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Extremal Multigraph and Digraph Problems

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We survey work done jointly with P. Erdős — some not yet published — which generalizes familiar "Turán-type" extremal theorems to digraphs and multigraphs. Some of the phenomena encountered here are direct generalizations of ordinary Turán type extremal problems but most of them have no counterpart in the theory for ordinary graphs. The second part, contains a short survey of some related fields.

1. Introduction

The Universes: $\mathbb{U}$, $\overrightarrow{\mathbb{D}}$, $\overrightarrow{\mathbb{O}}$

In this paper loops are always excluded! When speaking of an **extremal graph problem**, we shall have several settings, in each of which a "universe" of graphs is fixed, like

- $\mathbb{U}_1$: "simple graphs": no loops, no multiple edges,
- $\mathbb{U}_q$: "multigraphs" with maximum edge-multiplicity q,
- $\overrightarrow{\mathbb{D}}_s$: "digraphs" with maximum arc-multiplicity s,
- $\overrightarrow{\mathbb{O}}$: oriented graphs where between any two vertices there is at most one arc (directed edge).
- $\mathbb{Q}_{r,q}$: r-uniform "directed multihypergraphs" with maximum multiplicity q.

Language

Speaking about *multigraphs* we shall use q to denote the maximum multiplicity of parallel edges.

Below, to avoid cumbersome expressions, like submultigraphs, directed multihypergraphs, etc. we agree that sometimes we use the full, precise expression, but mostly we write subgraphs, "graphs", etc. Sometimes directed edges, hyperedges, directed hyperedges will be called arcs but often we will speak of "edges" even if we speak of digraphs or hypergraphs, etc. However, if multiplicities are involved, the "number of edges" will always refer to the sum of multiplicities.

In each case we fix a universe $\mathbb{U}$ and a family of "excluded" objects $\mathcal{L} \subseteq \mathbb{U}$ and consider an "object" ("graph") $U_n \in \mathbb{U}$ on n vertices, not containing any $L \in \mathcal{L}$, and try to maximize the possible number of edges, $e(U_n)$ — counted with multiplicities — under the conditions fixed. The most important universes of this survey are $\mathbb{U}_2$ (where two vertices can be joined by 0, 1, or 2 edges) and $\overrightarrow{\mathbb{D}}_1$ (the class of digraphs where any two vertices can be independent, or joined by one arc, or by two arcs of opposite directions). These are the universes which are much more general than the universe of simple graphs and yet many interesting results can be formulated for them.

Notation. Given a graph, (digraph, multigraph, multidigraph, hypergraph), the first subscript usually denotes the number of vertices, e.g., G_n is always a graph (digraph, ...) on n vertices[1]. The number of vertices is also denoted by $v(G)$, the number of edges of G by $e(G)$, where "edges" may also mean arcs, hyperedges, or, for multigraphs, multihypergraphs, the number of edges *always with multiplicity.* $\chi(G)$ is the chromatic number of G[2]. Given a multidigraph $\overrightarrow{L}$, we call M its **underlying graph** if M is obtained by forgetting the orientations[3].

Given two multigraphs (or digraphs) G and H, we shall denote by $G \otimes H$ the multigraph (digraph) created by joining each vertex in G to each one in H by q edges (or $s+s$ arcs of opposite directions).

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[1] There will be two typical exceptions: if a set $\{L_1, \dots, L_t\}$ of forbidden graphs is considered, and if a "structure" or a matrix generates a graph, say, a matrix A_1 generates a graph $A_1(n)$ that has n vertices.

[2] We shall use the chromatic number only for graphs, digraphs and multigraphs, without loops, so it is well defined.

[3] The new multiplicity $\mu(x,y) := \mu(\overrightarrow{xy}) + \mu(\overrightarrow{yx})$.

We discuss mainly results of Paul Erdős and ourselves, — some not yet published — which generalize "Turán-type" extremal results from ordinary graphs to digraphs and multigraphs. Other progress reports on these results have appeared in [7], [68, §15], [66, §11]. The aim of this survey is to list the most important results of the field, to give ample of background explanation of what is going on in this area, and to connect this area to the surrounding, strongly related fields. (A short but excellent description of the topic can be found in the important paper of Sidorenko [64].)

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1.1. Extremal problems

Much of graph theory can be described as *extremal graph theory.* We restrict ourselves to *Turán type extremal graph problems* where the general question can be described as follows.

Definition 1.1 (Extremal problems)**.** Fix a universe $\mathbb{U}$ and a family of forbidden "graphs" $\mathcal{L} \subseteq \mathbb{U}$.

- The maximum number of edges a "graph" $G_n \in \mathbb{U}$ on n vertices can have without containing some $L \in \mathcal{L}$ is denoted by $\mathbf{ext}\,(n, \mathcal{L})$, (or, if we wish to emphasize the universe, we may write $\mathbf{ext}_{\mathbb{U}}(n, \mathcal{L})$, or in case of $\mathbb{U}_q$ and $\overrightarrow{\mathbb{D}}_s$ we may use $\mathbf{ext}_q(n, \mathcal{L})$ or $\overrightarrow{\mathbf{ext}}_s(n, \mathcal{L})$).
- The "graphs" $U_n \in \mathbb{U}$ not containing any $L \in \mathcal{L}$ and attaining this maximum are called **extremal graphs** for $\mathcal{L}$, their family is $\mathbf{EXT}\,(n, \mathcal{L})$.
- The **extremal problem** corresponding to $\mathcal{L}$ is when the **goal** is to determine or estimate $\mathbf{ext}\,(n, \mathcal{L})$.

These definitions include the directed case as well.

We will make *ad hoc* adjustments to these notations to simplify readability[4] or to eliminate ambiguities.

[4] We employ the usual "abuse of notation" when $\mathcal{L}$ consists of a single graph L, and write $\mathbf{ext}\,(n, L)$ and $\mathbf{EXT}\,(n, L)$ rather than $\mathbf{ext}\,\big(n, \{L\}\big)$ and $\mathbf{EXT}\,\big(n, \{L\}\big)$.

1.1.1. Universe = Simple graphs. We shall call graphs without loops and multiple edges **simple graphs**.

The root of multigraph/digraph extremal problems is Turán's theorem[5]. Let us restrict ourselves to simple graphs, $\mathbb{U}_1$. Let $T_{n,p}$ denote the simple p-chromatic graph on n vertices with maximum number of edges. More explicitly, we partition n vertices into p classes as uniformly as possible, and join two vertices iff they belong to distinct classes. This is $T_{n,p}$, the *Turán graph* (on n vertices and p classes). The corresponding Turán number is $t_{n,p} := e(T_{n,p})$. Given a so called *sample graph* L, we shall call G_n L-free if $L \not\subseteq G_n$. More generally, G_n is $\mathcal{L}$-free if no $L \in \mathcal{L}$ is a (not necessarily induced) subgraph of G_n.

Theorem 1.2 (Turán, 1940 [74]). *Among all the K_{p+1}-free simple graphs G_n on n vertices there is exactly one having the maximum number of edges, namely, $T_{n,p}$.*

Clearly, $T_{n,p}$ — being p-chromatic — contains no K_{p+1}. The deep part of this theorem asserts that if $e(G_n) \geq e(T_{n,p})$ and $G_n \neq T_{n,p}$, then $K_{p+1} \subseteq G_n$.

Some of the most important goals in extremal graph theory are to find

(a) the extremal graphs for a given family $\mathcal{L}$ of forbidden graphs,

(b) good asymptotics for **ext** $(n, \mathcal{L})$ where exact results are hopeless,

(c) interesting applications of extremal graph results.

When we generalize extremal graph problems and results from *simple* graphs to digraphs, multigraphs, hypergraphs, etc, we seek to answer the above questions, and to clarify the similarities and differences: to describe the new phenomena.

1.1.2. Asymptotics for simple graphs. The asymptotic behavior of the extremal function **ext** $(n, \mathcal{L})$ is very simple for *simple graphs.* The extremal numbers depend primarily on the minimum chromatic number in $\mathcal{L}$.

Let us start with the Erdős–Stone theorem from 1946. Denote the complete d-partite graph with n_i vertices in its i^{th} class $(i = 1, 2, \ldots, d)$, by $K_d(n_1, \ldots, n_d)$

[5] To be precise, one should mention Mantel's theorem and Erdős' C_4-theorem as well.

Theorem 1.3 (Erdős–Stone [29]). *For fixed positive integers m, p,*

$$\text{(1)}\quad \mathbf{ext}\,\big(n, K_{p+1}(m,m,\dots,m)\big) = \left(1-\frac{1}{p}\right)\binom{n}{2} + o(n^2) \quad \textit{as} \quad n\to\infty.$$

This easily implies

Theorem 1.4 (Erdős–Simonovits [28]). *For any $\mathcal{L}$, if*

$$\text{(2)}\qquad p = p(\mathcal{L}) := \min_{L\in\mathcal{L}} \chi(L) - 1,$$

then

$$\text{(3)}\qquad \mathbf{ext}\,(n,\mathcal{L}) = \left(1-\frac{1}{p}\right)\binom{n}{2} + o(n^2) \quad \textit{as} \quad n\to\infty.$$

The meaning of this theorem is that if the minimum chromatic number in $\mathcal{L}$ is at least 3, then the corresponding Turán graph $T_{n,p}$ is "asymptotically extremal"[6] for $\mathcal{L}$. The next Erdős–Simonovits theorem asserts that for any $\mathcal{L}$ the extremal graphs also are "very similar" to the Turán graph.

Theorem 1.5 (Extremal graphs, [17, 18, 65]). *If $p = p(\mathcal{L})$ is defined by (2) and (S_n) is a sequence of extremal graphs, then one can change $o(n^2)$ edges in S_n to obtain $T_{n,p}$.*

The almost extremal graphs are also "very near" in structure to $T_{n,p}$.

Theorem 1.6 (Erdős–Simonovits, [18, 65]). *If $p = p(\mathcal{L})$ is defined by (2) and (G_n) is a sequence of $\mathcal{L}$-free graphs satisfying*

$$e(G_n) > \mathbf{ext}\,(n,\mathcal{L}) - o(n^2), \quad \textit{as} \quad n\to\infty,$$

then one can change $o(n^2)$ edges in G_n to obtain $T_{n,p}$.

One basic question in *multigraph/digraph extremal problems* is

> How do the above theorems generalize to multigraph extremal problems or digraph extremal problems?

We shall see in Section 3 that, for the simplest cases, $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$, Theorem 1.4 has a natural extension, but structural (asymptotic) **uniqueness** of extremal graphs (Theorem 1.5) extends to multigraphs or digraphs only under **very strict extra conditions** (which are natural and are necessary and sufficient but yet very restrictive). Further, if we proceed to *multigraphs* and $q \ge 3$, then we have only conjectures (and some counterexamples), but no real positive analogues of the above theorems.

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[6] For a precise definition see Section 1.6.

