A cardinal number connected to the solvability of systems of difference equations in a given function class

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Abstract

Let $\mathbb{R}^{\mathbb{R}}$ denote the set of real valued functions defined on the real line. A map $D : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ is said to be a *difference operator*, if there are real numbers a_i, b_i $(i = 1, \dots, n)$ such that $(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i)$ for every $f \in \mathbb{R}^{\mathbb{R}}$ and $x \in \mathbb{R}$. By a *system of difference equations* we mean a set of equations $S = \{D_i f = g_i : i \in I\}$, where I is an arbitrary set of indices, D_i is a difference operator and g_i is a given function for every $i \in I$, and f is the unknown function. One can prove that a system S is solvable if and only if every finite subsystem of S is solvable. However, if we look for solutions belonging to a given class of functions, then the analogous statement is no longer true. For example, there exists a system S such that every finite subsystem of S has a solution which is a trigonometric polynomial, but S has no such solution; moreover, S has no measurable solutions.

This phenomenon motivates the following definition. Let $\mathcal{F}$ be a class of functions. The *solvability cardinal* $\mathrm{sc}(\mathcal{F})$ of $\mathcal{F}$ is the smallest cardinal number κ such that whenever S is a system of difference equations and each subsystem of S of cardinality less than κ has a solution in $\mathcal{F}$, then S itself has a solution in $\mathcal{F}$. In this paper we will determine the solvability cardinals of most function classes that occur in analysis. As it turns out, the behaviour of $\mathrm{sc}(\mathcal{F})$ is rather erratic. For example, $\mathrm{sc}(\text{polynomials}) = 3$ but $\mathrm{sc}(\text{trigonometric polynomials}) = \omega_1$, $\mathrm{sc}(\{f : f \text{ is continuous}\}) = \omega_1$ but $\mathrm{sc}(\{f : f \text{ is Darboux}\}) = (2^\omega)^+$, and $\mathrm{sc}(\mathbb{R}^{\mathbb{R}}) = \omega$. We consistently determine the solvability cardinals of the classes of Borel, Lebesgue and Baire measurable functions, and give some partial answers for the Baire class 1 and Baire class α functions.

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1 Preliminaries

Difference operators occur in various branches of analysis. For example, it was shown in [6] that the existence of certain types of liftings is closely related to the solvability of systems of difference equations. The goal of this paper is to give necessary conditions under which systems of difference equations have solutions belonging to a given function class.

Notation 1.1 Let $\mathbb{R}^{\mathbb{R}}$ denote the set of real valued functions defined on the real line. The classes of polynomials and trigonometric polynomials are denoted by $\mathcal{P}$ and $\mathcal{TP}$. For every set H we shall denote by χ_H and $|H|$ the characteristic function and the cardinality of H. We denote the symmetric difference of the sets A and B by $A\Delta B$. If $A, B \subset \mathbb{R}$ and $x \in \mathbb{R}$ then we shall write $A + B = \{a + b : a \in A,\ b \in B\}$ and $A + x = \{a + x : a \in A\}$. If $A \subset \mathbb{R}$ then $\langle A \rangle$ denotes the additive group generated by A. The symbols κ^+ and $\mathrm{cf}(\kappa)$ denote the successor cardinal and the cofinality of the cardinal κ.

Definition 1.2 A *difference operator* is a mapping $D : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ of the form

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i),$$

where a_i and b_i are real numbers. The set of difference operators is denoted by $\mathcal{D}$.

Definition 1.3 For $b \in \mathbb{R}$ the difference operators T_b and Δ_b are defined by

$$(T_b f)(x) = f(x + b) \quad (x \in \mathbb{R}), \quad \text{and}$$

$$(\Delta_b f)(x) = f(x + b) - f(x) \quad (x \in \mathbb{R}).$$

Definition 1.4 A *difference equation* is a functional equation

$$Df = g,$$

where D is a difference operator, g is a given function and f is the unknown. A *system of difference equations* is

$$D_i f = g_i \quad (i \in I),$$

where I is an arbitrary set of indices. More formally, by a system of difference equations we mean a set $S \subset \mathcal{D} \times \mathbb{R}^{\mathbb{R}}$. A function $f : \mathbb{R} \to \mathbb{R}$ is a solution to S if $Df = g$ for every $(D, g) \in S$.

It was proved in [6, Thm. 2.2] that a system of difference equations is solvable iff each of its finite subsystems is solvable. However, if we are interested in solutions belonging to a given subclass of $\mathbb{R}^{\mathbb{R}}$ then this result is no longer true. This motivates the following.

Definition 1.5 Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The *solvability cardinal of* $\mathcal{F}$ is the minimal cardinal $\mathrm{sc}(\mathcal{F})$ with the property that if every subsystem of size less than $\mathrm{sc}(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$, then the whole system has a solution in $\mathcal{F}$.

For example, $\mathrm{sc}(\mathbb{R}^{\mathbb{R}}) \leq \omega$ is a reformulation of the above cited result. The next statement shows that the cardinal $\mathrm{sc}(\mathcal{F})$ actually exists, and also provides an upper bound.

Fact 1.6 *For every* $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ *we have* $\mathrm{sc}(\mathcal{F}) \leq (2^{\omega})^{+}$.

Proof. Note that the cardinality of $\mathcal{D}$ is 2^{ω}. Suppose $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, S is system of difference equations, and every subsystem of S of cardinality at most 2^{ω} is solvable in $\mathcal{F}$. In particular, every pair of equations of S is solvable, hence for every $D \in \mathcal{D}$ there is at most one $g \in \mathbb{R}^{\mathbb{R}}$ such that $(D, g) \in S$. Therefore the cardinality of S is at most 2^{ω}, and we are done. $\square$

We may add the following trivial estimate.

Fact 1.7 *For every* $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ *we have* $\mathrm{sc}(\mathcal{F}) \leq |\mathcal{F}|^{+}$.

