global error term $o(mn)$ is much larger than the one in Szemerédi's theorem. That explains the reasonable sizes ($O(1/\varepsilon^2)$ instead of tower functions).

The decomposition is applied to various standard graph algorithms such as the Max-Cut problem, the Minimum Linear Arrangement problem, and the Maximum Acyclic Subgraph problem, as well as to get quick approximate solutions to systems of linear equations and systems of linear inequalities (Linear Programming feasibility).

The results are also extended from 2-dimensional matrices to r-dimensional matrices.

6.2. A sparse-graph version of the Regularity Lemma

It would be very important to find extensions of the Regularity Lemma for sparse graphs, e.g., for graphs where we assume only that $e(G_n) > cn^{2-\alpha}$ for some positive constants c and α. Y. Kohayakawa [81] and V. Rödl [108] independently proved a version of the Regularity Lemma in 1993 that can be regarded as a Regularity Lemma for sparse graphs. As Kohayakawa puts it: "Our result deals with subgraphs of pseudo-random graphs." He (with co-authors) has also found some interesting applications of this theorem in Ramsey theory and in Anti-Ramsey theory, (see e.g. [75, 76, 77, 78, 84, 86, 83]).

To formulate the Kohayakawa-Rödl Regularity Lemma we need the following definitions.

Definition 6.1. A graph $G = G_n$ is (P_0, η)-uniform for a partition P_0 of $V(G_n)$ if for some $p \in [0, 1]$ we have

$$|e_G(U, V) - p|U||V|| \le \eta p|U||V|,$$

whenever $|U|, |V| > \eta n$ and either P_0 is *trivial*, U, V are disjoint, or U, V belong to different parts of P_0.

Definition 6.2. A partition $Q = (C_0, C_1, \ldots, C_k)$ of $V(G_n)$ is (ε, k)-equitable if $|C_0| < \varepsilon n$ and $|C_1| = \ldots = |C_k|$.

Notation.

$$d_{H,G}(U, V) = \begin{cases} e_H(U, V)/e_G(U, V) & \text{if } e_G(U, V) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6.3. We call a pair (U, V) (ε, H, G)-regular if for all $U' \subset U$ and $W' \subset W$ with $|U'| \ge \varepsilon|U|$ and $|W'| \ge \varepsilon|W|$, we have

$$|d_{H,G}(U, W) - d_{H,G}(U', W')| \le \varepsilon.$$

Theorem 6.4 (Kohayakawa 1993 [81]). *Let ε and $k_0, \ell > 1$ be fixed. Then there are constants $\eta > 0$ and $K_0 > k_0$ with the following properties. For any (P_0, η)-uniform graph $G = G_n$, where $P_0 = (V_i)_i^\ell$ is a partition of $V = V(G)$, if $H \subset G$ is a spanning subgraph of G, then there exists an (ε, H, G)-regular, (ε, k)-equitable partition of V refining P_0, with $k \le k_0 \le K_0$.*

For more information, see Kohayakawa 1997 [82].

7. Algorithmic questions

The Regularity Lemma is used in two different ways in computer science. Firstly, it is used to prove the existence of some special subconfigurations in given graphs of positive edge-density. Thus by turning the lemma from an existence-theorem into an algorithm one can transform many of the earlier existence results into relatively efficient algorithms. The first step in this direction was made by Alon, Duke, Leffman, Rödl and Yuster [2] (see below). Frieze and Kannan [63] offered an alternative way for constructing a regular partition based on a simple lemma relating non-regularity and largeness of singular values.

In the second type of use, one takes advantage of the fact that the regularity lemma provides a random-like substructure of any dense graph. We know that many algorithms fail on randomlike objects. Thus one can use the Regularity Lemma to prove lower bounds in complexity theory, see e.g., W. Maass and Gy. Turán [72]. One of these randomlike objects is the expander graph, an important structure in Theoretical Computer Science.

7.1. Two applications in computer science

A. Hajnal, W. Maass and Gy. Turán applied the Regularity Lemma to estimate the communicational complexity of certain graph properties [72]. We quote their abstract:

"We prove $\Theta(n \log n)$ bounds for the deterministic 2-way communication complexity of the graph properties CONNECTIVITY, s,t-CONNECTIVITY and BIPARTITENESS. ... The bounds imply improved lower bounds for the VLSI complexity of these decision problems and sharp bounds for a generalized decision tree model that is related to the notion of evasiveness."