1.2. Equivalence of digraph and multigraph problems

Claim 1.7 (Transfer Principle). *For any family $\mathcal{M} \subseteq \mathbb{U}_{2s}$ of multigraphs, define the family $\vec{\mathcal{L}} = \vec{\mathcal{L}}(\mathcal{M}) \subseteq \vec{\mathbb{D}}_s$ of forbidden digraphs by taking for each $M \in \mathcal{M}$ each orientation $\vec{L}$ of M belonging to $\vec{\mathbb{D}}_s$: with at most s parallel arcs. Then* $\mathbf{ext}\,(n, \mathcal{M}) = \overrightarrow{\mathbf{ext}}\,(n, \vec{\mathcal{L}})$[7]

Indeed,

(a) if $\vec{Q}_n$ is an oriented graph and M_n is obtained from $\vec{Q}_n$ by suppressing the orientation, then $\vec{Q}_n$ contains an $\vec{L} \in \vec{\mathcal{L}}$ if and only if M_n contains some $M \in \mathcal{M}$;

(b) clearly, by (a), $M_n \in \mathbf{EXT}\,(n, \mathcal{M})$ iff $\vec{Q}_n \in \overrightarrow{\mathbf{EXT}}\,(n, \vec{\mathcal{L}})$.

This is the sense in which the digraph extremal problems are more general than the multigraph extremal problems. Yet, in practice, the extremal problems for $\vec{\mathbb{D}}_s$ and $\mathbb{U}_{2s}$ are equivalent: mostly, if we can handle the multigraph problems, then we can handle the corresponding digraph problems as well.

Hence we shall often formulate our results only either for digraphs or only for multigraphs, when the corresponding other result immediately follows from the above "transfer principle".

1.3. Why do we need a bound on the multiplicities?

Consider the following generalization of $T_{n,p}$:

Definition 1.8 (Generalized Turán graph). $T_{n,p}^{q,h}$ is the multigraph obtained by partitioning n vertices into p classes V_i $(i = 1, \dots, p)$, as equally as possible, and joining two vertices by q edges iff they belong to distinct classes, by h edges iff they belong to the same class.

Given a forbidden multigraph L, if we have no restriction on the edge-multiplicity, then an L-free M_n can have arbitrary many edges. So, e.g., if $p+1 := \chi(L) > 2$, then $L \not\subseteq T_{n,p}^{q,0}$ and therefore

$$\mathbf{ext}_q(n, L) \geq q \cdot e(T_{n,p}) \to \infty \quad \text{as} \quad q \to \infty, \quad \text{even if } n \text{ is fixed.}$$

[7] Sometimes we use $\overrightarrow{\mathbf{ext}}$ to emphasize that we speak of the directed case; on the other hand, we often drop it.

[8] This is why to get a *finite extremum,* either explicitly or implicitly, we have to assume that

> a q is fixed and between any two vertices x and y the number of edges is bounded by q.

1.4. The effect of the multiplicity bound

Here, for the sake of simplicity, we restrict ourselves to the case of multigraphs. The extremal numbers $\mathbf{ext}_q(n,\mathcal{L})$ generally do depend on q as well: let L be a multigraph with maximum edge-multiplicity κ. Clearly, for $q<\kappa$,

$$\mathbf{ext}_q(n,L)=q\binom{n}{2}.$$

On the other hand, if $\kappa=1$, i.e., all the multiplicities are 0 or 1 in L, then

$$\mathbf{ext}_q(n,L)=q\cdot\mathbf{ext}_1(n,L).$$

1.5. Some early results on multigraphs and digraphs

The systematic investigation of *digraph and multigraph extremal problems* was initiated by Brown and Harary in [11]. In a first foray into this territory Brown and Harary were able to determine the extremal numbers and extremal digraphs for 3 cases:

(a) when $\vec{L}$ is a tournament (i.e., any orientation of K_n),

(b) if $\vec{L}=\vec{D}_1\otimes\vec{D}_2$, direct sums of two tournaments, i.e., the digraph obtained by joining each vertex of $\vec{D}_1$ to each vertex of $\vec{D}_2$ by two arcs of opposite directions;

(c) for any $\vec{L}$ of at most 4 vertices where any two vertices are joined by at least one arc. Their other results were mostly the multigraph analogues of these cases.

These results were all "exact", and all involved the "Turán" numbers $t_{n,p}$.

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[8] More generally, unless L is a multigraph where all the edges join two fixed vertices, then $\mathbf{ext}_q(n,L)\geq q\ \to\infty$.

Theorem 1.9 (Brown–Harary, [11]). *Let $\overrightarrow{C}_3$ be the cyclically oriented K_3, $\overrightarrow{L}_1 := \overrightarrow{C}_3 \otimes \overrightarrow{C}_3$. If $\overrightarrow{D}_1$ and $\overrightarrow{D}_2$ are two tournaments, $\overrightarrow{L} := \overrightarrow{D}_1 \otimes \overrightarrow{D}_2$ and $r := v(\overrightarrow{L}) \geq 5$, but $\overrightarrow{L} \neq \overrightarrow{L}_1$, then $\overrightarrow{T}^{1,0}_{n,r-1}$ is the (only) extremal graph for $\overrightarrow{L}$.*

All of the extremal numbers found in [11] — even in the general cases considered — were either of the forms $2t_{n,p} + O(1)$ or $\binom{n}{2} + t_{n,p} + O(1)$, because **some of** the extremal graphs were (basically) $T^{2,1}_{n,p}$ or $T^{2,0}_{n,p}$, or some directed versions of these graphs.

We shall not give here a detailed description of the Brown–Harary paper but remark that it is a very long paper (more than 60 pages, though not too densely printed) and contains all extremal multigraph and digraph results where the forbidden graphs have at most 4 vertices and the underlying graph is complete: any two of their vertices are joined by at least one edge. The paper also contains a general result, and a complete description of the non-uniqueness of extremal graphs for several sample graphs, see below in Sections 7 and [10].

1.6. Multigraph and digraph extremal problems

Many results from extremal graph theory can easily be extended to such other universes as multigraphs, weighted graphs, digraphs or to multidigraphs.

Most of our results were developed originally for $\mathbb{U}_2$ or $\overrightarrow{\mathbb{D}}_1$. We shall mostly (but not always) confine our discussion to these two universes. Some results generalize without difficulty to digraphs and multigraphs with higher multiplicities of edges; or to hypergraphs, even "directed multihypergraphs" (see Section 9 and [12]).

Definition 1.10 (Asymptotically extremal sequences). If the universe $\overrightarrow{\mathbb{D}}_s$ and $\overrightarrow{\mathcal{L}}$ are fixed and $\overrightarrow{D}_n \in \overrightarrow{\mathbb{D}}_s$ is a sequence of $\overrightarrow{\mathcal{L}}$-free digraphs and

$$e(\overrightarrow{D}_n) \geq \overrightarrow{\mathbf{ext}}\,(n, \overrightarrow{\mathcal{L}}) - o(n^2) \quad \text{as} \quad n \to \infty,$$

then $(\overrightarrow{D}_n)$ is called an **asymptotically extremal sequence** for $\overrightarrow{\mathcal{L}}$.

Basically there are two important reasons to speak of asymptotically extremal sequences (instead of extremal sequences):

(a) Often we cannot determine the extremal graphs but we can find asymptotically extremal graph sequences.

(b) In some other cases we have a fairly good but rather complicated description of the extremal graphs: they have complicated structures but they are obtained from some much simpler structures by adding $o(n^2)$ edges.

Fundamental problem:
Can we provide asymptotically extremal sequences of fairly simple structure for a given $\mathcal{L}$?

1.7. Some motivation for multigraph/digraph extremal problems

Extremal graph theory being interesting on its own, yet it is worth remarking that some of its roots are coming from applications. One of these roots comes from [14] where Erdős solved the problem $\mathbf{ext}\,(n, C_4)$ to apply it in number theory.[9]

As Turán used to emphasize, the applicability of extremal graph theorems derives from the fact that, in some sense, extremal graph results are generalizations of the *Pigeon Hole Principle*[10]. Ordinary extremal graph results are applicable

- to distance distribution in geometry (Erdős, [15],..., see also the book of Pach and Agarwal, [57])
- estimating convergence of some potential-like integrals in analysis, Turán [75], Erdős, Meir, T. Sós, Turán [25]
- to Probability theory, (Katona, Sidorenko, [46],...)

1.7.1. Ramsey–Turán theory. Sometimes applications of extremal graph theory provide satisfactory results, but often they give only crude estimates. This is why one has two distinct approaches to improve these results: either one forgets extremal graph theory and attacks the problem in a completely different way, or one improves the extremal graph theoretical tools: to get tools more fitting to the given situation. This (improving the tools) happened, e.g., when Komlós, Pintz and Szemerédi disproved the old conjecture

[9] For a more detailed description of the story, see [68]. For applications of extremal graph theory in number theory see several papers of Erdős, and among the latest ones, see [27] or the survey paper of V. T. Sós on the interaction between Graph Theory and Number Theory, [71] in this volume.

[10] Of course, Ramsey theorems are also generalizations of the Pigeon Hole Principle, and very applicable but the two applications have different characters.

on the Heilbronn problem [47]. Also this led to the Ramsey–Turán theory. For a longer survey on this topic see the paper of Simonovits and T. Sós [69].

This second approach: "improving the tools" is one of the main motivations to investigate Turán type extremal graph results for weighted graphs (which include multigraphs as well).

An important graph theoretical application of *multigraph* extremal theorems is a result of Erdős, Hajnal, Szemerédi and T. Sós, [24], on Ramsey–Turán problems. Let $\alpha(G_n)$ denote the stability number, i.e., the cardinality of the largest independent vertex set in G_n. Consider the following problem:

Problem 1.11 (Ramsey–Turán problem, $o(n)$ scope)**.** Given a family of forbidden graphs, $\mathcal{L}$, and a sequence (G_n) of (ordinary) graphs for which

(i) G_n contains no $L \in \mathcal{L}$,

(ii) $\alpha(G_n) = o(n)$.

What is the maximum of $e(G_n)$ under these conditions?

In many important cases Ramsey–Turán theorems reduce to solving some multigraph extremal problem described by Theorem 2.3 below. One case reduced to extremal problems in $\mathbb{U}_2$ is the following theorem of Erdős–Hajnal–T. Sós–Szemerédi theorem on the Ramsey–Turán problem of $K_{2\ell}$.

Theorem 1.12 (Erdős–Hajnal–T. Sós–Szemerédi, [24])**.** *If $K_{2\ell} \not\subseteq G_n$ and $\alpha(G_n) = o(n)$ then*

$$e(G_n) \le \frac{1}{2}\frac{3\ell-5}{3\ell-2}n^2 + o(n^2)$$

and this is sharp.

This deep theorem is highly nontrivial and even the simplest case of K_4 was difficult to solve (see Szemerédi, [72], Bollobás–Erdős [4]) and after that it took several years to settle the general case.