Proof. Let S be a system of difference equations such that every subsystem of S of cardinality at most $|\mathcal{F}|$ is solvable in $\mathcal{F}$. Suppose S is not solvable in $\mathcal{F}$. Then for every $f \in \mathcal{F}$ there is a $(D_f, g_f) \in S$ such that $D_f f \neq g_f$. Then $S' = \{(D_f, g_f) : f \in \mathcal{F}\}$ has no solution in $\mathcal{F}$ and $|S'| \leq |\mathcal{F}|$, a contradiction. $\square$

Remark 1.8 Fact 1.7 can be improved if we take into consideration the product topology on $\mathbb{R}^{\mathbb{R}}$. Namely, if $Df \neq g$ for some $f \in \mathbb{R}^{\mathbb{R}}$ then f has a neighbourhood U in the product topology such that $Df' \neq g$ for every $f' \in U$. Combining this observation with the proof of Fact 1.7 we obtain the estimate $\mathrm{sc}(\mathcal{F}) \leq L(\mathcal{F})^{+}$, where $L(X)$ is the Lindelöf number of the topological space X; that is, the smallest cardinal κ such that each open cover of X contains a subcover of cardinality at most κ. This sharper inequality implies Fact 1.6 since the space $\mathbb{R}^{\mathbb{R}}$ has a base of cardinality 2^{ω} and thus $L(X) \leq 2^{\omega}$ for every subspace $X \subset \mathbb{R}^{\mathbb{R}}$.

It is natural to ask whether or not every cardinal $2 \leq \kappa \leq (2^{\omega})^{+}$ equals $\mathrm{sc}(\mathcal{F})$ for some $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. As we shall see in Theorem 2.1, ω is such a cardinal. The following result gives a positive answer for successor cardinals.

Theorem 1.9 *For every cardinal* $1 \leq \kappa \leq 2^{\omega}$ *there exists an* $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ *such that* $\mathrm{sc}(\mathcal{F}) = \kappa^{+}$.

Proof. Let $B \subset \mathbb{R}$ be linearly independent over the rationals with $|B| = \kappa$. For every $b \in B$ we denote by f_b the characteristic function of the group $\langle B \setminus \{b\} \rangle$. Then f_b is periodic mod each element of $B \setminus \{b\}$, but f_b is not periodic mod b.

We claim that the solvability cardinal of the class $\mathcal{F} = \{f_b : b \in B\}$ equals κ^{+}. The inequality $\mathrm{sc}(\mathcal{F}) \leq \kappa^{+}$ is clear from Fact 1.7. In order to prove $\mathrm{sc}(\mathcal{F}) \geq \kappa^{+}$ we have to construct a system S such that every subsystem $S' \subset S$ of size less than κ is solvable in $\mathcal{F}$, while S is not. We show that $S = \{(\Delta_b, 0) : b \in B\}$ is such a system. If $b \in B$ then, as f_b is not periodic mod b, we have $\Delta_b f_b \neq 0$ showing that S is not solvable in $\mathcal{F}$. On the other hand, if S' is a proper subsystem of S and $(\Delta_b, 0) \notin S'$ then f_b solves S' completing the proof. $\square$

Question 1.10 *Is it true (in ZFC) that for every $2 \leq \kappa \leq (2^\omega)^+$ there exists an $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ such that $\mathrm{sc}(\mathcal{F}) = \kappa$? Is there (in ZFC) an $\mathcal{F}$ with $\mathrm{sc}(\mathcal{F}) = 2^\omega$? Is it consistent with ZFC that $\mathrm{sc}(\mathcal{F})$ can be an uncountable limit cardinal?*

In the first part of the paper (Sections 2, 3, 4, and 5) we determine the exact value of $\mathrm{sc}(\mathcal{F})$ for several classes $\mathcal{F}$; see Theorems 2.1, 3.2, 3.3, 4.1, Corollaries 5.4, 5.5 and Theorem 5.6. As it turns out, the behaviour of $\mathrm{sc}(\mathcal{F})$ is rather erratic. For example, $\mathrm{sc}(\mathcal{P}) = 3$, but $\mathrm{sc}(\mathcal{TP}) = \omega_1$; $\mathrm{sc}(\mathbb{R}^\mathbb{R}) = \omega$, but $\mathrm{sc}(\{f : f \text{ is Darboux}\} = (2^\omega)^+$.

In the second part (Sections 6, 7 and 8) we give estimates of $\mathrm{sc}(\mathcal{F})$ for subclasses of Borel, Lebesgue and Baire measurable functions. The estimates for Borel, Lebesgue and Baire measurable functions provide the exact value of the solvability cardinals consistently. The result $\mathrm{sc}(\{f : f \text{ is Lebesgue measurable}\}) > \omega_1$ answers Problem 3 of [6].

2 Arbitrary functions

The nontrivial direction of the next theorem was proved in [6, Thm. 2.2], but we reformulate this result using the notation introduced in the present paper.

Theorem 2.1 $\mathrm{sc}(\mathbb{R}^\mathbb{R}) = \omega$.

Proof. $\mathrm{sc}(\mathbb{R}^\mathbb{R}) \leq \omega$ is [6, Thm. 2.2], so we only need to show that $\mathrm{sc}(\mathbb{R}^\mathbb{R}) \neq n$ for every $n \in \mathbb{N}$. Let $n \geq 2$, let $a_1, \ldots, a_{n-1} \in \mathbb{R}$ be linearly independent over the rationals, and put $a_n = -\sum_{i=1}^{n-1} a_i$. Then any $n-1$ of the numbers $a_1, \ldots, a_n$ are linearly independent over the rationals. Define the following system of n equations:

$$\Delta_{a_i} f = 1, \quad i = 1, \ldots, n.$$

It is easy to see that each subsystem of cardinality at most $n-1$ is solvable (consider the factor group of $\mathbb{R}$ modulo the additive group generated by the corresponding linearly independent a_i's). On the other hand, if f were a solution to the whole system, then $f(0) + n = f(a_1 + \ldots + a_n) = f(0)$ would hold, which is impossible. This shows $\mathrm{sc}(\mathbb{R}^\mathbb{R}) > n$ and, as n was arbitrary, the proof is complete. $\square$

3 Bounded functions

Definition 3.1 We say that the difference equation (D, g) is *deducible from* the system S if there are $A_1, \ldots, A_n \in \mathcal{D}$ and $(D_1, g_1), \ldots, (D_n, g_n) \in S$ such that $(D, g) = (\sum_{i=1}^n A_i D_i, \sum_{i=1}^n A_i g_i)$.

