Less than 2^ω many translates of a compact nullset may cover the real line

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Abstract

We answer a question of Darji and Keleti by proving that there exists a compact set $C_0 \subset \mathbb{R}$ of measure zero such that for every perfect set $P \subset \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $(C_0 + x) \cap P$ is uncountable. Using this C_0 we answer a question of Gruenhage by showing that it is consistent with ZFC (as it follows e.g. from $\mathrm{cof}(\mathcal{N}) < 2^\omega$) that less than 2^ω many translates of a compact set of measure zero can cover $\mathbb{R}$.

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Introduction

The behaviour of the classical cardinal invariants in the Cichoń diagram is very well described. See e.g. the monograph [BJ] for definitions and details. We also follow the terminology of [BJ] throughout the paper.

The invariant we are interested in is $cov(\mathcal{N})$, that is the least cardinal κ for which it is possible to cover $\mathbb{R}$ by κ many nullsets (sets of Lebesgue measure zero), and also some variants of $\mathrm{cov}(\mathcal{N})$. There are two natural ways to modify this definition. (See [BJ] Chapter 2.6 and 2.7.) First, $cov^*(\mathcal{N})$ is the least cardinal κ for which it is possible to cover $\mathbb{R}$ by κ many translates of some nullset. In other words, $\mathrm{cov}^*(\mathcal{N}) = \min\{|A| \mid A \subset \mathbb{R}, \exists N \in \mathcal{N}, A + N = \mathbb{R}\}$, where $|A|$ is the cardinality of A, $\mathcal{N}$ is the σ-ideal of nullsets and $A + N = \{a + n \mid a \in A, n \in N\}$. The other possible modification is $cov(c\mathcal{N})$, that is the least cardinal κ for which it is possible to cover $\mathbb{R}$ by κ many compact nullsets. (At this point we depart from the terminology of [BJ] as this notion is denoted by $\mathrm{cov}(\mathcal{E})$ there.) It can be found in these two chapters of this monograph that both $\mathrm{cov}^*(\mathcal{N}) < 2^\omega$ and $\mathrm{cov}(c\mathcal{N}) < 2^\omega$ are consistent with ZFC.

G. Gruenhage posed the natural question whether $cov^*(c\mathcal{N}) < 2^\omega$ is also consistent, that is, whether we can consistently cover $\mathbb{R}$ by less than continuum many translates of a compact nullset.

The main goal of this paper is to answer this question in the affirmative via an answer (in ZFC) to a question of U. B. Darji and T. Keleti that is also interesting in its own right.

We remark here that under CH (the Continuum Hypothesis, or more generally under $\mathrm{cov}(\mathcal{N}) = 2^\omega$) the real line obviously cannot be covered by less than 2^ω many nullsets. Therefore it is consistent that the type of covering we are looking for does not exist.

So the interesting case is when the consistent inequality $\mathrm{cov}^*(\mathcal{N}) < 2^\omega$ holds. The nullset in this statement can obviously be chosen to be G_δ. So the content of Gruenhage's question actually is whether this can be an F_σ or closed or compact nullset. We formulate the strongest version.

Question 0.1 *(Gruenhage) Is it consistent that there exists a compact set $C \subset \mathbb{R}$ of Lebesgue measure zero and $A \subset \mathbb{R}$ of cardinality less than 2^ω such that $C + A = \mathbb{R}$?*

For example Gruenhage showed that no such covering is possible if C is the usual ternary Cantor set (see [DK] and for another motivation of this question see [GL]).

Working on this question Darji and Keleti [DK] introduced the following notion.

Definition 0.2 (Darji - Keleti) Let $C \subset \mathbb{R}$ be arbitrary. A set $P \subset \mathbb{R}$ is called a *witness for C* if P is perfect and for every translate $C + x$ of C we have that $(C + x) \cap P$ is countable.

Obviously, if there is a witness P for C then less than 2^ω many translates of C cannot cover P, so they cannot cover $\mathbb{R}$. Motivated by a question of D. Mauldin, who asked what can be said if C is of Hausdorff dimension strictly less than 1, Darji and Keleti proved the following.