It uses the solution of a multigraph extremal problem in $\mathbb{U}_2$ where, for given t, we take $\mathcal{L}_t$ consisting of those (forbidden) subgraphs L where, for some $\tau \le t$, we have a graph on $u_1, \dots, u_\tau$; the edge-multiplicities are 1 or 2 and the the multiplicity of u_iu_j is 2 iff $i, j \le t - \tau$.

For motivations, applications and further details on Ramsey–Turán results, again, see the survey of Simonovits and V. T. Sós [69]. For some further generalizations and details see also the papers of Erdős, Hajnal, Simonovits, T. Sós and Szemerédi, [22, 23].

1.7.2. An application to hypergraphs. There are not too many satisfactory extremal hypergraph results. V. T. Sós asked the following question [70]: if we restrict ourselves to 3-uniform hypergraphs and F_7 is the Fano hypergraph, i.e., the 3-uniform hypergraph defined on 7 points by the 7 triples (=lines) of the Fano plane, what is the extremal function $\mathbf{ext}\,(n, F_7)$?

The hypergraph F_7 is 3-chromatic but if we delete any triple from it, we obtain a 2-colorable graph. This is one of the motivations of

Conjecture 1.13 (V. T. Sós). *If we partition $n > n_0$ vertices into two classes A and B and consider all the triplets containing at least one vertex from both A and B, then the 3-uniform hypergraph obtained is extremal for F_7.*

Recently de Caen and Füredi proved [13] that

Theorem 1.14.

$$\mathbf{ext}\,(n, F_7) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

The proof uses a corresponding Dirac type multigraph extremal theorem:

Theorem 1.15 (Füredi-Kündgen, [33]). *If M_n is an arbitrary multigraph (without restriction on the edge multiplicities, except that they are nonnegative) and all the 4-vertex subgraphs of M_n have at most 20 edges, then*

$$e(M_m) \le 3\binom{n}{2} + O(n).$$

This is a very special case of a much more general theorem, see [33] and also Section 10.

1.7.3. Codes and digraphs. Many combinatorial or algebraic conditions can be expressed through excluding some subgraphs in digraphs. One such example comes from Coding Theory. A paper of Ball and Cummings uses extremal digraph theorems to estimate the maximum number of possible "comma-free codewords" over a given alphabet [1]. Generally, a set $\mathcal{C}$ consisting of sequences of length k is a comma-free code if whenever it contains two sequences $a_1, \dots, a_k$ and $b_1, \dots, b_k$, then it contains *no* other k-segment of $a_1 a_2 \dots a_k b_1 \dots b_k$. (This means that in a sequence of codes one does not have to use commas to separate the codewords.) Being a comma-free code for $k = 2$ means that if ab and cd are such words then bc is not a codeword. Such a code describes a directed graph on n vertices (the symbol alphabet), the codewords being the arcs. This graph contains no directed path of length 3. The maximum possible number of codewords is $[n^2/3]$. (This is not so surprising: an even stronger Lemma states that these digraphs are 3-colorable.) The authors describe how to construct all graphs corresponding to maximal codes.

1.7.4. An application to geometry. Erdős, Harczos, and Pach [26], investigating a distance distribution problem in $\mathbb{R}^3$, needed and proved the following digraph extremal result. Let $\overrightarrow{K}_3(1,3,3)$ denote the digraph on 7 vertices $\{x, y_1, y_2, y_3, z_1, z_2, z_3\}$, and taking the 12 arcs of the form $\overrightarrow{xy_i}$ and $\overrightarrow{y_iz_j}$.

Theorem 1.16 (Excluded $\overrightarrow{K}_3(1,3,3)$, [26]). *If the outdegree of each vertex of $\overrightarrow{G}_n$ is at least $2n^{2/3}$ then $\overrightarrow{K}_3(1,3,3) \subseteq \overrightarrow{G}_n$.*

Observe, that this is not a standard extremal graph theorem, here we have a condition on the degrees, not on the number of edges. For ordinary graphs this is a slim difference, here $\mathbf{ext}\left(n, \overrightarrow{K}_3(1,3,3)\right) \geq \left\lfloor \frac{n^2}{4} \right\rfloor$, since the appropriate orientation of $T_{n,2}$ contains no $\overrightarrow{K}_3(1,3,3)$.

(Some interesting related applications of graph theory to geometry can be found in several papers of Erdős, and also in Füredi [31] and Füredi-Hajnal [32].)

2. Some Important Examples

Here we restrict ourselves to $\mathbb{U}_2$.

Let K_3^* denote the multigraph on 3 vertices a, b, c where a is joined to b, c by double edges and (b,c) is a single edge. One can learn much about the *multigraph* extremal problems from the case of K_3^*.

Theorem 2.1 (Brown–Harary).

$$\mathbf{ext}\,(n, K_3^*) = 2\left\lfloor \frac{n^2}{4} \right\rfloor.$$

In the corresponding Figure 2 we see K_3^* for which $T_{n,2}^{2,0}$ is one of the extremal graphs, but the complete graphs, K_n also form an asymptotically extremal graph sequence, and we can mix these two asymptotically extremal graph sequences in infinitely many ways. A mixed sequence can be seen on Figure 2: take ℓ copies of the double Turán graph $T^{2,0}_{n,2}$ in ℓ (possibly different) sizes: $H_1 = T^{2,0}_{n_1,2}$, $H_2 = T^{2,0}_{n_2,2}, \dots,$ $H_\ell = T^{2,0}_{n_\ell,2}$. Join each vertex of H_i to each vertex of H_j for every $1 \le i < j \le \ell$, by single edges. The resulting graphs contain no K_3^* and are asymptotically extremal. This shows that *one cannot hope for structural uniqueness* or stability of the (asymptotically) extremal graph sequences, not even in the simplest cases.

Remark 2.2. (K_n) is not an extremal sequence for K_3^* but if we add to K_n $[n/2]$ independent edges (to form $[n/2]$ independent edges of multiplicity 2) then we get extremal graphs. This is a subcase of the "mixing" described above.

2.1. A general multigraph result

In this section we restrict ourselves to $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$.[11]

> **Maximum multiplicity 2.** For a given family $\mathcal{M}$ of multigraphs, consider the largest p for which $T^{2,0}_{n,p}$ contains no $M \in \mathcal{M}$. Consider also the largest $\tilde{p}$ for which $T^{2,1}_{n,\tilde{p}}$ contains no $M \in \mathcal{M}$. Clearly,
>
> $$\textbf{ext}\,(n, \mathcal{M}) \ge \max\left\{ e(T^{2,0}_{n,p}), e(T^{2,1}_{n,\tilde{p}})\right\}. \tag{4}$$
>
> **Question:** for which families $\mathcal{M}$ is (4) sharp up to $o(n^2)$ (…or sharp without any error-term?)?

Theorem 2.3 (Brown, Erdős, Simonovits, [7]). *Consider $\mathbb{U}_2$. If M is complete (i.e., the multiplicities are 1 and 2), then (4) is sharp:*

Let $p = v(M) - 1$ and $\tilde{p} + 1$ be the chromatic number of the graph defined by the edges of multiplicity at least 2. Then

$$\textbf{ext}\,(n, M) = \max\left\{ e(T^{2,0}_{n,p}), e(T^{2,1}_{n,\tilde{p}})\right\} + o(n^2). \tag{5}$$

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[11] We have seen that these two cases are almost equivalent.

3. The Main Theorem

In our investigations with Erdős, we soon found that the general case was considerably more interesting and involved, even in the asymptotic behavior of the extremal numbers. The situation can be described in terms of certain matrices that we call "dense". We formulate here and explain below the main result of [6]:

> For any family $\mathcal{L}$, there exists an **asymptotically extremal** sequence of **optimal matrix digraphs** $A(n)$ associated with a **dense** matrix A.

The matrices are used here to encode some graph structures in a compact form. Below we first formulate our results without matrices, then we describe the encoding, and finally formulate some of our results in matrix-encoded form.

3.1. Main Theorem without matrices

The main question in the asymptotic theory of multigraph and digraph extremal problems is if, for every $\overrightarrow{\mathcal{L}}$, there is an asymptotically extremal sequence $(\overrightarrow{Z}_n)$ of digraphs/multigraphs that can be described in some simple way, by a small number of parameters. More explicitly, (restricting ourselves, e.g., to digraphs),

Question:
For a given family $\overrightarrow{\mathcal{L}}$ of excluded digraphs, can one always find a sequence $(\overrightarrow{Z}_n)$ of digraphs, not containing any $\overrightarrow{L} \in \overrightarrow{\mathcal{L}}$, with

$$e(\overrightarrow{Z}_n) = \overrightarrow{\mathbf{ext}}\,(n, \overrightarrow{\mathcal{L}}) + o(n^2),$$

for which $V(\overrightarrow{Z}_n)$ can be partitioned into a bounded number of classes, $V_1, \dots, V_r$ so that, if $x \in V_i$ and $y \in V_j$ then the direction and multiplicity of an arc $(x \to y)$ depend only on (i, j)?

Perhaps (switching to multigraphs) the simplest such non-trivial structures are $(T^{q,\ell}_{n,p})$. Somewhat more complicated structures are described in Figure 3. The "subdivided double Turán graph" is obtained from a $T^{2,0}_{n,k}$

where we put some ordinary (simple) Turán graphs in the classes of $T^{2,0}_{n,k}$ and then (perhaps) change the sizes of the classes. In the undirected version, for given $p_1, \dots, p_k$, take a partition $n = n_1 + \dots + n_k$ and let

$$S_n = \mathbb{S}^{p_1,\dots,p_k} < n_1, \dots, n_k > := T_{n_1,p_1} \otimes \dots \otimes T_{n_k,p_k}.$$

Take a partition $(n_1, \dots, n_k)$ for which $e(\mathbb{S}^{p_1,\dots,p_k} < n_1, \dots, n_k >)$ is the maximum possible. This will be denoted by $S_n = \mathbb{S}^{p_1,\dots,p_k}(n)$. In the directed case, take any permitted orientation of this S_n.

These sequences (S_n) still have a fairly transparent structure. There are many much more involved ones. Figure 4 describes two structures that are not covered by the above patterns, the "4-path" structure, $\mathbb{P}^4(n)$ and the "pentagon" structure $\mathbb{C}_5(n)$[12].

The Main Theorem asserts that for $\mathbb{U}_2$ or $\overrightarrow{\mathbb{D}}_1$, for any $\mathcal{L}$, there exist simple asymptotically extremal graph sequences.