Theorem 3.2 *Let $K > 0$ be a real number. Then* $\mathrm{sc}(\{f \in \mathbb{R}^{\mathbb{R}} : |f| \leq K\}) = \omega$.

Proof. We may assume $K = 1$. First we show $\mathrm{sc}(\{f \in \mathbb{R}^{\mathbb{R}} : |f| \leq 1\}) \leq \omega$. The proof is a modification of the proof of [6, Thm. 2.1], the new ingredient is the Hahn-Banach Theorem. Let S be a system such that all finite subsystems are solvable by functions of absolute value at most 1. Define

$$\mathcal{A} = \{D \in \mathcal{D} : \exists g \ (D, g) \text{ is deducible from } S\}.$$

Then $\mathcal{A}$ is a linear subspace of $\mathcal{D}$. Put

$$L(D) = g(0) \quad (D \in \mathcal{A}),$$

where (D, g) is deducible from S. Clearly, if (D, g) is deducible from S then it is also deducible from a finite subsystem of S, hence it is solvable. Moreover, any pair of equations deducible from S has a common solution. Therefore the map $L : \mathcal{A} \to \mathbb{R}$ is well defined. Note that L is clearly linear.

Now we define a norm on $\mathcal{D}$. It is easy to see that every $D \in \mathcal{D}$ has a unique representation of the form $D = \sum_{i=1}^{n} a_i T_{b_i}$, where the a_i's are nonzero and the b_i's are different. Using this representation set

$$||D|| = \sum_{i=1}^{n} |a_i|.$$

The function $||.|| : \mathcal{D} \to \mathbb{R}$ is easily seen to be a norm.

We claim that for every $D \in \mathcal{A}$ we have $|L(D)| \leq ||D||$. Let (D, g) be deducible from S. Then there is a function f such that $|f| \leq 1$ and $Df = g$. If $D = \sum_{i=1}^{n} a_i T_{b_i}$ then

$$|L(D)| = |g(0)| = \left| \sum_{i=1}^{n} a_i f(b_i) \right| \leq \sum_{i=1}^{n} |a_i| \cdot 1 = ||D||.$$

Hence by the Hahn-Banach Theorem (see e.g. [10, Thm. 3.3]) there exists a linear map $L^* : \mathcal{D} \to \mathbb{R}$ extending L such that

$$|L^*(D)| \leq ||D|| \text{ for every } D \in \mathcal{D}.$$

We claim that the function defined by

$$f(x) = L^*(T_x) \quad (x \in \mathbb{R})$$

is a solution to S such that $|f| \leq 1$. This last inequality is obvious, as $|f(x)| = |L^*(T_x)| \leq ||T_x|| = 1$. So we need to prove that f solves S. First we show that

$$(Df)(0) = L^*(D) \text{ holds for every } D \in \mathcal{D}. \tag{1}$$

Since L^* is linear, it is enough to check this for $D = T_x$ $(x \in \mathbb{R})$. Now $(T_x f)(0) = f(x) = L^*(T_x)$ by the definition of f, which proves (1). Let $(D, g) \in S$ and $x \in \mathbb{R}$ be given. Then $T_x D \in \mathcal{A}$, and thus (1) and the definition of L imply

$$(Df)(x) = (T_x Df)(0) = L^*(T_x D) = L(T_x D) = (T_x g)(0) = g(x).$$

Now we prove $\mathrm{sc}(\{f \in \mathbb{R}^{\mathbb{R}} : |f| \leq 1\}) \geq \omega$. Let $n \geq 2$ be an integer and let $a_1, \dots, a_n$ be linearly independent reals. Define a system as follows.

$$\Delta_{a_i} f = \frac{2}{n-1}\, \chi_{\langle\{a_1,\dots,a_{i-1},a_{i+1},a_n\}\rangle}, \quad (i = 1, \dots, n).$$

A simple induction shows that if f solves the whole system, then $f(a_1 + \ldots + a_n) - f(0) = \frac{2n}{n-1} > 2$, hence $|f| \leq 1$ cannot hold.

On the other hand, let $J \subset \{1, \dots, n\}$ be a set of at most $n-1$ elements. Every $x \in \langle\{a_1, \dots, a_n\}\rangle$ can be uniquely written in the form $x = k_1(x)a_1 + \ldots + k_n(x)a_n$, where the $k_i(x)$'s are integers. Define

$$f(x) = \begin{cases} 0 & \text{if } x \notin \langle\{a_1, \dots, a_n\}\rangle \\ -1 + \frac{2}{n-1}\, |\{i \in J : k_i(x) > 0\}| & \text{if } x \in \langle\{a_1, \dots, a_n\}\rangle. \end{cases}$$

Clearly, $|f| \leq 1$. It is easy to see that f solves the i^{th} equation for every $i \in J$, which yields $\mathrm{sc}(\{f \in \mathbb{R}^{\mathbb{R}} : |f| \leq 1\}) > n$. As n was arbitrary, the proof is complete. $\square$

In contrast to Theorem 3.2 we have the following.

Theorem 3.3 $\mathrm{sc}(\{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is bounded}\}) = \omega_1$.