Another place where the Regularity Lemma is used in estimating communicational complexity is an (electronic) paper of Pudlák and Sgall [102]. In fact, they only use the (6,3)-problem, i.e., the Ruzsa-Szemerédi theorem.

7.2. An algorithmic version of the Regularity Lemma

The Regularity Lemma being so widely applicable, it is natural to ask if for a given graph G_n and given $\varepsilon > 0$ and m one can find an ε-regular partition of G in time polynomial in n. The answer due to Alon, Duke, Lefmann Rödl and Yuster [2] is surprising, at least at first: Given a graph G, we can find regular partitions in polynomially many steps, however, if we describe this partition to someone else, he cannot verify in polynomial time that our partition is really ε-regular: he has better produce his own regular partition. This is formulated below:

Theorem 7.1. *The following decision problem is co-NP complete: Given a graph G_n with a partition $V_0, V_1, \ldots, V_k$ and an $\varepsilon > 0$. Decide if this partition is ε-regular in the sense guaranteed by the Regularity Lemma.*

Let $Mat(n)$ denote the time needed for the multiplication of two $(0,1)$ matrices of size n.

Theorem 7.2 (Constructive Regularity Lemma). *For every $\varepsilon > 0$ and every positive integer $t > 0$ there exists an integer $Q = Q(\varepsilon, t)$ such that every graph with $n > Q$ vertices has an ε-regular partition into $k+1$ classes for some $k < Q$ and such a partition can be found in $O(Mat(n))$ sequential time. The algorithm can be made parallel on an EREW with polynomially many parallel processors, and it will have $O(\log n)$ parallel running time.*

7.3. Counting subgraphs

Duke, Lefmann and Rödl [30] used a variant of the Regularity Lemma to design an efficient approximation algorithm which, given a labelled graph G on n vertices and a list of all the labelled graphs on k vertices, provides for each graph H in the list an approximation to the number of induced copies of H in G with small total error.

8. Regularity and randomness

8.1. Extremal subgraphs of random graphs

Answering a question of P. Erdős, L. Babai, M. Simonovits and J. Spencer [6] described the Turán type extremal graphs for random graphs:

> Given an excluded graph L and a probability p, take a random graph R_n of edge-probability p (where the edges are chosen independently) and consider all its subgraphs F_n not containing L. Find the maximum of $e(F_n)$.

Below we formulate four theorems. The first one deals with the simplest case.

We will use the expression "almost surely" in the sense "with probability $1-o(1)$ as $n \to \infty$". In this part a p-random graph means a random graph of edge-probability p where the edges are chosen independently.

Theorem 8.1. *Let $p = 1/2$. If R_n is a p-random graph and F_n is a K_3-free subgraph of R_n containing the maximum number of edges, and B_n is a bipartite subgraph of R_n having maximum number of edges, then $e(B_n) = e(F_n)$. Moreover, F_n is almost surely bipartite.*

Definition 8.2 (Critical edges). Given a k-chromatic graph L, an edge e is **critical** if $L - e$ is $k-1$-chromatic.

Many theorems valid for complete graphs were generalized to arbitrary L having critical edges (see e.g., [116]). Theorem 8.1 also generalizes to every 3-chromatic L containing a critical edge e, and for every probability $p > 0$.

Theorem 8.3. *Let L be a fixed 3-chromatic graph with a critical edge e (i.e., $\chi(L-e) = 2$). There exists a function $f(p)$ such that if $p \in (0,1)$ is given and $R_n \in \mathbf{G}(p)$, and if B_n is a bipartite subgraph of R_n of maximum size and F_n is an L-free subgraph of maximum size, then*

$$e(B_n) \le e(F_n) \le e(B_n) + f(p)$$

almost surely, and almost surely we can delete $f(p)$ edges of F_n so that the resulting graph is already bipartite. Furthermore, there exists a $p_0 < 1/2$ such that if $p \ge p_0$, then F_n is bipartite: $e(F_n) = e(B_n)$.

Theorem 8.3 immediately implies Theorem 8.1. The main point in Theorem 8.3 is that the observed phenomenon is valid not just for $p = 1/2$, but for slightly smaller values of p as well.

If $\chi(L) = 3$ but we do not assume that L has a critical edge, then we get similar results, having slightly more complicated forms. Here we formulate only some weaker results.