Theorem 3.1 (Main Theorem). *Consider $\overrightarrow{\mathbb{D}}_1$. For any excluded digraph family $\overrightarrow{\mathcal{L}}$ there exists a sequence $(\overrightarrow{S}_n)$ of asymptotically extremal digraphs for which $V(\overrightarrow{S}_n)$ can be partitioned into a fixed number $r = r(\overrightarrow{\mathcal{L}})$ of classes, $V_1, \dots, V_r$ so that each V_i spans either a transitive tournament or an empty set and for each $1 \le i < j \le r$ (depending on (i,j)) either each $x \in V_i$ and $y \in V_j$ are joined by two arcs of opposite directions or each $x \in V_i$ is joined to each $y \in V_j$ by an arc directed toward V_i or vice versa.*

The case of independent $x \in V_i$, $y \in V_j$ can be excluded, see Section 3.2. This is what will be expressed below by saying that we may restrict ourselves to *dense* matrices. We do not know Theorem 3.1 for higher multiplicities: not even for $\mathbb{U}_3$, nor for $\overrightarrow{\mathbb{D}}_2$.

3.2. Encoding graph structures with matrices

These **structures** can be encoded by $r \times r$ matrices $A = (a_{ij})$, where a_{ij} corresponds to the connection between V_i and V_j. This is why we shall introduce the matrix graphs and some related notions. This encoding can be done in various ways and we shall choose one where the quadratic form, xAx^* corresponds to the number of edges in the corresponding structure.[13]

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[12] The names are invented just to make them easier to distinguish.

[13] We use an asterisk to denote the transpose of a vector or matrix.

Definition 3.2 (Matrix digraphs).[14] Let $A = (a_{ij})$ be an $r \times r$ matrix, $a_{ij} = 0$ or 2 if $i \neq j$, and $a_{ii} = 0$ or 1, $\mathbf{x} = (x_1, \ldots, x_r)$ a vector with nonnegative integer coordinates, and $n = x_1 + \ldots + x_r$. We define a digraph $A\langle \mathbf{x} \rangle$ as follows: n vertices are divided into classes $V_1, \ldots, V_r$, where the i^{th} class contains x_i vertices, and for $i \neq j$ we join each vertex of V_i to each vertex of V_j by $\frac{1}{2}a_{ij}$ arcs directed toward V_j. If $a_{ii} = 0$, then the vertices of V_i are independent; if $a_{ii} = 1$, then they form a transitive tournament.

Claim 3.3. *For any $r \times r$ matrix $A = (a_{ij})_{i,j=1,2,\ldots,r}$, and any vector $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ of non-negative integers,*

$$e(A\langle \mathbf{x} \rangle) = \frac{1}{2}\mathbf{x}A\mathbf{x}^* - \frac{1}{2}\sum_{i=1}^{r} a_{ii}x_i, \tag{6}$$

Definition 3.4 (Density of matrices). Denote by Δ_r the simplex

$$\left\{ \mathbf{u} : \sum_i u_i = 1,\ u_1 \geq 0, \ldots, u_r \geq 0. \right\}. \tag{7}$$

The density of an $r \times r$ matrix A is

$$g(A) := \max_{\mathbf{u} \in \Delta_r} \mathbf{u}A\mathbf{u}^*. \tag{8}$$ [15]

The vectors $\mathbf{u}$ for which $\mathbf{u}A\mathbf{u}^* = g(A)$ in (8) are called the *optimum vectors* of A.[16]

The meaning of the next definition is that if we have a structure providing asymptotically extremal graphs but we can delete some classes of this structure and still have an asymptotically extremal graph sequence, then we delete these classes. Finally we obtain some minimal (i.e. simplest) structures. These structures and the corresponding matrices will be called **dense**.

---

[14] In introducing the analogous concept for multigraphs, the authors chose to symmetrize the matrices by; so the matrix we chose to use to represent the multigraph underlying a digraph $A\langle \mathbf{x} \rangle$ is the symmetric matrix $\frac{1}{2}(A + A^*)$.

[15] Note that the value of $\mathbf{u}A\mathbf{u}^*$ depends only on $A + A^*$.

[16] Our approach to these Turán-type problems by considering the quadratic forms associated with an adjacency-type matrix was motivated, in part, by the work of Motzkin and Straus [56], which is not unrelated to the methods of Zykov [78].

Definition 3.5 (Dense matrices). A matrix A is *dense* if deleting any of its rows and the corresponding columns yields an A' with $g(A') < g(A)$.[17]

The dense matrices play important role in our investigation. An $r \times r$ integer matrix A is *dense* if the maximum in (8) is attained only at interior points of the simplex Δ_r. (For a dense A there is only one optimum vector, by Lemma 3.9.)

Definition 3.6 (Optimal Matrix graph). *$A(n)$ is an optimal matrix graph for a matrix A if $e\big(A(n)\big)$ is the maximum for all $A\langle \mathbf{z}\rangle$, when $\sum z_i = n$.*[18]

Claim 3.7.

$$e\big(A(n)\big) = g(A)\binom{n}{2} + O(n). \tag{9}$$

3.3. Main Theorem with matrices

What we proved is the following:

Theorem 3.8 (Main Theorem, [6, Theorem 1]). *Consider $\mathbb{U}_2$ or $\vec{\mathbb{D}}_1$. For any finite or infinite family $\mathcal{L}$ there exists a dense matrix A such that $\big(A(n)\big)$ is asymptotically extremal for $\mathcal{L}$.*

The matrix A is *extremal* for the family $\mathcal{L}$ if the sequence $\big(A(n)\big)$ is asymptotically extremal for $\mathcal{L}$.

One of the lemmas we proved was

Lemma ([6, 9]). *Let $\mathbf{j} = (1, \ldots, 1)$. If A is a dense matrix then*

(a) $a_{ij} + a_{ji} > a_{ii} + a_{jj}$, *for $i \neq j$.*

(b) *A is non-singular and $A\mathbf{x} = \mathbf{j}$ has only one solution, where each $x_i > 0$.*

(c) *For this unique solution of $A\mathbf{x} = \mathbf{j}$, $\sum_i x_i = \frac{1}{g(A)}$.*

The "normalized" vector $u := g(A)\mathbf{x} \in \Delta_r$ is the "optimum vector".

---

[17] Equivalently, a matrix A is *dense* if no principal proper submatrix A' is such that the sequences $\big(A(n)\big)$ and $\big(A'(n)\big)$ have, asymptotically, numbers of edges which differ by $o(n^2)$ as $n \to \infty$.

[18] $A(n)$ is not uniquely defined, we take one of the possible optimal graphs. By the way, for a dense A there are only $O_A(1)$ optimal graphs.

This lemma corresponds to the symmetrization method of Zykov [78].

A. Sidorenko extended our necessary conditions on a matrix to be dense [6] [9]. He proved the following elegant characterization of dense matrices:

Theorem 3.10 ([64, Theorem 2]). *Let* $\mathbf{e} = (1, 1, \dots, 1)$. *A matrix* A *is dense iff both of the following conditions hold:*

1. *A is non-singular, and all components of the vector $\mathbf{e}A^{-1}$ are positive.*
2. *A is "of negative type", i.e., for any non-zero vector $\mathbf{x}$ for which $\mathbf{x}\mathbf{e}^* = 0$, $\mathbf{x}A\mathbf{x}^* < 0$.*

3.4. Examples: Directed case

Here we are working in $\overrightarrow{\mathbb{D}}_1$, but, as we have observed, this is just a minor difference: most of the statements of this kind can easily be "translated" into theorems in $\mathbb{U}_2$.

Consider the following simple examples:

We shall use below as forbidden graphs the 2-cycle, the 3-cycle and the transitive triangle, shown on Figure 5.

Example 1. Clearly, $\mathbf{ext}\,(n, \mathbb{D}K_2) = \binom{n}{2}$, and the extremal matrices are all tournaments on n vertices. Further, any asymptotically extremal sequence for $\mathbb{D}K_2$ will consist of subdigraphs of tournaments containing almost all edges. The (unique) asymptotically extremal matrix is $A_1 = (1)$ here. Indeed, this matrix is *extremal,* in the sense that the sequence $\big(A_1(n)\big)$ consists exclusively of extremal graphs.[19]

Example 2. Now exclude $\overrightarrow{T}_3$, the transitive tournament on 3 vertices. As shown in [11], $\mathbf{ext}\,(n, \overrightarrow{T}_3) = 2\left\lfloor \frac{n^2}{4} \right\rfloor$, and the extremal digraphs are the digraphs $T^{2,0}_{n,2}$ [11, (7.5)]. The only extremal matrix is $A_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$; the only other dense matrix of density 1 is A_1, and it is not asymptotically extremal for $\overrightarrow{T}_3$.

Example 3. In this case we "repeat" in matrix form what was already explained in Section 2 about mixing extremal structures for K_3^*. $\mathbf{ext}\,(n, \overrightarrow{C}_3) =$

[19] But not all extremal graphs are of the form $A_1(n)$.

$2\left\lfloor \frac{n^2}{4} \right\rfloor$, $T^{2,0}_{n,2}$ is extremal, but there are other extremal digraphs. As seen in [11],

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \left(\begin{array}{cc|cc} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{array}\right), \quad \left(\begin{array}{cc|cc|cc} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 & 2 \\ \hline 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 & 2 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right) \ldots$$

are among the possible extremal matrices, where the 2×2 submatrices along the main diagonal represent copies of $T^{2,0}_{n_\ell,2}$. The extremal digraphs represented by these matrices are composed of copies of "double" Turán graphs, between any two of which all edges are present, all directed the same way, so that the digraph has no cyclic triangle. But, in fact, there are other extremal digraphs. Of the matrices listed above, only the first is *dense.*

3.5. Matrix coloring

For ordinary graphs the chromatic number of the excluded graph governs the asymptotic structure of the extremal graphs. Several results concerning this assertion can be generalized to $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$. The first thing one has to do is to generalize the chromatic number to multigraphs and multidigraphs.

Definition 3.11. Given a matrix A, with non-negative entries, we call the multidigraph $\overrightarrow{U}$ A-colorable if $\overrightarrow{U} \subseteq A(m)$ for some sufficiently large m.

If we wish to generalize the chromatic number to $\overrightarrow{\mathbb{D}}_s$, we can use

$$\gamma(\overrightarrow{U}) := \inf\left\{ g(A) \ : \ \overrightarrow{U} \text{ is } A\text{-colorable}\right\}.$$

Or, if we use $\chi(\overrightarrow{U}) := 1 - \frac{1}{\gamma(\overrightarrow{U})}$, then we get back the original chromatic number for ordinary graphs.

Now, many extremal graph results, connected to Erdős–Stone–Simonovits Theorem or its structural versions, can be generalized to $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$. We skip the details.

4. Inverse Theorems

We speak of "inverse theorems" when we fix a sequence of graphs (digraphs, ...) (S_n) and try to find out if there are families $\mathcal{L}$ for which these graphs are the (asymptotically) extremal graphs. If yes, which are these $\mathcal{L}$'s? The meaning of the next inverse theorem is that the Main Theorem is sharp: one cannot narrow the family of dense matrices.