Proof. First we prove $\mathrm{sc}(\{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is bounded}\}) \leq \omega_1$. Let S be a system such that every countable subsystem of S is solvable by a bounded function. For a countable $S' \subset S$ let $K_{S'}$ be the minimal integer for which S' has a solution in $\{f \in \mathbb{R}^{\mathbb{R}} : |f| \leq K_{S'}\}$. The set $\{K_{S'} : S' \subset S,\ |S'| \leq \omega\}$ is bounded in $\mathbb{N}$, otherwise we could easily find a countable subsystem of S with no bounded solutions. Fix an upper bound K of the above set. Then every countable, in particular, every finite subsystem of S is solvable in $\{f \in \mathbb{R}^{\mathbb{R}} : |f| \leq K\}$, hence by the previous theorem S is solvable in $\{f \in \mathbb{R}^{\mathbb{R}} : |f| \leq K\}$, hence S has a bounded solution.

Now we prove $\mathrm{sc}(\{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is bounded}\}) > \omega$. Similarly to the previous theorem, let $a_1, a_2, \dots$ be a linearly independent sequence of reals. Define a system by

$$\Delta_{a_i} f = \chi_{\langle\{a_1,\dots,a_{i-1},a_{i+1},\dots\}\rangle}, \quad (i \in \mathbb{N}^+).$$

A simple induction shows that if f solves the whole system, then $f(a_1 + \ldots + a_n) - f(0) = n$ for every n, hence f cannot be bounded.

On the other hand, let $J \subset \mathbb{N}^+$ be a finite set. Every $x \in \langle\{a_1, a_2, \dots\}\rangle$ can be uniquely written in the form $x = k_1(x)a_1 + k_2(x)a_2 + \ldots$, where the $k_i(x)$'s are integers, and only finitely many of them are nonzero. Similarly to the proof of the previous theorem one can check that

$$f(x) = \begin{cases} 0 & \text{if } x \notin \langle\{a_1, a_2, \dots\}\rangle \\ |\{i \in J : k_i(x) > 0\}| & \text{if } x \in \langle\{a_1, a_2, \dots\}\rangle \end{cases}$$

is a bounded solution to the finite subset of S corresponding to J. $\square$

4 Darboux functions

Theorem 4.1 $\mathrm{sc}(\{f : f \textit{ is Darboux}\}) = (2^\omega)^+$.

Proof. $\mathrm{sc}(\{f : f \text{ is Darboux}\}) \leq (2^\omega)^+$ follows from Fact 1.6. In order to prove the other inequality we have to construct a system S such that every subsystem of cardinality less than continuum is solvable by a Darboux function but S has no Darboux solution. We define S as

$$\Delta_b f = \Delta_b \chi_{\{0\}} \quad (b \in \mathbb{R}).$$

The whole system clearly has no Darboux solution, for if f is a solution to S then there exists a $c \in \mathbb{R}$ such that $f = \chi_{\{0\}} + c$, which is not Darboux. On the other hand, let S' be a subset of S such that $|S'| < 2^\omega$, and let $B \subset \mathbb{R}$ be the corresponding set of indices with $|B| < 2^\omega$. By enlarging B if necessary, we may assume that B is an additive subgroup of $\mathbb{R}$, and also that B is dense.

As $|B| < 2^\omega$, the factor group $\mathbb{R}/B$ consists of 2^ω cosets. Fix a bijection $\varphi : \mathbb{R}/B \to \mathbb{R}$ and define

$$f(x) = \varphi(B + x) + \chi_{\{0\}}(x) \quad (x \in \mathbb{R}).$$

As B is dense, f attains every value on every interval, hence it is Darboux. In addition, it is easy to see that f solves S'. $\square$

Remark 4.2 The same system can be used to demonstrate that for the class $\mathcal{F}$ of functions with connected graphs we also have $\mathrm{sc}(\mathcal{F}) = (2^\omega)^+$. With a more elaborate version of the argument above it can be shown that if $|B| < 2^\omega$ then the system $\{(\Delta_b, \Delta_b \chi_{\{0\}}) : b \in B\}$ has a solution with a connected graph.

5 Subclasses of Lebesgue measurable and Baire measurable functions

In this section our aim is to prove that $\mathrm{sc}(\mathcal{F}) = \omega_1$ for many classes including the classes of trigonometric polynomials, continuous functions, Lipschitz functions, C^n, C^∞, analytic functions, derivatives, approximately continuous functions etc.

Let $\mathcal{N}$ denote the σ-ideal of Lebesgue nullsets of $\mathbb{R}$ and $\mathcal{M}$ denote the σ-ideal of first category (= meager) subsets of $\mathbb{R}$. In the rest of the section let $\mathcal{I}$ stand for either $\mathcal{N}$ or $\mathcal{M}$. The term $\mathcal{I}$-almost everywhere will be abbreviated by $\mathcal{I}$-a.e. Instead of 'Lebesgue measurable' and 'with the Baire property' we will use the term $\mathcal{B}_\mathcal{I}$*-measurable*, where $\mathcal{B}_\mathcal{I}$ is the σ-algebra generated by the Borel sets and $\mathcal{I}$.

First we show that if we do not distinguish between $\mathcal{I}$-almost everywhere equal functions, then the value of this modified solvability cardinal is at most ω_1 for all subclasses of both Lebesgue measurable functions and functions with the property of Baire.