Theorem 8.4. *Let L be a given 3-chromatic graph. Let $p \in (0,1)$ be fixed and let R_n be a p-random graph. Let $\omega(n) \to 0$ as $n \to \infty$. If B_n is a bipartite subgraph of R_n of maximum size and F_n contains only $\omega(n) \cdot n^{v(L)}$ copies of L and has maximum size under this condition, then almost surely*

$$e(B_n) \le e(F_n) \le e(B_n) + o(n^2)$$

and we can delete $o(n^2)$ edges of F_n so that the resulting graph is already bipartite.

The above results also generalize to r-chromatic graphs L.

Some strongly related important results are hidden in the paper of Haxell, Kohayakawa and Łuczak [77].

8.2. Quasirandomness

Quasi-random structures have been investigated by several authors, among others, by Thomason [129], Chung, Graham, Wilson, [23]. For graphs, Simonovits and T. Sós [118] have shown that quasi-randomness can also be characterized by using the Regularity Lemma. Fan Chung [22] generalized their results to hypergraphs.

Let $N_G^*(L)$ and $N_G(L)$ denote the number of induced and not necessarily induced copies of L in G, respectively. Let $S(x,y)$ be the set of vertices joined to both x and y in the same way. First we formulate a theorem of Chung, Graham, and Wilson, in a shortened form.

Theorem 8.5 (Chung-Graham-Wilson [23]). *For any graph sequence (G_n) the following properties are equivalent:*

$\mathbf{P}_1(\nu)$: *for fixed ν, for all graphs H_ν*

$$N_G^*(H_\nu) = (1+o(1))n^\nu 2^{-\binom{\nu}{2}}.$$

$\mathbf{P}_2(t)$: *Let C_t denote the cycle of length t. Let $t \geq 4$ be even.*

$$e(G_n) \geq \frac{1}{4}n^2 + o(n^2) \qquad \text{and} \qquad N_G(C_t) \leq \left(\frac{n}{2}\right)^t + o(n^t).$$

$\mathbf{P}_5$: *For each subset $X \subset V$, $|X| = \lfloor \frac{n}{2} \rfloor$ we have $e(X) = \left(\frac{1}{16}n^2 + o(n^2)\right)$.*

$\mathbf{P}_6$: $\sum_{x,y\in V} \left| |S(x,y)| - \frac{n}{2} \right| = o(n^3)$.

Graphs satisfying these properties are called **quasirandom**. Simonovits and T. Sós formulated a graph property which proved to be equivalent with the above properties.

$\mathbf{P}_S$: For every $\varepsilon > 0$ and κ there exist two integers, $k(\varepsilon,\kappa)$ and $n_0(\varepsilon,\kappa)$ such that for $n \geq n_0$, G_n has a regular partition with parameters ε and κ and k classes $U_1,\dots,U_k$, with $\kappa \leq k \leq k(\varepsilon,\kappa)$, so that

$$(U_i,U_j) \text{ is } \varepsilon-\text{regular, and} \qquad \left| d(U_i,U_j) - \frac{1}{2} \right| < \varepsilon$$

holds for all but at most εk^2 pairs (i,j), $1 \leq i,j \leq k$.

It is easy to see that if (G_n) is a random graph sequence of probability $1/2$, then $\mathbf{P}_S$ holds for (G_n), almost surely. Simonovits and T. Sós [118] proved that $\mathbf{P}_S$ is a quasi-random property, i.e. $\mathbf{P}_S \Longleftrightarrow \mathbf{P}_i$ for all the above properties P_i.

8.3. Hereditarily extended properties

Randomness is a hereditary property: large subgraphs of random graphs are fairly random-like. In [119] and [120] Simonovits and T. Sós proved that some properties which are not quasi random, become quasirandom if one extends them to hereditary properties. This "extension" means that the properties are assumed not only for the whole graph but for all sufficiently large subgraphs. Their most interesting results were connected with counting some small subgraphs $L \subseteq G_n$.

Obviously, $\mathbf{P}_1(\nu)$ of Theorem 8.5 says that the graph G_n contains each subgraph with the same frequency as a random graph.