Theorem 4.1 (Inverse Theorem [8, Theorem 1]). *For every universe $\overrightarrow{\mathbb{D}}_s$ and for every dense matrix A (with entries described in Definition 3.2) there exists a finite family $\overrightarrow{\mathcal{L}}$ of digraphs such that*

1. *A is extremal for $\overrightarrow{\mathcal{L}}$; moreover,* $\mathbf{EXT}\,(n,\mathcal{L}) = \big(A(n)\big)$ *for all n.*
2. *For any asymptotically extremal sequence (G_n), G_n may be obtained from $A(n)$ by adding/deleting/redirecting $o(n^2)$ edges as $n \to \infty$.*
3. *A is unique with this property, up to like permutations of rows and columns.*

Of course, for different excluded families $\overrightarrow{\mathcal{L}}$ we may have the same A. Theorem 4.1 would be easy if we allowed infinite $\overrightarrow{\mathcal{L}}$'s.

Observe the gap: the inverse theorem is proved for all universes, the Main Theorem only for $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$.

For $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$, if A is extremal for $\mathcal{L}$, then

$$g(A) = 2 \lim_{n\to\infty} \frac{\mathbf{ext}\,(n,\mathcal{L})}{n^2}. \tag{10}$$

Thus the set of limits[20] $\left\{ \lim\limits_{n\to\infty} \frac{2\,\mathbf{ext}\,(n,\mathcal{L})}{n^2} \right\}$ is contained in the set of densities of dense matrices. And so, by virtue of Theorem 4.1, the two sets coincide.

4.1. One excluded graph

One could ask if the inverse extremal theorem changes when (instead of excluding a finite family of graphs) we exclude just one L. We shall discuss this case somewhere else. Here we mention just two results regarding the simplest non-trivial cases.

[20] known to exist, even without Theorem 3.8, by an argument of Katona, Nemetz and Simonovits [45, Corollary to Theorem 1], [12, Lemma 2].

Theorem 4.2. *Let $W_4 \in \mathbb{U}_2$ be the multigraph on $\{a,b,c,x\}$ where ax, bx are the double edges and abc is a triangle of single edges, finally, cx is also a single edge. Then $\mathbb{S}^{1,2}(n)$ is an asymptotically extremal sequence for W_4.*

Theorem 4.3. *Consider again $\mathbb{U}_2$. If the double edges form a connected* **spanning subgraph** *of the sample graph L, then $\mathbb{S}^{1,2}(n)$ cannot be asymptotically extremal for L.*

5. Augmentation of Structures

To prove our results, mostly we use a special procedure, called **augmentation**. This means (in a nutshell): "having a nice substructure of a graph G_n we build up an even nicer substructure ... and we iterate this if needed". Here the "nice substructure" is a subgraph $A(m)$ where $g(A)$ large. Augmentation has two forms: sometimes we think of augmenting a "structure", sometimes we encode this procedure and get "augmentation of matrices". Here, for the sake of clarity we shall be brief and restrict ourselves to the matrices.

5.1. Augmentation of Matrices

First we describe the augmentation formally, then explain what is really

Definition 5.1. The $(r+1)\times(r+1)$ matrix $B = (a_{ij})_{i,j=1,2,\dots,r+1}$ is an augmentation of its principal submatrix $A = (a_{ij})_{i,j=1,2,\dots,r}$ if A is dense, having optimum vector $\mathbf{u} = (u_1, u_2, \dots, u_r)$, and

$$\gamma := \frac{1}{2}\sum_{j=1}^{r}(a_{j,r+1} + a_{r+1,j})u_j > g(A). \tag{11}$$

The next lemma asserts that the augmented matrix is "better" than the original.

Lemma 5.2. *If A' is an augmentation of A then $g(A') > g(A)$.*

Finally, the lemma below asserts that in an iterated augmentation the densities tend to the density given on the LHS of (11).

Lemma 5.3. *Let γ^* be fixed. If $A_0, \dots, A_t, \dots$ is a sequence of dense matrices where A_0 is arbitrary and A_{r+1} is obtained from A_r by augmentation, for every $r > 0$, and $\gamma \ge \gamma^*$ in (11), then $\lim_{r\to\infty} g(A_r) \ge \gamma^*$.*

5.2. How do we augment?

We keep using the well-known technical lemma (deletion of small degrees):

Lemma 5.4. *If $\varepsilon > 0$ is fixed and $G_n \in \mathbb{U}_q$ is a graph sequence with*

$$e(G_n) \geq (\alpha+\varepsilon)\binom{n}{2}, \tag{12}$$

then there is an $H_{\nu_n} \subseteq G_n$ with minimum degree

$$\mathbf{d}_{\min}(H_{\nu_n}) \geq \left(\alpha + \frac{1}{2}\varepsilon\right)\nu_n \quad \text{where} \quad \nu_n \to \infty \quad \text{as} \quad n \to \infty.$$

Using the lemma, (and changing the notation) we may always replace (12) by

$$\mathbf{d}_{\min}(G_n) \geq \left(\alpha + \frac{1}{2}\varepsilon\right) n. \tag{13}$$

Consider Figure 5. We split G_n into $A_\ell(m_\ell)$ and $G_n - A_\ell(m_\ell)$. By (13)

$$e\big(A_\ell(m_\ell), G_n - A_\ell(m_\ell)\big) \geq \left(\alpha + \frac{1}{2}\varepsilon\right) m_\ell(n-m_\ell) - e\big(A_\ell(m_\ell)\big).$$

Therefore we may find a set $W \subseteq G_n - A_\ell(m_\ell)$ joined to $A_\ell(m_\ell)$ by

$$e\big(A_\ell(m_\ell), W\big) \geq \left(\alpha + \frac{1}{2}\varepsilon\right) m_\ell|W| \tag{14}$$

edges, and in the same way: for each $w \in W$, $N(w) \cap A_\ell(m_\ell)$ is the same. The vertices of W will correspond to the new row of the augmented matrix and (14) ensures (11), the meaning of which is that the vertices of W are joined with many edges to $A_\ell(m_\ell)$.

We use this augmentation often with excluding some family $\mathcal{L}$ of graphs and whenever we get an augmentation B of a matrix A, we check if

"is $B(n)$ always $\mathcal{L}$-free?"

and if NO, then we discard this B since it is irrelevant in the extremal problem of $\mathcal{L}$. The real issue is to prove that in those cases when we use this augmentation procedure, it stops in finitely many steps: after a while we cannot get new, relevant matrices B. When the augmentation stops, we have the extremal matrices. We skip the details, which can be found, e.g., in [9].[21]

[21] We ignored here the serious issue of non-zero diagonal entries, see Section 8.

5.3. Examples in $\mathbb{U}_2$

Here we shall discuss some examples on the asymptotic behavior of $\mathbf{ext}_2(n, \mathcal{L})$, illustrating the augmentation.

Claim 5.5. *For any fixed m,*

$$\mathbf{ext}_2(n, T^{2,0}_{m,2}) = \frac{1}{2}n^2 + o(n^2).$$

Proof. (a) $K_n \not\supseteq T^{2,0}_{m,2}$. Hence

$$\mathbf{ext}_2(n, T^{2,0}_{m,2}) \geq \binom{n}{2}.$$ [22]

(b) If $G_n \in \mathbb{U}_2$ and

$$e(G_n) > \left(\frac{1}{2} + \varepsilon\right) n^2,$$

then G_n has more than εn^2 edges of multiplicity 2. Applying the Kővári–T. Sós–Turán theorem [48] to these edges we have that, for sufficiently large n, $T^{2,0}_{m,2} \subseteq G_n$. ■

Surprisingly, the following stronger result also holds:

Claim 5.6.

$$\mathbf{ext}_2\left(n, K_1(m) \otimes K_2(m,m)\right) = \frac{1}{2}n^2 + o(n^2).$$

Here we can use $\mathbb{S}^{2,1}(\mu)$ instead of $K_1(m) \otimes K_2(m,m)$, as well.

Proof. (a) Neither (K_n) nor $(T^{2,0}_{n,2})$ contains $K_1(m) \otimes K_2(m,m)$, showing that

$$\mathbf{ext}_2\left(n, K_1(m) \otimes K_2(m,m)\right) \geq \frac{1}{2}n^2 + O(1), \tag{15}$$

for any fixed m.

[22] One can see that

$$\mathbf{ext}_2(n, T^{2,0}_{m,2}) \geq \binom{n}{2} + \mathbf{ext}_1\left(n, K\left([m/2], [m/2]\right)\right).$$

(b) Take a graph sequence (G_n) with

$$e(G_n) \geq \frac{1}{2}n^2 + \varepsilon n^2.$$

Applying Lemma 5.4, we may assume that

$$\mathbf{d}_{\min}(G_n) \geq (1+\varepsilon)n.$$

Put $h = \left\lceil \frac{4}{\varepsilon}m \right\rceil$, and fix a $H := T^{2,0}_{2h,2} \subseteq G_n$, for $n > n_0(m,\varepsilon)$. Thus

$$e(H, G_n - H) \geq 2h(1+\varepsilon)n - e(H).$$

So, there exists a vertex set $W \subseteq G_n - H$, with $|W| \geq m$, for which each $w \in W$ is connected to H by more than $(2 + \frac{1}{2}\varepsilon)h$ edges, and in exactly the same way. This yields a $K_2(m,m) \otimes K_1(m) \subseteq G_n$, if n is sufficiently large. ∎

Similarly,

$$\text{(16)} \qquad \mathbf{ext}_2\left(n, K_3(m,m,m)\right) = \frac{1}{2}n^2 + o(n^2) \quad \text{as} \quad n \to \infty.$$

Equation (15) is an example of a result leading to a "jumping" constant, see Section 6.1. The presence of $\left(\frac{1}{2}+\varepsilon\right)n^2$ edges in a 2-multigraph implies the presence of $K_1(3m) \otimes K_2(3m,3m)$, which is a structure of density $\frac{4}{7} > \frac{1}{2}$.

5.3.1. Degenerate multigraph problems. For simple graphs

$$\mathbf{ext}\,(n, \mathcal{L}) = o(n^2)$$

iff $\mathcal{L}$ contains a bipartite subgraph. These problems are called **degenerate** and in some sense most extremal graph problems can be reduced to degenerate extremal graph problems [67] but most of the degenerate extremal graph problems seem to be hopelessly difficult.

For multigraphs the problem of degenerate multigraph extremal problems seems to disappear. Indeed, if e.g., a forbidden multigraph L contains at least one double edge, then $L \not\subseteq K_n$:

$$\mathbf{ext}\,(n, L) \geq e(K_n) = \binom{n}{2}.$$

On the other hand, if L has no double edges, then all edges in the extremal graphs are of multiplicity q:

$$\mathbf{ext}_q(n, L) = q \cdot \mathbf{ext}_1(n, L),$$

showing that for multigraphs $\mathbf{ext}\,(n, L) = o(n^2)$ occurs iff L is a simple bipartite graph.

5.3.2. Complicated cases? Generally we have two approaches: either we fix the class of forbidden graphs or we investigate the partial order of dense matrices given by the augmentation where we write $B > A$ if either B is an augmentation of A or (building up the transitive closure) if there is a sequence $B = A_r, A_{r-1} \dots, A_2, A_1 = A$ where A_i is an augmentation of A_{i-1}.