Theorem 0.3 *(Darji - Keleti) If $C \subset \mathbb{R}$ is a compact set of packing dimension $\dim_p(C) < 1$ then there is a witness for C, and consequently less than 2^ω translates of C cannot cover $\mathbb{R}$.*

They posed the following question, an affirmative answer to which would also answer the original question of Gruenhage in the negative.

Question 0.4 *(Darji - Keleti) Is there a witness for every compact set $C \subset \mathbb{R}$ of Lebesgue measure zero?*

We will answer this question in the negative, which still leaves the original question of Gruenhage open. Then we will show that using the same ideas it is also possible to give an affirmative answer to Gruenhage's question.

1 Answer to the question of Darji and Keleti

The following set is fairly well known in geometric measure theory, as it is probably the most natural example of a compact set of measure zero but of Hausdorff and packing dimension 1. It was investigated for example by Erdős and Kakutani [EK].

Definition 1.1 Denote
$$C_0 = \left\{ \sum_{n=2}^{\infty} \frac{d_n}{n!} \;\middle|\; d_n \in \{0, 1, \dots, n-2\} \ \forall n \right\}.$$

Think of d_n as digits with "increasing base"; then all but countably many $x \in [0,1]$ have a unique expansion
$$x = \sum_{n=2}^{\infty} \frac{x_n}{n!},$$

where $x_n \in \{0, 1, \ldots, n-1\}$ for every $n = 2, 3, \ldots$

The set of real numbers with prescribed first n digits is a closed interval. Let us call these *basic intervals of level* n and denote this collection by $\mathcal{B}_n$. Let $\mathcal{B} = \cup_{n=0}^{\infty} \mathcal{B}_n$ be the set of all *basic intervals*. So $\mathcal{B}$ forms a tree under inclusion, the n^{th} level of which is $\mathcal{B}_n$, which consists of $n!$ nonoverlapping intervals of length $\frac{1}{n!}$.

Using the above expansion it is easy to see that $C_0 \subset \mathbb{R}$ is a compact set of Lebesgue measure zero.

The following theorem answers the question of Darji and Keleti.

Theorem 1.2 *For every perfect set $P \subset \mathbb{R}$ there exists a translate $C_0 + x$ of the compact nullset C_0 such that $(C_0 + x) \cap P$ is uncountable.*

Proof. We will show that there exists a perfect set $Q \subset P$ and $y \in \mathbb{R}$ such that $Q + y \subset C_0$. This is clearly sufficient, as then for $x = -y$ we get $Q \subset (C_0 + x) \cap P$, so this intersection is uncountable. First we will construct Q via a dyadic tree of basic intervals, then we will construct the "digits" of y.

By translating P if necessary we can assume that P intersects $(0, \frac{1}{5!})$. Instead of P we may as well work with any perfect subset of it, for if we find the set Q inside this subset then this Q also works for P. So we may find a perfect subset of P in $[0, \frac{1}{5!}]$ and therefore we can assume that P itself is inside $[0, \frac{1}{5!}]$. Moreover, as P is uncountable and the endpoints of the basic intervals form a countable set, we can find a perfect subset of P that is disjoint from the set of endpoints (we used here twice the well known fact that every uncountable Borel set contains a perfect set). Therefore we can assume that P itself is disjoint from the endpoints.

Now we recursively pick an increasing sequence of levels $(l_k)_{k=0}^{\infty}$ and for every k choose a set $\mathcal{I}_{l_k} \subset \mathcal{B}_{l_k}$ of size 2^k such that

1. for each $I \in \mathcal{I}_{l_k}$ there are exactly two intervals in $\mathcal{I}_{l_{k+1}}$ (at level l_{k+1}) that are contained in I, the so called *successors* of I

2. for each $I \in \mathcal{I}_{l_k}$ we have $I \cap P \neq \emptyset$

3. $l_k \geq 2^{k+2} + 1$.