Theorem 5.1 *Let $\mathcal{F} \subset \mathcal{B}_{\mathcal{I}}$, and suppose that for every countable subsystem S' of a system of difference equations S there exists an $f' \in \mathcal{F}$ such that $Df' = g$ $\mathcal{I}$-a.e. for every $(D,g) \in S'$. Then there is an $f \in \mathcal{F}$ such that $Df = g$ $\mathcal{I}$-a.e. for every $(D,g) \in S$.*

Proof. Let S be a system satisfying the assumptions. Every $D \in \mathcal{D}$ can be written in a unique way as $D = \sum_{i=1}^{n} a_i T_{b_i}$. Define $\varphi : \mathcal{D} \to \bigcup_{n\in\mathbb{N}} \mathbb{R}^{2n}$ by

$$\varphi(D) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \quad (D \in \mathcal{D}).$$

Set $S_n = \{(D,g) \in S : D \text{ has } n \text{ terms}\}$. For every $n \in \mathbb{N}$ choose a countable $S_n' \subset S_n$ such that $\{\varphi(D) : (D,g) \in S_n'\} \subset \mathbb{R}^{2n}$ is dense in $\{\varphi(D) : (D,g) \in S_n\} \subset \mathbb{R}^{2n}$.

Let $f \in \mathcal{F}$ be a function '$\mathcal{I}$-a.e.' solving $\bigcup_{n\in\mathbb{N}} S_n'$. We claim that it '$\mathcal{I}$-a.e.' solves the whole S. Let $(D,g) \in S_n$, and choose $(D_i, g_i) \in S_n'$ such that $\varphi(D_i) \to \varphi(D)$ in $\mathbb{R}^{2n}$.

Suppose first $\mathcal{I} = \mathcal{N}$. It is well known that for every measurable h if $t_n \to 0$ $(n \to \infty)$ then $T_{t_n} h \to h$ in measure (which means that it converges in measure on every bounded interval; see e.g. [11] or [2] for the definitions and basic facts). Hence $D_i h \to Dh$ in measure. Let f' be an a.e. solution to $\bigcup_{n\in\mathbb{N}} S_n' \cup \{(D,g)\}$. Then

$$Df = \lim_{i\to\infty} D_i f = \lim_{i\to\infty} g_i = \lim_{i\to\infty} D_i f' = Df' = g$$

a.e., where lim stands for limit in measure.

Suppose now $\mathcal{I} = \mathcal{M}$. We claim that for every h with the Baire property if $t_n \to 0$ $(n \to \infty)$ then $T_{t_n} h \to h$ pointwise on a residual set. Indeed, if H is a residual set on which h is continuous then $H \cap \bigcap_{n\in\mathbb{N}} (H - t_n)$ is such a set. Therefore $D_i h \to Dh$ pointwise on a residual set. Let f' be an $\mathcal{M}$-a.e. solution to $\bigcup_{n\in\mathbb{N}} S_n' \cup \{(D,g)\}$. Then

$$Df = \lim_{i\to\infty} D_i f = \lim_{i\to\infty} g_i = \lim_{i\to\infty} D_i f' = Df' = g$$

on a residual set. $\square$

Theorem 5.2 *Let $\mathcal{F} \subset \tilde{\mathcal{F}} \subset \mathcal{B}_{\mathcal{I}}$, where $\tilde{\mathcal{F}}$ is a translation invariant linear subspace of $\mathcal{B}_{\mathcal{I}}$ such that whenever $f \in \tilde{\mathcal{F}}$ and $f = 0$ $\mathcal{I}$-a.e. then $f = 0$ everywhere. Then $\mathrm{sc}(\mathcal{F}) \leq \omega_1$.*

Proof. Suppose that every countable subsystem of S has a solution in $\mathcal{F}$. Then obviously $g \in \tilde{\mathcal{F}}$ whenever $(D,g) \in S$.

By Theorem 5.1, there is an $f \in \mathcal{F}$ such that $Df = g$ $\mathcal{I}$-a.e. for every $(D,g) \in S$. Since $Df - g \in \tilde{\mathcal{F}}$ and $Df - g = 0$ $\mathcal{I}$-a.e., we have $Df = g$, which proves $\mathrm{sc}(\mathcal{F}) \leq \omega_1$. $\square$

It is clear that the class $C(\mathbb{R})$ of continuous functions satisfies the conditions imposed on $\tilde{\mathcal{F}}$. The same is true for the classes of derivatives and approximately continuous functions (see [1]).

We shall denote by $\mathcal{TP}$ the set of trigonometric polynomials.

Theorem 5.3 *If $\mathcal{TP} \subset \mathcal{F} \subset \mathcal{B}_{\mathcal{I}}$ then* $\mathrm{sc}(\mathcal{F}) \geq \omega_1$.

Proof. We shall construct a system S such that every finite subsystem of S has a solution which is a trigonometric polynomial, but S itself does not have a $\mathcal{B}_{\mathcal{I}}$-measurable solution. We shall repeat the construction of [6, Thm. 4.4] with a small modification.

Let $C(x) = \cos 2\pi x$ and $E_{j,n}(x) = \Delta_{2^{-n}} C\left(2^j x\right)$, then $E_{j,n} \in \mathcal{TP}$ for every $j, n \in \mathbb{N}$. Also, $E_{j,n} = 0$ if $j \geq n$ and, if $j < n$ then $E_{j,n}$ is a continuous function periodic mod 1 with finitely many roots in $[0,1]$.

Let c_j $(j = 0, 1, \ldots)$ be a sequence of real numbers, and consider the system S of the equations

$$\Delta_{2^{-n}} f = h_n, \quad \text{where} \quad h_n = \sum_{j=0}^{n-1} c_j E_{j,n} \quad (n = 1, 2, \ldots).$$

Then the trigonometric polynomial $\sum_{j=0}^{n-1} c_j C\left(2^j x\right)$ is a solution to the first n equations of S. On the other hand, we shall choose the numbers c_j in such a way that S does not have $\mathcal{B}_{\mathcal{I}}$-measurable solutions.

First suppose $\mathcal{I} = \mathcal{N}$. If $f : \mathbb{R} \to \mathbb{R}$ is measurable then the sequence of functions $\Delta_{2^{-n}} f$ converges to zero in measure on $[0,1]$. Therefore, if S has a measurable solution, then h_n should converge to zero in measure on $[0,1]$. But we can prevent this by a suitable choice of the sequence c_j. We shall define c_j inductively. If c_j has been defined for every $j < n-1$, then we choose c_{n-1} so large that $\lambda(\{x \in [0,1] : |h_n(x)| > 1\}) > 1/2$ holds. This is possible, since $E_{n-1,n} \neq 0$ a.e. in $[0,1]$. Therefore, with this choice, h_n does not converge (in measure) to zero on $[0,1]$, and thus S cannot have measurable solutions.