Above we considered the partial order, here we give two non-trivial examples, showing how augmentation works.

Let $W_5 \in \mathbb{U}_2$ be the multigraph, where the double edges form a path of 5 vertices, P_5 on $axbyc$, abc is a triangle of single edges and the other pairs are independent. Let $Z_4 = K_2 \otimes K_2$.

Claim 5.7. $\mathbf{ext}_2(n, W_5) = \frac{5}{8}n^2 + o(n^2)$.

Claim 5.8. $\mathbf{ext}_2\left(n, \{W_5, Z_4\}\right) = \frac{4}{7}n^2 + o(n^2)$.

Proof of Claims 5.7, 5.8. (a) Observe, that $W_5 \not\subseteq \mathbb{S}^{2,2}(n)$. Further, denote the 4-path structure (on Figure 4) by $\mathbb{P}^4(n)$. Then $\mathbb{P}^4(n) \subseteq \mathbb{S}^{2,2}(2n)$ does not contain Z_4 either. These graph sequences yield the lower bounds.

(b) To prove the upper bounds, observe first that if

$$e(U_n) > \frac{4}{7}n^2 + \varepsilon n^2,$$

then U_n contains an $\mathbb{S}^{1,2}(\mu)$, assumed that $n > n_0(\varepsilon, \mu)$, by Claim 5.6. By the augmentation procedure we also have some augmentation of this structure $\mathbb{S}^{1,2}(\mu)$. Since the "double triangle" structure, $T^{2,0}_{3\mu,3}$ (identical with $\mathbb{S}^{1,1,1}(3\mu)$) contains all the 3-chromatic graphs and $\chi(W_5) = 3$, therefore the double triangle is excluded in our augmentations. It is not too difficult to see that we have 3 possibilities: either we get the $\mathbb{P}^4(m)$-structure, or the $\mathbb{S}^{2,2}(m)$ structure, or $\mathbb{S}^{1,3}(m)$. The last structure contains W_5. In case of Claim 5.8 (by $Z_4 \subseteq \mathbb{S}^{2,2}(4)$) $\mathbb{S}^{2,2}(m)$ is also excluded. So we get the $\mathbb{P}^4(m)$-structure in Claim 5.8, and the $\mathbb{S}^{2,2}(m)$ structure, in Claim 5.7.

(c) We still have to check the augmentations of the remaining structures: in principle, some augmentations could be better and still $\mathcal{L}$-free. One can easily see that all the augmentations of $\mathbb{P}^4(m)$ and all the augmentations of $\mathbb{S}^{2,2}$ contain W_5. ∎

5.4. Examples: directed case

The augmentation "does not feel" the orientations. Therefore, speaking of the general theory, we do not see too much difference between the directed and undirected cases. If A is a matrix corresponding to a directed structure, $\frac{1}{2}(A + A^*)$ will correspond to the multigraph case and this "mapping" $A \mapsto \frac{1}{2}(A + A^*)$ sends dense matrices into dense matrices and if B is an augmentation of A then $\frac{1}{2}(B + B^*)$ is an augmentation of $\frac{1}{2}(A + A^*)$.

So, e.g., (15) and (16) can be shown to imply analogous results for digraphs.

5.4.1. Degenerate directed problems. Consider $\overrightarrow{\mathbb{D}}_1$. First we answer the following question: when is $\mathbf{ext}\,(n, \overrightarrow{L}) = o(n^2)$? If $\chi(\overrightarrow{L}) \geq 3$ then $\mathbf{ext}\,(n, \overrightarrow{L}) > \left\lfloor \frac{n^2}{4} \right\rfloor$. So we may assume that $\chi(\overrightarrow{L}) = 2$. If in all 2-colorings of $\overrightarrow{L}$ there are arcs in both directions between the color classes, then taking the bipartite graph $T_{n,2}$ and orienting all the edges from the first class towards the second class we see that

$$\overrightarrow{\mathbf{ext}}\,(n, \overrightarrow{L}) \geq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

(This is the case, e.g., if $\overrightarrow{L}$ contains a directed path of 3 vertices.) On the other hand, if $\overrightarrow{L}$ has a 2-coloring where all the arcs go from the first class to the second one, then (using the Kővári–T. Sós–Turán theorem, [48]) one can easily see that

$$\mathbf{ext}\,(n, \overrightarrow{L}) = O(n^{2-c}),$$

for some $c > 0$ depending only on $v(\overrightarrow{L})$.

5.5. The algorithmic solution

Theorem 5.9 (Approximation [8, Theorem 2]). *For any universe $\mathbb{U}_q$ or $\overrightarrow{\mathbb{D}}_s$ every family $\mathcal{L}$, and every $\varepsilon > 0$, there exists a finite subfamily $\mathcal{L}^* \subseteq \mathcal{L}$ for which, for n sufficiently large,*

$$\mathbf{ext}\,(n,\mathcal{L}) \le \mathbf{ext}\,(n,\mathcal{L}^*) \le \mathbf{ext}\,(n,\mathcal{L}) + \varepsilon n^2. \tag{17}$$

For $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$ the Approximation Theorem was superseded[23] by

Theorem 5.10 (Compactness [9, Theorem 3]). *For $\mathbb{U}_2$ (or $\overrightarrow{\mathbb{D}}_1$) for every infinite family $\mathcal{L}$ of forbidden subdigraphs there exists a finite subfamily $\mathcal{L}^* \subset \mathcal{L}$ for which*

$$\mathbf{ext}\,(n,\mathcal{L}) = \mathbf{ext}\,(n,\mathcal{L}^*) + o(n^2) \quad \text{as} \quad n \to \infty,$$

and such that any dense matrix A is asymptotically extremal for $\mathcal{L}$ iff it is asymptotically extremal for $\mathcal{L}^*$.

Eventually we developed an algorithm to find, for any such family, all dense matrices A that are asymptotically extremal for $\mathcal{L}$:

Theorem 5.11 (Algorithmic solution [9, Theorem 4]). *Consider the universe $\mathbb{U}_2$ or $\overrightarrow{\mathbb{D}}_1$. Given a subroutine ("oracle") for deciding, for a given family $\mathcal{L}$ and any dense matrix A, whether some $A(n)$ contains some $L \in \mathcal{L}$, there exists a finite algorithm (independent of $\mathcal{L}$ except that it uses the subroutine) that determines all dense matrices A that are asymptotically extremal for $\mathcal{L}$.*

The set of attainable densities is interesting in itself. We proved

Theorem 5.12 ([9, Theorem 1]). *Consider the universe $\mathbb{U}_2$ or $\overrightarrow{\mathbb{D}}_1$. For any $\gamma \ge 0$ there exist only finitely many dense matrices A such that $g(A) = \gamma$,*

and

Theorem 5.13 ([9, Theorem 2]). *The set of attained densities is well ordered under the usual ordering of the reals.*

However, the order type of the set of densities of matrices[24] is definitely not that of the natural numbers.

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[23] However, Theorem 5.9 was proved for all multiplicities, while Theorem 5.10 is known only for digraphs of multiplicity 1.

[24] of the type investigated in this paper: with off-diagonal entries 0 or 2, and main diagonal entries 0 or 1.

6. Sidorenko's Solution

We conjectured [6] that if A is an extremal matrix for $\overrightarrow{\mathcal{L}}$ then its size can be bounded by some function of the graphs in $\overrightarrow{\mathcal{L}}$. Sidorenko proved this for $\mathbb{U}_2$ and $\overrightarrow{\mathbb{D}}_1$. This immediately gave a new, much simpler algorithm for solving the extremal problem: the entries of A are bounded, so if one has a bound on the size, then one can list the (boundedly many) matrices A for which it has to be checked if $A(n)$ contains forbidden subgraphs: then we take those matrices which have maximum density among the remaining ones. That provides a complete solution.

Theorem 6.1 (Sidorenko, [64]). *Consider $\mathbb{U}_2$. Let $R(k,\ell)$ be the Ramsey number for (k,ℓ).*[25] *Put $k=\min_{L\in\mathcal{L}} v(L)$ and $k'=\max_{L\in\mathcal{L}} v(L)$. Then any dense extremal matrix A for $\mathcal{L}$ has at most $R(k,k+6)-1)(k'-l)$ rows (and columns).*

Before continuing, let us introduce the notion of a **dense multigraph**. If G is a multigraph (or digraph) and $A:=A(G)$ is its adjacency matrix, we shall call G dense if A is dense. Sidorenko also disproved some of our conjectures, by

Construction (Sidorenko). *Let G be a connected, $q-1$-regular* **simple** *graph and join any two vertices of G by $q-1$ additional (parallel) edges. The obtained $G^q\in\mathbb{U}_q$ is a dense q-multigraph and $g(G^q)=q-1$.*

This shows that (already) for $q=3$ there are infinitely many dense matrices of density 2.

6.1. The "jumping constants"

We conjectured that the set $\mathcal{D}_q$ of the densities of multigraphs in $\mathbb{U}_q$ is **well-ordered**: there are neither densities of infinite multiplicity, nor a sequence of strictly decreasing densities. The "infinite multiplicity" part of this conjecture was disproved by Sidorenko, [64] and the "decreasing densities" part, for $q\geq 4$ by Rödl and Sidorenko, [62].

Let us explain some details about this phenomenon. For $q=1$ the conjecture follows easily from the Erdős–Stone–Simonovits theorem. It was

[25] i.e. the maximum order of a graph not containing a complete k-graph, neither an independent ℓ-tuple.

proved for $q = 2$ in [9] and is still open for $q = 3$. (For hypergraphs it was a famous conjecture of Erdős and disproved in some sense[26], by Frankl and Rödl, [30].)

The meaning of this conjecture would have been that for each $\gamma > 0$ it asserts the existence of a $\gamma^* > \gamma$ for which for each sequence of multigraphs G_n with

$$e(G_n) > (\gamma + \varepsilon)\binom{n}{2}$$

there is a sequence $H_m \subseteq G_n$ with $m = m_n \to \infty$ and

$$e(H_m) > \gamma^* \binom{m}{2}.$$

Then we could have said: the constant jumped up from γ to γ^*. This "jumping" for $\gamma = 0$ follows from a theorem of Erdős, [16].

One can easily see that (in any universe) the "Jumping Constant" conjecture and the Approximation theorem imply the "Compactness theorem".

7. Extremal Digraphs and Multigraphs: The Problem of Uniqueness

More difficult than the asymptotic determination of the extremal numbers is the determination of the structure of the extremal digraphs and multigraphs for a given family $\mathcal{L}$ of forbidden subgraphs. We have seen that even in very simple cases — for example where $\mathcal{L}$ consists only of the cyclic triangle — it can happen that, as $n \to \infty$, there will exist extremal digraphs differing from one another in more than cn^2 edges [11]. We do have in preparation a paper which characterizes situations where this cannot happen [10]; i.e. where two "almost" extremal digraphs with n vertices will differ from one another by $o(n^2)$ directed edges as $n \to \infty$. Below we formulate the main result of this paper. We also characterized the extremal multigraphs in certain general situations [7], [10].