The recursion is carried out as follows. Fix $p_0 \in P$. Let $l_0 = 5$, and at level 5 we pick the (unique) basic interval I containing p_0 in its interior. Let $\mathcal{I}_{l_0} = \mathcal{I}_5 = \{I\}$. The recursion step is as follows. As P is disjoint from the endpoints of the basic intervals, each interval $I \in \mathcal{I}_{l_k}$ (at level l_k) contains some point $p_I \in P$ in its interior by condition 2. As P is perfect, we can choose a distinct point $p'_I \in P$ in I. We can find a large enough n such that the 2^{k+1} distinct points p_I and p'_I $(I \in \mathcal{I}_{l_k})$ are all separated by $\mathcal{B}_n$. Define

$$l_{k+1} = \max\{n, 2^{k+3} + 1\}.$$

Clearly, condition 3 is also satisfied.

Let $\mathcal{I}_{l_{k+1}}$ be the subcollection of $\mathcal{B}_{l_{k+1}}$ consisting of the 2^{k+1} basic intervals containing all the points p_I and p'_I $(I \in \mathcal{I}_{l_k})$. This recursion clearly provides a system of intervals satisfying the required properties.

Now we can define

$$Q = \bigcap_{k=0}^{\infty} \bigcup \mathcal{I}_{l_k}.$$

Let us extend this tree of intervals to the intermediate levels in the natural way, that is, for every $I \in \mathcal{I}_{l_k}$ and successor $J \in \mathcal{I}_{l_{k+1}}$ and every $n \in (l_k, l_{k+1})$ let us add to the tree the unique basic interval of level n that is contained in I and contains J. For $n = 2, 3, 4, 5$ define $\mathcal{I}_n = \{[0, \frac{1}{n!}]\}$. Hence we get $\mathcal{I}_n$ for every $n = 2, 3, \dots$ so that

$$\bigcap_{n=2}^{\infty} \bigcup \mathcal{I}_n = \bigcap_{k=0}^{\infty} \bigcup \mathcal{I}_{l_k} = Q.$$

Our next goal is to define $y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$ so that $Q + y \subset C_0$. Define $y_2 = y_3 = y_4 = y_5 = 0$. For every $n \geq 6$ there exists k such that $l_k < n \leq l_{k+1}$. Clearly, the size of $\mathcal{I}_n$ is 2^{k+1}, and $Q \subset \cup \mathcal{I}_n$. This means that there are at most 2^{k+1} possible values for q_n, where $q \in Q$ and $q = \sum_{n=2}^{\infty} \frac{q_n}{n!}$ (we do not have to worry about nonunique expansions, as $Q \subset P$ so Q is disjoint from the endpoints of the basic intervals). For every such q_n there are at most two values of m such that $q_n + m \in \{n-2, n-1\}$. Hence altogether there are at most $2 \cdot 2^{k+1}$ such "bad" values, so if $n - 1 > 2 \cdot 2^{k+1}$ then we can fix a $y_n \in \{0, 1, \dots, n-2\}$ such that $q_n + y_n \notin \{n-2, n-1\}$ for every possible q_n. But our requirement on n and k, namely $n - 1 > 2 \cdot 2^{k+1}$, is clearly satisfied as $n > l_k \geq 2^{k+2} + 1$ by condition 3.

So we can define

$$y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$$

so that for every $n \geq 6$ we have $y_n \in \{0, 1, \dots, n-2\}$ and that for every $q \in Q$ with $q = \sum_{n=2}^{\infty} \frac{q_n}{n!}$ we have $q_n + y_n \notin \{n-2, n-1\}$. We claim that

$Q+y\subset C_0$, which will complete the proof. Fix $q\in Q$ with $q=\sum_{n=2}^{\infty}\frac{q_n}{n!}$, then

$$q+y=\sum_{n=2}^{\infty}\frac{q_n+y_n}{n!}=\sum_{n=2}^{\infty}\frac{q_n+y_n-n\varepsilon_n}{n!}+\sum_{n=2}^{\infty}\frac{n\varepsilon_n}{n!},$$

where ε_n is the "carried digit", so

$$\varepsilon_n=\begin{cases}0 & \text{if } q_n+y_n\le n-1\\ 1 & \text{otherwise.}\end{cases}$$