Next suppose $\mathcal{I} = \mathcal{M}$. If $f : \mathbb{R} \to \mathbb{R}$ is Baire measurable then the sequence of functions $\Delta_{2^{-n}} f$ converges to zero pointwise on a residual subset of $[0,1]$. Again, we shall choose the constants c_j such that $h_n \not\to 0$ on a second category set. Namely, we shall define c_j in such a way that each function h_n satisfies the following condition: for every interval $I \subset [0,1]$ of length $1/n$ the inequality $|h_n| > 1$ holds on a subinterval of I. (In the course of the proof by an interval we shall mean a closed nondegenerate interval, and by $|I|$ we shall mean the length of the interval I.)

We put $c_0 = 1$. Then $h_1(x) = C(x + \frac{1}{2}) - C(x) = -2\cos 2\pi x$ has the required property with $n = 1$, since there is a subinterval of $[0,1]$ on which $|h_1| > 1$. Let $n > 1$ and suppose that $c_0, \ldots, c_{n-2}$ have been chosen. Since $E_{n-1,n}$ only has a finite number of roots in $[0,1]$, the function

$$h_n = \left(\sum_{j=0}^{n-2} c_j E_{j,n}\right) + c_{n-1} E_{n-1,n}$$

clearly has the required property if c_{n-1} is large enough.

We show that the set $A = \{x \in [0,1] : h_n(x) \to 0\}$ is not residual. Suppose the contrary, and let $\bigcap_{k=1}^{\infty} U_k \subset A$, where each U_k is dense open. Let $I_1 \subset U_1$ be an interval. If $1/n_1 < |I_1|$ then there is a subinterval $J_1 \subset I_1$ such that $|h_{n_1}| > 1$ on J_1. Since U_2 is dense open, there is an interval $I_2 \subset U_2 \cap J_1$. If $1/n_2 < |I_2|$ then there is a subinterval $J_2 \subset I_2$ such that $|h_{n_2}| > 1$ on J_2. Continuing this process we find the nested sequence of intervals J_k such that $\bigcap_{k=1}^{\infty} J_k \subset \bigcap_{k=1}^{\infty} U_k \subset A$ and $|h_{n_k}| > 1$ on J_k for every k. This implies $h_n(x) \not\to 0$ for every $x \in \bigcap_{k=1}^{\infty} J_k$ which contradicts $x \in A$. $\square$

Corollary 5.4 *Suppose $\mathcal{TP} \subset \mathcal{F} \subset \tilde{\mathcal{F}} \subset \mathcal{B}_{\mathcal{I}}$, where $\tilde{\mathcal{F}}$ is a translation invariant linear subspace of $\mathcal{B}_{\mathcal{I}}$ such that whenever $f \in \tilde{\mathcal{F}}$ and $f = 0$ $\mathcal{I}$-a.e. then $f = 0$ everywhere. Then $\mathrm{sc}(\mathcal{F}) = \omega_1$.*

It is clear that the class $C(\mathbb{R})$ of continuous functions satisfies the conditions imposed on $\tilde{\mathcal{F}}$. The same is true for the classes of derivatives and approximately continuous functions (see [1]). Thus we have the following.

Corollary 5.5 *If $\mathcal{F}$ equals any of the classes $\mathcal{TP}$, $C(\mathbb{R})$, the class of Lipschitz functions, $C^n(\mathbb{R})$, $C^\infty(\mathbb{R})$, the class of real analytic functions, derivatives, approximately continuous funcions, then $\mathrm{sc}(\mathcal{F}) = \omega_1$. The same is true for the subclasses $\{f \in \mathcal{F} : f \text{ is bounded}\}$ where $\mathcal{F}$ is any of the classes listed above.*

We remark that the class $\mathcal{P}$ of polynomials behaves quite differently from $\mathcal{TP}$. Indeed, [6, Thm. 4.5] states that $\mathrm{sc}(\mathcal{P}) \leq 3$. Since $\mathrm{sc}(\mathcal{P}) \geq 3$ is obvious, we have the following.

Theorem 5.6 $\mathrm{sc}(\mathcal{P}) = 3$.

6 Borel functions

First we prove an auxiliary lemma.

Lemma 6.1 *There exist non-empty perfect subsets $\{P_\alpha : \alpha < 2^\omega\}$ of $\mathbb{R}$ and real numbers $\{p_\alpha : \alpha < 2^\omega\}$ such that*

$$(P_\alpha + G_{\alpha+1}) \cap (P_\beta + G_{\beta+1}) = \emptyset \quad (\alpha \neq \beta),$$

and for every $\alpha < 2^\omega$

$$(P_\alpha + g_1) \cap (P_\alpha + g_2) = \emptyset \quad (g_1, g_2 \in G_{\alpha+1},\ g_1 \neq g_2),$$

where $G_\alpha = \langle \{p_\beta : \beta < \alpha\} \rangle$.

Proof. Let $P \subset \mathbb{R}$ be a non-empty perfect set that is linearly independent over the rationals (see e.g. [9] or [8]). We can choose nonempty perfect sets $P_\alpha \subset P$ and $p_\alpha \in P$ $(\alpha < 2^\omega)$ such that $P_\alpha \cap P_\beta = \emptyset$ for every $\alpha \neq \beta$ and such that $p_\alpha \notin P_\beta$ for every $\alpha, \beta < 2^\omega$. It is a straightforward calculation to check that all the requirements are fulfilled. $\square$

Theorem 6.2 $\mathrm{sc}(\{f : f \text{ is Borel}\}) \geq \omega_2$.