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[26] They disproved what Erdős asked but not what Erdős really meant: Erdős wanted to know if the smallest possible density $\frac{1}{27}$ is a jumping constant or not?

7.1. Weak maximality condition, $\mathcal{WMC}$

We have seen in connection with Theorem 2.1 that even in the simplest cases there may be many different extremal graphs that cannot be transformed into each other just by changing $o(n^2)$ edges: this happens even for $\mathbb{U}_2$ and K_3^*. The reason of this phenomenon is that various extremal structures can be combined into new extremal structures. If we exclude the simplest combination of extremal structures than all the extremal structures will be asymptotically the same. This is the meaning of the theorem below. To formulate it we need some definitions.

Sometimes we may have many different extremal matrices, say, $A_1, \dots, A_k$ for an $\mathcal{L}$ and they can be combined into some further extremal patterns, (matrices that are not dense!). So next we introduce a condition that rules out the possibility of combining different extremal structures derived from dense matrix graphs.

Definition 7.1 (Equivalent matrices)**.** The matrices A and B are called equivalent if $\{ A\langle x_1, \dots, x_a\rangle \}$ and $\{ B\langle y_1, \dots, y_b\rangle \}$ coincide.[27]

There are two basic examples of this: (a) if B is obtained by permuting the rows and columns of A in the same way, or (b) if we split each class V_i of $(A\langle x_1, \dots, x_a\rangle)$ into two classes V_i' and V_i'' and B describes the "corresponding structure": that does not changes the graph but it changes the representation.

Definition 7.2 (Weak maximality condition $\mathcal{WMC}$)**.** Given a family $\mathcal{L}$ of forbidden subgraphs, we shall say that $\mathcal{L}$ satisfies the **weak maximality condition**, if there are only finitely many extremal matrices $A_1, \dots, A_k$ and

for every i and j if $s(A_i) = a_i$ and $s(A_j) = a_j$, then choosing the $a_i \times a_j$ matrix Q and the $a_j \times a_i$ matrix R (with the appropriate entries) arbitrarily, the $(a_i + a_j) \times (a_i + a_j)$ matrix

$$B = \begin{pmatrix} A_i & Q \\ R & A_j \end{pmatrix} \tag{18}$$

satisfies one of the following conditions:

(i) B is forbidden in the sense that if h is sufficiently large, then $B\langle h\mathbf{e}\rangle$ contains some $L \in \mathcal{L}$.

[27] i.e., the sets of A-colorable graphs and B-colorable graphs coincide.

(ii) if v_i and v_j are the optimum vectors of A_i and A_j and w is obtained by concatenating v_j to v_i, then

$$wBw^* < 4g(A_i), \tag{19}$$

or

(iii) B is equivalent to A (as described in Definition 7.1).

(Clearly, (i) asserts that B cannot be extremal, since the B-structures contain some forbidden subgraphs; (ii) says that $g(B)$ is small for B to be extremal and (iii) says that we did not really obtain new structures.)

Theorem 7.3 (Uniqueness, multigraphs). *Assume that for $\mathcal{L}$ $A_1, \dots, A_k$ are the* **extremal** *matrices, and there are* **no other ones**. *Then the following statements are equivalent:*

(#) *$\mathcal{L}$ satisfies the weak maximality condition,*

(##) *For any asymptotically extremal multigraph sequence (U_n), for every $n > 0$ for some $i = i_n$ one can change $o(n^2)$ edges of U_n to get $A_i(n)$.*

Remark 7.4. For a finite $\mathcal{L}$ one can check the $\mathcal{WMC}$-condition by a polynomial algorithm.

Watch out: (##) excludes all extremal graphs "far from the listed dense matrix graphs": not only those obtained from some dense or not necessarily dense matrices. For digraphs the assertion is slightly more complicated because of the transitive tournaments corresponding to the diagonal elements $a_{ii} > 0$ of the matrices.

8. Difficulties in the General Case, or: Why is the Case $q = 2$ Simpler?

One of the question is, why is the case of $\mathbb{U}_2$ simpler than the case of $\mathbb{U}_q$ for $q \geq 3$. There is a simple answer for this. Namely, most of the problems in our proofs are created by the positive diagonal matrix-entries: by classes of the structures forming complete graphs. Generally, by Lemma 3.9, if we have a *dense* structure H, where in some classes $V_1, \dots, V_d$ any two vertices are joined by $q-1$ edges, then $H = T^{q,q-1}_{m,d} \otimes H^*$ for some H^* having maximum diagonal elements $\leq q-2$. For $q=2$ the diagonal elements of H^* can be only 0, there are no middle elements between 0 and $q-1$: this makes the difference.

9. The "Most General Case"

A universe of graphs where Turán type extremal problems do make sense is the class of *directed multihypergraphs.* This means that r and q are fixed, a set of vertices V is given and a family $\mathcal{F}$ of some subsequences $(a_1,\dots,a_r)\in V^r$ with multiplicity $\mu(a_1,\dots,a_r)\leq q$. The triple $(V,\mathcal{F},\mu)$ can be regarded as a **directed r-uniform q-multihypergraph**. Loops are excluded: $a_i\neq a_j$ if $i\neq j$.[28]

The extremal problems, numbers and the extremal directed multihypergraphs can be defined in the obvious ways and many general results from ordinary extremal graph theory can be generalized to this universe.

We were discussing such problems in [12]. Below we shall give some illustrations of these results.

The basic "extremal graph problem" immediately can be generalized to these objects. The Katona–Nemetz–Simonovits observation [45] that

$$\frac{\mathbf{ext}\,(n,\mathcal{L})}{\binom{n}{r}} \searrow c_{\mathcal{L}}$$

(i.e., this ratio is monotone decreasing and therefore converges to some constant $c_{\mathcal{L}}$) also easily generalizes to this case. To describe another general principle, take the simplest case, when $\mathcal{L}=\{L\}$.

Definition 9.1 (Blown-up directed q-multihypergraph)**.** Given L and an integer t, $L(t)$ has $t\cdot v(L)$ vertices: each vertex $x\in V(L)$ is replaced by a t-tuple $T(x)$, and these t-tuples are disjoint. The ordered r-tuple

$$(\tilde{x}_1,\dots,\tilde{x}_r)\qquad \tilde{x}_i\in T(x_i)\quad\text{for}\quad i=1,\dots,r$$

is an ordered hyperedge of $L(t)$ of multiplicity ν iff $(x_1,\dots,x_r)$ has multiplicity ν in L.

[28] With multiplicity functions we always have two choices: either we take all the r-sequences and allow $\mu=0$ or we take only some of them and assume that $\mu\in[1,q]$: this ambiguity does not really matter.

Erdős has proved [19] that if $K_\ell^{(r)}$ is the r-uniform complete graph on ℓ vertices and t is arbitrary, then

$$\mathbf{ext}\left(n, K_\ell^{(r)}(t)\right) - \mathbf{ext}\left(n, K_\ell^{(r)}\right) = o(n^r), \tag{20}$$

i.e. the extremal edge density does not change if we blow up $K_\ell^{(r)}$. (This is a generalization of the Erdős–Stone theorem.) We observed [12] that Erdős' original theorem extends to any directed multihypergraph:

$$\mathbf{ext}\left(n, \overrightarrow{L}^{(r)}(t)\right) - \mathbf{ext}\left(n, \overrightarrow{L}^{(r)}\right) = o(n^r).$$

We also proved

Theorem 9.2 (Brown–Simonovits, [12]). *In the above setting, let*

$$\gamma_L := \lim_{n\to\infty} \frac{\mathbf{ext}\,(n, L)}{\binom{n}{r}},$$

For every $\varepsilon > 0$ *there exists a* $c_L(\varepsilon) > 0$ *such that if*

$$e(H_n) > (\gamma_L + \varepsilon)\binom{n}{r}$$

then H_n *contains at least* $c_L(\varepsilon)\cdot n^{v(L)}$ *copies of* L.

By the Erdős hypergraph theorem [16] this second assertion immediately implies (20), (see Brown–Simonovits, [12] for more details).

10. "Excluded Density" Problems

Here we consider the following problem:

> Fix two integers k, ℓ. Given a multigraph M_n, how many edges can it have without containing k vertices and ℓ edges (counted with multiplicities) among these k vertices.

Obviously, this is a Turán type extremal graph problem. For some related ordinary extremal graph problems see e.g., Griggs, Simonovits and Thomas, [36]. Here we do not assume that the multiplicities are bounded by some q, yet, if there is just one edge with multiplicity $\geq \ell$ then we have a forbidden subgraph. So we could say that

$$\boxed{\text{The multiplicities are implicitly bounded by } \ell - 1.}$$

Well, this is only more or less so. This problem has two versions. In the first version we assume that the multiplicities are nonnegative, in the second one we allow negative multiplicities as well. That means that if, e.g., we have an M_n where one edge has multiplicity 100 and all the other edges have multiplicities -1, then the resulting graph has an edge of high multiplicity, yet it has no 20-vertex subgraph with at least one edge, assumed that the number of edges is calculated by adding up the multiplicities.

OK, one could say, *forget about the negative multiplicities.* Again, this is not so simple, since the solution of the case of non-negative weights very strongly depends on the solution of the case of the negative weights.

For multigraphs this field was (perhaps) started by Bondy and Tuza, [5] and in some sense completely solved by Füredi and Kündgen [33].

A *weighting* of a graph G is a function $w : E(G) \to \mathbb{Z}$ that assigns an integer weight to each edge. The *weight of a subgraph* H of G is just the total weight of its edges, $w(H) := \sum \{ w(e) \, : \, e \in E(H) \}$. The *weight of a vertex set* A is the weight of the subgraph induced by A, $w(A) := w\big(G|_A\big)$. A weighted graph (G, w) is (k, r)*-dense* if every set of k vertices has weight at most r. In this language we define the weighted Turán numbers as

$$\mathbf{ext}_{\mathbb{Z}}(n, k, r) := \max \big\{ w(G) \, : \, \big|V(G)\big| = n, \ (G, w) \text{ is } (k, r) \text{ -dense} \big\} .$$

Since every non-edge can be considered to be an edge of weight 0, it will suffice to consider $G = K_n$. We can refer to a graph by the corresponding $\{0, 1\}$-weighting, where edges are weight 1 and non-edges are weight 0. Multigraphs can also be viewed as weightings of K_n.

By [45], $\mathbf{ext}_{\mathbb{Z}}(n, k, r)/\binom{n}{2}$ is a monotone non-increasing sequence in n. Therefore the *asymptotic density*

$$\alpha(k, r) := \lim_{n \to \infty} \frac{\mathbf{ext}_{\mathbb{Z}}(n, k, r)}{\binom{n}{2}}$$

exists for all k and $r \geq 0$. Füredi and Kündgen gave a simple method [29] to find $\alpha(k, r)$ for all (k, r). Furthermore, they gave an exact answer for

[29] but we skip it here.