Continuing the above calculation we get

$$q+y=\sum_{n=2}^{\infty}\frac{q_n+y_n-n\varepsilon_n}{n!}+\sum_{n=2}^{\infty}\frac{\varepsilon_n}{(n-1)!}=\sum_{n=2}^{\infty}\frac{q_n+y_n-n\varepsilon_n}{n!}+\sum_{n=1}^{\infty}\frac{\varepsilon_{n+1}}{n!}=$$

$$=\varepsilon_2+\sum_{n=2}^{\infty}\frac{q_n+y_n-n\varepsilon_n+\varepsilon_{n+1}}{n!}=\sum_{n=2}^{\infty}\frac{q_n+y_n-n\varepsilon_n+\varepsilon_{n+1}}{n!},$$

since $\varepsilon_2=0$ by e.g. $y_2=0$. We now check that for every $n\ge 2$ the numerator $q_n+y_n-n\varepsilon_n+\varepsilon_{n+1}\in\{0,1,\dots,n-2\}$, which shows that $q+y\in C_0$. For $n<6$ this is clear, as $y_n=0$ and also $q_n=0$ by the assumption $P\subset[0,\frac{1}{5!}]$. For $n\ge 6$ recall that $q_n\le n-1$, $y_n\le n-2$, so $q_n+y_n\le 2n-3$ and also that $q_n+y_n\notin\{n-2,n-1\}$. We separate the cases $\varepsilon_n=0$ and $\varepsilon_n=1$. If $\varepsilon_n=0$, then $q_n+y_n\le n-1$, but then also $q_n+y_n\le n-3$. Therefore $q_n+y_n-n\varepsilon_n+\varepsilon_{n+1}=q_n+y_n+\varepsilon_{n+1}\le n-2$, and we are done. On the other hand, if $\varepsilon_n=1$, then $q_n+y_n-n\varepsilon_n\le n-3$, so $q_n+y_n-n\varepsilon_n+\varepsilon_{n+1}\le n-2$, so this case is also done. This completes the proof. $\square$

2 Answer to the question of Gruenhage

Now we answer the original question of Gruenhage. Recall that *cof*($\mathcal{N}$) is the minimal cardinality of a family $\mathcal{F} \subset \mathcal{N}$ for which every nullset is contained in some member of $\mathcal{F}$.

Theorem 2.1 *$\mathbb{R}$ can be covered by cof($\mathcal{N}$) many translates of C_0, consequently cov***($c\mathcal{N}$) $\leq$ cof($\mathcal{N}$).*

Proof. It is clearly sufficient to cover the unit interval.

Fix $f : \omega \setminus \{0,1\} \to \omega \setminus \{0\}$. A set of the form $S = \Pi_{n=2}^{\infty} A_n$, where $A_n \subset \{0, 1, \ldots, n-1\}$ is of cardinality at most $f(n)$ for every n is called an *f-slalom.* Suppose $\lim_{n\to\infty} f(n) = \infty$, $f(2) = f(3) = f(4) = f(5) = 1$ and also that $f(n) < \frac{n-1}{2}$ for every $n \geq 6$. Combining [BJ] Thm 2.3.9 and [GL] Thm 2.10 we obtain a cover of $\Pi_{n=2}^{\infty}\{0, 1, \ldots, n-1\}$ by cof($\mathcal{N}$) many f-slaloms $\{S_\alpha \mid \alpha < \mathrm{cof}(\mathcal{N})\}$. (We actually obtain a cover of ω^ω first, which is a larger space, so it is trivial to restrict this cover to get a cover of $\Pi_{n=2}^{\infty}\{0, 1, \ldots, n-1\}$. Moreover, [BJ] works with inclusion mod finite, but that makes no difference, as we can replace each slalom by countably many slaloms to get around this difficulty.) For a slalom S define

$$S^* = \left\{ \sum_{n=2}^{\infty} \frac{s_n}{n!} \,\middle|\, (s_n)_{n=2}^{\infty} \in S \right\}.$$

Clearly $\{S_\alpha^* \mid \alpha < \mathrm{cof}(\mathcal{N})\}$ covers the unit interval. The following lemma will complete the proof of the theorem.