Proof. Let P_α and p_α be as in the previous lemma. For every $\alpha < \omega_1$ let $B_\alpha \subset P_\alpha$ be a Borel set of class α (that is, not of any smaller class). Define $A_\alpha = B_\alpha + G_\alpha$, and consider the system of difference equations:

$$\Delta_{p_\alpha} f = \Delta_{p_\alpha} \left(\sum_{\beta<\omega_1} \chi_{A_\beta} \right) \quad (\alpha < \omega_1).$$

Note that the A_β's are disjoint. We claim that every countable subsystem of this system has Borel solution, but the whole system does not.

To prove the first statement we have to check that for every $\alpha < \omega_1$ the first α equations have a common Borel solution. We show that the Borel function

$$\sum_{\beta\leq\alpha} \chi_{A_\beta}$$

will do. If $\gamma < \beta$ then A_β is periodic mod p_γ, so $\Delta_{p_\gamma} \chi_{A_\beta} = 0$. Therefore, in view of the properties required in Lemma 6.1, we obtain that for $\gamma < \alpha$

$$\Delta_{p_\gamma} \left(\sum_{\beta<\omega_1} \chi_{A_\beta} \right) = \Delta_{p_\gamma} \left(\sum_{\beta\leq\alpha} \chi_{A_\beta} \right),$$

which proves this part of the claim.

In order to show that the whole system has no Borel solution it is sufficient to check that the functions on the right hand side of the equations are of unbounded Baire class. But this is not hard to see, as $\Delta_{p_\alpha}(\sum_{\beta<\omega_1} \chi_{A_\beta})$ restricted to P_α equals $-\chi_{B_\alpha}$. $\square$

Using Fact 1.6 we obtain the following.

Corollary 6.3 *The Continuum Hypothesis implies that* $\mathrm{sc}(\{f : f \text{ is Borel}\}) = \omega_2 = (2^\omega)^+$.

Question 6.4 *Can we omit the use of the Continuum Hypothesis? Is it true that* $\mathrm{sc}(\{f : f \text{ is Borel}\}) = \omega_2$*? Is it true that* $\mathrm{sc}(\{f : f \text{ is Borel}\}) = (2^\omega)^+$*?*

Remark 6.5 In order to prove $\mathrm{sc}(\{f : f \text{ is Borel}\}) = \omega_2$ it would be sufficient to prove $\mathrm{sc}(\{f : f \text{ is Baire class } \alpha\}) \leq \omega_2$ for every $\alpha < \omega_1$. Indeed, assume that every subsystem of cardinality at most ω_1 of a system has a Borel solution. Let us assign to every such subsystem the minimal $\alpha < \omega_1$ for which it has a Baire class α solution. We claim that the set of these α's is bounded in ω_1. Otherwise, the union of ω_1-many appropriate subsystems would itself be a subsystem of cardinality ω_1 without a Borel solution, which proves our statement.

So if every subsystem of cardinality at most ω_1 of a system has a Borel solution, then there exists an $\alpha < \omega_1$ such that every such subsystem has a Baire class α solution.

Remark 6.6 For $2 \leq \alpha < \omega_1$ the idea of the proof of Theorem 6.2 probably gives $\mathrm{sc}(\{f : f \text{ is Baire class } \alpha\}) \geq \omega_2$. If we had an appropriate notion of rank for Baire class α functions, sharing the properties of the well known ranks on Baire class 1, it would yield $\mathrm{sc}(\{f : f \text{ is Baire class } \alpha\}) \geq \omega_2$. Unfortunately, according to [4] no such rank is known.

For Baire class 1 these ranks exist, but do not give $\mathrm{sc}(\{f : f \text{ is Baire class } 1\}) \geq \omega_2$. The proof breaks down, as $\sum_{\beta \leq \alpha} \chi_{A_\beta}$ is not Baire class 1.

Question 6.7 *Is there a rank on Baire class α with the usual properties?*

Remark 6.5 shows why we are particularly interested in the solvability cardinals of the individual Baire α classes. The simplest case, namely $C(\mathbb{R})$ is solved already. So we take one step further.

7 Baire class 1 functions

It is clear from Theorem 5.3 that $\mathrm{sc}(\{f : f \text{ is Baire class } 1\}) \geq \omega_1$. As opposed to the case $2 \leq \alpha < \omega_1$ we conjecture that, in fact, $\mathrm{sc}(\{f : f \text{ is Baire class } 1\}) = \omega_1$. Unfortunately, we only can prove this in a special case. What makes this case interesting is that it covers the usual situation in which every difference operator D is of the form $D = \Delta_b$.

First we need two lemmas.

Lemma 7.1 *Let $a, b \in \mathbb{R} \setminus \{0\}$. The solutions to the equation*

$$f(x + b) - af(x) = 0$$

are the functions of the form

$$f(x) = \varphi(x)(|a|^{1/b})^x,$$

where φ is an arbitrary function periodic mod b if $a > 0$, and an arbitrary function anti-periodic mod b (that is, $\varphi(x + b) = -\varphi(x)$ for every $x \in \mathbb{R}$) if $a < 0$.

In addition, f is Baire class 1 iff φ is Baire class 1.

Proof. Straightforward calculations. □

Lemma 7.2 *Let $a_1, a_2, b_1, b_2 \in \mathbb{R} \setminus \{0\}$. Suppose that the equations $f(x + b_1) - a_1 f(x) = 0$ and $f(x + b_2) - a_2 f(x) = 0$ have a common Baire class 1 solution which is not identically zero. Then $|a_1|^{1/b_1} = |a_2|^{1/b_2}$.*

Proof. Suppose this is not true. Then by the previous lemma there exist two Baire class 1 functions φ_1 and φ_2 such that

$$\varphi_1(x)(|a_1|^{1/b_1})^x = \varphi_2(x)(|a_2|^{1/b_2})^x,$$

where φ_1 and φ_2 are periodic (or anti-periodic) mod b_1 and b_2, respectively. We may assume that both functions are periodic, otherwise we could consider $\psi_i(x) = \varphi_i(2x)$ for $i = 1, 2$. We can also assume that $|a_1|^{1/b_1} < |a_2|^{1/b_2}$, and therefore

$$\varphi_1(x) = \varphi_2(x)c^x, \text{ where } c > 1. \tag{2}$$

Finally, as φ_2 is not identically zero, we can also suppose (by applying an appropriate translation if needed) that $\varphi_2(0) \neq 0$.