$\mathbf{ext}_{\mathbb{Z}}(n,k,r)$ for "at least half the pairs" (k,r) for all n [30]. Surprisingly, here the error term is always only $O(n)$:

$$\mathbf{ext}_{\mathbb{Z}}(n,k,r) = \alpha(k,r)\binom{n}{2} + O(n).$$

10.1. The Häggkvist–Thomassen theorem

Below we list a few digraph results, not in our main scope, yet connected to it.

Beside the Brown–Harary paper and the first Brown–Erdős–Simonovits paper, one of the first important papers on digraph extremal problems is that of Häggkvist and Thomassen [37] on "Pancyclic digraphs". (A graph/digraph is pancyclic if it contains cycles of all possible lengths.) Let $\overrightarrow{K}_{p,p}$ denote the digraph obtained from $L := K_{p,p}$ by replacing each edge of L by two arcs of opposite directions.

Theorem 10.1 (R. Häggkvist–C. Thomassen [37]). *If $\overrightarrow{D}_n$ is a strongly connected digraph with minimum degree*[31] $\geq n$, *then $\overrightarrow{D}_n$ is pancyclic, unless it is a $\overrightarrow{K}_{p,p}$.*

The paper contains many further results that guarantee that if $e(\overrightarrow{D}_n)$ is sufficiently large, then the oriented cycle $\overrightarrow{C}_k \subseteq \overrightarrow{D}_n$. There are two important differences between the corresponding simple graph problem and the digraph problem:

(a) Since the transitive tournament $\overrightarrow{T}_n$ contains no directed cycle, our conditions on $\overrightarrow{D}_n$ have to eliminate somehow $\overrightarrow{D}_n = \overrightarrow{T}_n$. Mostly Häggkvist and Thomassen assume also that $\overrightarrow{D}_n$ is strongly connected.

(b) The other difference is that while in the Erdős–Gallai theorem for simple graphs the extremal graphs have $O(n)$ edges, here the extremal numbers are always around $\binom{n}{2}$ if k is small and we wish to ensure a $\overrightarrow{C}_k$ in $\overrightarrow{D}_n$.

We again skip the details.

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[30] Speaking of countably many cases, the expression "at least half the pairs" needs some clarification.

[31] Here degree = the sum of indegree and outdegree.

10.2. A "path"-problem strongly related to Turán's theorem

Let $\mathcal{G}_k$ be the set of oriented graphs, i.e. digraphs with no 2-cycles and no loops, such that for any two vertices x and y there are at most k directed paths from x to y. Howalla, Dabboucy, and Tout [40, 41] determined the maximum size of $\overrightarrow{D}_n \in \mathcal{G}_k$, for $k = 1, 2, 3$, and characterized the extremal graphs. For $k = 1$ this coincides with $\mathbf{ext}_1(n, K_3)$.

10.3. Directed Trees

For simple graphs, if T is a tree, then $\mathbf{ext}\,(n, T) = O(n)$. For directed graphs, P. Erdős conjectured and R. L. Graham proved [35] that if $\overrightarrow{T}$ is a tree without directed path of length 2, then

$$\overrightarrow{\mathbf{ext}}\,(n, \overrightarrow{T}) = O(n). \tag{21}$$

It is worth noting that this is among the early digraph extremal theorem. We have seen that the assumption that $\overrightarrow{T}$ contains no directed path of length ≥ 2 is necessary for (21).

10.4. Hamiltonicity and extremal graph problems

We do not know of too many early results on digraph extremal problems. Let us mention yet some nice early ones.

For ordinary graphs the important extremal graph theorem of Erdős and Gallai [21] on paths and cycles has an important subcase, namely, Dirac's theorem, on the Hamiltonicity of graphs under certain degree conditions.

For digraphs, in some sense, the generalization of this Erdős–Gallai theorem is missing, yet, there are many results connected to Hamiltonicity of digraphs.

We have already mentioned the Häggkvist–Thomassen theorem. We should mention here (among others) some results of Bondy, Ghouila-Houri, Meyniel and Woodall. For the sake of brevity, we shall not formulate all of them. Let us start with a result of Woodall [77], who proved, among others, a strengthening of a theorem of Ghouila–Houri [34]:

Theorem 10.2 (Woodall). *If $\overrightarrow{D}$ is a directed graph on n vertices in which $\rho_{out}(a) + \rho_{in}(b) \geq n$, for every pair of distinct vertices a and b such that a*

is not joined to b (by an edge of $\vec{D}$), then $\vec{D}$ has a (directed) Hamiltonian circuit.

A strengthening of this is

Theorem 10.3 (M. Meyniel [55]). *If $\vec{D}$ is a strongly connected directed graph on n vertices in which $\rho(a)+\rho(b) \geq 2n-1$ for every pair of nonadjacent vertices a and b, then $\vec{D}$ has a directed Hamiltonian circuit.*

Another result, of extremal character is the theorem of Heydemann, Sotteau, and Thomassen [39], proving a conjecture of A. Benhocine and A. P. Wojda:

Theorem 10.4. *If $\vec{D}_n$ is a digraph with at least $(n-1)(n-2)+3$ directed edges, then $\vec{D}_n$ contains all the orientations of C_n but the directed circuit $\vec{C}_n$, assuming that $\vec{D}$ has no parallel arcs.*

There are many results connected with the existence of even directed cycles in digraphs, see, e.g.,

Theorem 10.5 (Thomassen [73]). *If $\vec{D}$ is a strongly connected digraph of minimum in- and outdegree at least 3 then $\vec{D}$ contains a directed cycle of even length.*

10.5. Unavoidable Subgraphs

Linial, Saks, and T. Sós [49] called a digraph $\overrightarrow{L}$ n-**unavoidable** if every tournament $\overrightarrow{T}_n$ contains $\overrightarrow{L}$. Why does the description of unavoidable graphs fit into our survey? The universe $\overrightarrow{\mathbb{O}}$ of oriented graphs, i.e., where between any two vertices there is at most one arc, is a subuniverse of $\overrightarrow{\mathbb{D}}_1$: considering problems in $\overrightarrow{\mathbb{D}}_1$ where $\mathbb{D}K_2$ is also excluded, we get problems on $\overrightarrow{\mathbb{O}}$.

One could ask the following "inverse extremal problem": For which $\overrightarrow{L}_k$ is it true that

$$\mathbf{ext}\left(n, \{\mathbb{D}K_2, \overrightarrow{L}_k\}\right) = \binom{k}{2}?$$

In other words, which are those k-vertex digraphs that occur in each k-vertex tournament?

Linial, Saks, and T. Sós investigated (among others) how large $e(\overrightarrow{L}_k)$ can be if $\overrightarrow{L}_k$ is unavoidable. Denote the maximum by $f(n)$.

Theorem 10.6 (Linial, Saks, and T. Sós). *There exist positive constants c_1 and c_2 such that for all positive integers n,*

$$n \log_2 n - c_1 n \geq f(k) \geq n \log_2 n - c_2 n \log\log n.$$

Some classical examples of n-unavoidable digraphs include Hamiltonian paths, [58], anti-directed Hamiltonian paths [59] and more.

Saks and T. Sós also proved [60] that if $n \geq 2$ then every tournament $\overrightarrow{D}_n$ contains a rooted directed tree $\overrightarrow{T}_n$ in which every branch is a path. Further, they proved that every tournament $\overrightarrow{D}_n$ contains a rooted directed tree $\overrightarrow{T}_n$ of height at most 3.

Again, we cut this part short and refer the reader (among others) to the papers of Bloom and Burr, [2], Xiaoyun Lu [50] (on unavoidable rooted 2-trees), Petrovic, [61] (completely forgetting about the strongly related undirected case).

10.6. Topological Subgraphs

If A is a topological subgraph of B, we shall use the notation $A \prec B$.

Motivated partly by Kuratowski theorem, partly by other results, one may ask for ordinary graphs:

- How many edges ensure $L \prec G_n$?
- How many edges ensure a subcontraction of L in G_n?

An important result of Mader [51] shows that for every p there exists a constant c_p such that

$$e(G_n) > c_p n \quad \text{and} \quad n > n_0$$

imply that $K_p \prec G_n$.

The above results can be asked for digraphs as well.

A subdivision of a digraph $\overrightarrow{D}$ arises from $\overrightarrow{D}$ by subdividing any arc of $\overrightarrow{D}$ by an arbitrary number of distinct new vertices.

Given a digraph $\overrightarrow{D}$, let $\mathbf{Top}\,(\overrightarrow{D})$ denote the family of digraphs formed by subdividing the arcs of $\overrightarrow{D}$, or, in other words, by replacing each arc by a directed path in the same direction.

Theorem 10.7 (Mader, [52]). *Every finite digraph of minimum outdegree 3 contains a subdivision of the transitive tournament on 4 vertices.*

For some related results see, e.g., the survey of W. Mader, [53], primarily on undirected problems but also touching on the directed graph variants.

In a series of 3 papers, [42,43,44] Chris Jagger proved results concerning both earlier results of Bollobás and Thomason and also some others, related to the directed extremal graph problem where the excluded graphs are either the topological versions of some complete digraphs ($\overrightarrow{\mathcal{L}} = \mathbf{Top}\,(\overrightarrow{K}_p)$) or (in the other case) those digraphs that can be subcontracted onto $\overrightarrow{K}_p$.

10.7. Some further problems, results

As ordinary extremal graph theory has many less well known branches, the theory digraph extremal problems also has many generalizations of these problems. Many of these problems are connected to some posets, ... Without going into details, we just mention some papers related to such problems.

There are many papers on the "competition graphs", introduced by J. E. Cohen. A related notion, the competition number $k(G)$ of an undirected graph G was introduced by Roberts: this is the minimum number of isolated vertices to be added to G to obtain a competition graph of an acyclic directed graph. In [38] Harary, Kim and Roberts prove $k(G_n) \le \lfloor n^2/4 \rfloor - n+2$. They also prove that there are exactly two graphs whose competition numbers achieve this bound. This is a generalization of Turán's theorem.

The paper of Schelp and Thomason [63] is connected, e.g., with some subgraphs of the n-dimensional cube.

A paper of Maurer, Rabinovitch and Trotter [54] considers the subgraphs $\overrightarrow{D}_n$ of a transitive tournament $\overrightarrow{T}_n$, defined on $V(\overrightarrow{T}_n) = \{1, \dots, n\}$ and satisfying the following condition: for any m consecutive integers, $M := \{h, h+1, \dots, h+m-1\}$, for any two vertices $x, y \in M$, the induced subgraph $\overrightarrow{D}[M]$ has at most one directed path joining x to y. What is the maximum of $e(\overrightarrow{D}_n)$ under this condition? This problem is again, related to Turán's theorem.

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