Let $\nu = v(L)$, $E = e(L)$. Denote by $\beta_L(p)$ and $\gamma_L(p)$ the "densities" of **labelled induced** and **labelled not necessarily induced** copies of L in a p-random graph:

$$\beta_L(p) = p^E(1-p)^{\binom{\nu}{2}-E} \quad \text{and} \quad \gamma_L(p) = p^E. \tag{1}$$

Theorem 8.6 (Simonovits-Sós). *Let L_ν be a fixed sample-graph, $e(L) > 0$, $p \in (0,1)$ be fixed. Let (G_n) be a sequence of graphs. If (for every sufficiently large n) for every induced $F_h \subseteq G_n$,*

$$\mathbf{N}(L_\nu \subseteq F_h) = \gamma_L(p)h^\nu + o(n^\nu), \tag{2}$$

then (G_n) is p-quasi-random.

Observe that in (2) we used $o(n^\nu)$ instead of $o(h^\nu)$, i.e., for small values of h we allow a relatively much larger error-term. As soon as $h = o(n)$, this condition is automatically fulfilled.

For "Induced Copies" the situation is much more involved, because of the lack of monotonicity. Below we shall always exclude $e(L_\nu) = 0$ and $e(\overline{L}_\nu) = 0$. One would like to know if for given (L_ν, p) the following is true or not:

(#) *Given a sample graph L_ν and a probability p, if for a graph graph sequence (G_n) for every induced subgraph F_h of G_n*

$$\mathbf{N}^*(L_\nu \subseteq F_h) = \beta_L(p)h^\nu + o(n^\nu) \tag{3}$$

then (G_n) is p-quasi-random.

(#) is *mostly false* in this form, for two reasons:

- the probabilities are in **conjugate pairs**;
- There may occur strange **algebraic coincidences**.

Clearly, $\beta_L(p)$ (in (1)) is a function of p which is monotone increasing in $\big[0, e(L_\nu)/\binom{\nu}{2}\big]$, monotone decreasing in $\big[e(L_\nu)/\binom{\nu}{2}, 1\big]$ and vanishes in $p = 0$ and in $p = 1$. For every $p \in \big(0, e(L_\nu)/\binom{\nu}{2}\big)$ there is a *unique* probability $\overline{p} \in \big(e(L_\nu)/\binom{\nu}{2}, 1\big)$ yielding the same expected value. Therefore the hereditarily assumed number of induced copies does not determine the probability uniquely, unless $p = e(L_\nu)/\binom{\nu}{2}$. Given a graph L_ν, the probabilities p and $\overline{p}$ are called **conjugate** if $\beta_L(p) = \beta_L(\overline{p})$. We can mix two such sequences: (G_n) obviously satisfies (3) if

(*) (G_n) is the union of a p-quasi random graph sequence and a $\overline{p}$-quasi random graph sequence.

One can create such sequences for P_3 or its complementary graph. Simonovits and Sós think that there are no other real counterexamples:

Conjecture 8.7. *Let L_ν be fixed, $\nu \geq 4$ and $p \in (0,1)$. Let (G_n) be a graph sequence satisfying (3). Then (G_n) is the union of two sequences, one being p-quasi-random, the other $\overline{p}$-quasi-random (where one of these two sequences may be finite, or even empty).*

To formulate the next Simonovits-Sós theorem we use

Construction 8.8 (Two class generalized random graph). *Define the graph $G_n = G(V_1, V_2, p, q, s)$ as follows: $V(G_n) = V_1 \cup V_2$. We join independently the pairs in V_1 with probability p, in V_2 with probability q and the pairs (x,y) for $x \in V_1$ and $y \in V_2$ with probability s.*

Theorem 8.9 (Two-class counterexample). *If there is a sequence (G_n) which is a counterexample to Conjecture 8.7 for a fixed sample graph L and a probability $p \in (0,1)$, then there is also a 2-class generalized random counterexample graph sequence of form $G_n = G(V_1, V_2, p, q, s)$ with $|V_1| \approx n/2$, $p \in (0,1)$, $s \neq p$. (Further, either $q = p$ or $q = \overline{p}$.)*

This means that if there are counterexamples then those can be found by solving some systems of algebraic equations. The proof of this theorem heavily uses the regularity lemma.

Theorem 8.10. *If L_ν is* **regular***, then Conjecture 8.7 holds for L_ν and any $p \in (0,1)$.*

For some further results of Simonovits and T. Sós for induced subgraphs see [120].

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