Lemma 2.2 *Let f be as above and S be an f-slalom. Than there exists $y \in \mathbb{R}$ such that $S^* + y \subset C_0$.*

Proof. The proof is based on the ideas used in Theorem 1.2. S^* plays the role of Q. Our goal is to define

$$y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$$

with $y_n \in \{0, 1, \ldots, n-2\}$ so that for every $(s_n)_{n=2}^{\infty} \in S$ and for every $n \geq 6$ we have $s_n + y_n \notin \{n-2, n-1\}$. But this is clearly possible by our assumptions on f, as there are at most $f(n) < \frac{n-1}{2}$ possibilities for s_n, hence there are two consecutive values excluded, and so we can find a suitable $y_n \in \{0, 1, \ldots, n-2\}$.

Then by the same calculation as in the last part of the proof of Theorem 1.2 we check that

$$\sum_{n=2}^{\infty} \frac{s_n + y_n}{n!} \in C_0.$$

This completes the proof of the lemma. $\square$

Hence for every $\alpha < \mathrm{cof}(\mathcal{N})$ there exists y_α such that $S_\alpha^* + y_\alpha \subset C_0$, but than for $x_\alpha = -y_\alpha$ we have $S_\alpha^* \subset C_0 + x_\alpha$, so we obtain a cover of the unit interval by $\mathrm{cof}(\mathcal{N})$ many translates of C_0 and therefore the proof of Theorem 2.1 is also complete. $\square$

Corollary 2.3 *It is consistent that less than continuum many translates of a compact set of measure zero cover the real line, that is, cov*$(c\mathcal{N}) < 2^\omega$ is consistent.*

Proof. $\mathrm{cof}(\mathcal{N})$ is consistently less than the continuum [BJ] p. 388. $\square$

3 Remarks and open problems

There are alternative ways to prove Theorem 2.3. Using Theorem 1.2 one can directly use Sacks forcing to show that the translates of C_0 by the ground model reals cover $\mathbb{R}$. Another method is to use the so called CPA axiom (see [CP]). However, in all these cases $\mathrm{cof}(\mathcal{N})$ is less than the continuum. This is not necessary, as in the Laver model $\mathrm{cof}(\mathcal{N}) = 2^\omega = \omega_2$ but it can be shown that the above slalom argument still works, that is, using the above f the space $\Pi_{n=2}^{\infty}\{0, 1, \ldots, n-1\}$ can be covered by ω_1-many f-slaloms in the Laver model. (This can be derived from the so called *Laver property*, see [BJ].) Therefore $\mathrm{cov}^*(c\mathcal{N})$ is not the same as $\mathrm{cof}(\mathcal{N})$.

Question 3.1 *Is $cov^*(c\mathcal{N})$ equal to one of the known cardinal invariants?*

Another natural question is the following. In most cases the values of the classical cardinal invariants in the Cichoń diagram remain the same if we replace $\mathbb{R}$ with an arbitrary (uncountable) Polish space. However, in case of $\mathrm{cov}^*(c\mathcal{N})$ the situation is not clear. The authors were able to reprove the results of this paper in case of the Cantor group or more generally for countable products of finite discrete groups equipped with the Haar measure, but not in the general case.

Question 3.2 *Can it be shown (without resorting to extra set theoretic axioms) that there is an uncountable locally compact Polish group G with Haar measure μ such that for every compact set $C \subset G$ with $\mu(C) = 0$ and every $A \subset G$ of cardinality less than 2^ω we have $C + A \neq G$?*

As for the cardinal invariants, even the following is open.

Definition 3.3 If G is an uncountable locally compact Polish group with Haar measure μ then $cov^*_G(c\mathcal{N})$ is the smallest cardinal κ for which it is possible find a compact set of μ-measure zero and cover G by κ many translates of it.

Question 3.4 *Is it true that for every G_1 and G_2 uncountable locally compact Polish groups $cov^*_{G_1}(c\mathcal{N}) = cov^*_{G_2}(c\mathcal{N})$?*

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