Suppose that $b_1/b_2 \in \mathbb{Q}$. Then φ_1 and φ_2 are periodic mod a common value p. But this is impossible, since $c^x \neq 1$ when $x \neq 0$.

Therefore $b_1/b_2 \notin \mathbb{Q}$. Then for every (nondegenerate) interval $I \subset \mathbb{R}$ there exist integers $n, k \in \mathbb{Z}$ with k arbitrarily large such that $nb_1 + kb_2 \in I$. By substituting kb_2 into (2) we get $\varphi_1(kb_2) = \varphi_2(kb_2)c^{kb_2}$ for every $k \in \mathbb{Z}$, thus $\varphi_1(kb_2) = \varphi_2(0)c^{kb_2}$ for every $k \in \mathbb{Z}$. Therefore $\varphi_1(nb_1 + kb_2) = \varphi_2(0)c^{kb_2}$ for every $n, k \in \mathbb{Z}$, which yields that φ_1 is unbounded on I. As I was arbitrary, φ_1 is unbounded on every subinterval of $\mathbb{R}$. But φ_1 is of Baire class 1, so it has a point of continuity (see e.g. [3, 24.15]), hence it must be bounded on some interval, a contradiction. $\square$

Remark 7.3 The impossibility of (2) is closely related to the well known statement that the identity function is not the sum of two measurable periodic functions (though it is surprisingly the sum of two periodic functions; see e.g. [7] and [5]). Indeed, taking the logarithm of (2), we would obtain a representation of the identity function as the sum of two Baire class 1 periodic functions; the only problem is that our functions can vanish at certain points.

Theorem 7.4 *Let $D_i f = g_i$ ($i \in I$) be a system of difference equations, and suppose that every difference operator consists of at most two terms; that is for every $i \in I$ the i^{th} equation is of the form*

$$a_i^{(1)} f\left(x + b_i^{(1)}\right) + a_i^{(2)} f\left(x + b_i^{(2)}\right) = g_i(x).$$

Then if every countable subsystem has a Baire class 1 solution, then the whole system has one as well.

Proof. If any of the equations consists of a single term, then it has a unique solution, so we are clearly done. Thus, by applying a translation and multiplying by a real number, we may assume that every equation is of the form

$$f(x + b_i) - a_i f(x) = g_i(x).$$

First suppose that $|a_{i_1}|^{1/b_{i_1}} \neq |a_{i_2}|^{1/b_{i_2}}$ for some $i_1, i_2 \in I$. Then it easily follows from Lemma 7.2 that the two corresponding equations have a unique common Baire class 1 solution. This clearly solves the whole system, as every triple of equations is solvable.

So we can assume that there exists a $c > 0$ such that $|a_i|^{1/b_i} = c$ for every $i \in I$. If we now replace $f(x)$ by $f(x)/c^x$ (and modify g_i appropriately), then our equations will be of the form $\Delta_{b_i} f(x) = f(x+b_i) - f(x) = g_i$ ($i \in I^-$) or $f(x+b_i) + f(x) = g_i$ ($i \in I^+$), where $I = I^- \cup I^+$. We put $B^- = \{b_i : i \in I^-\}$ and $B^+ = \{b_i : i \in I^+\}$. There are countable subsets $J^- \subset I^-$ and $J^+ \subset I^+$ such that $E^- = \{b_i : i \in J^-\}$ is relatively dense in B^-, and $E^+ = \{b_i : i \in J^+\}$ is relatively dense in B^+.

By assumption, there exists a common Baire class 1 solution f to the equations with indices $J^- \cup J^+$. We claim that f is a solution to the whole system.

First let $i \in I^-$. As $J^- \cup \{i\}$ is also countable, we can choose a Baire class 1 function f^- such that $\Delta_{b_j} f^- = g_j$ for every $j \in J^-$ and $\Delta_{b_i} f^- = g_i$.

Put $f' = f - f^-$. Then for every $j \in J^-$ we have

$$\Delta_{b_j} f' = \Delta_{b_j}(f - f^-) = \Delta_{b_j} f - \Delta_{b_j} f^- = g_j - g_j = 0,$$

thus f' is periodic mod b_j for each $j \in J^-$. Let $G^- = \langle E^- \rangle$; then f' is periodic mod each element of G^-.

We distinguish between two cases. If G^- is dense in $\mathbb{R}$, then f' must be a constant function c, for otherwise it would attain two distinct values on dense sets, so it would have no point of continuity, which is impossible as f' is Baire class 1.

Thus

$$\Delta_{b_i} f = \Delta_{b_i}(f^- + c) = \Delta_{b_i} f^- + \Delta_{b_i} c = g_i + 0 = g_i,$$

which completes the proof in the first case.

If, on the other hand, G^- is not dense in $\mathbb{R}$ then $G^- = \mathbb{Z}d$ for some $d \in \mathbb{R}$. In particular, G^- is discrete. Then so is E^- and thus $E^- = B^-$ as E^- is dense in B^-. Since $b_i \in B^- = E^-$, there is a $j \in J^-$ with $b_i = b_j$ which obviously implies $g_i = g_j$. Therefore, f satisfies $\Delta_{b_i} f = \Delta_{b_j} f = g_j = g_i$.

Let now $i \in I^+$. Choose a Baire class 1 function f^+ such that $f^+(x+b_j) + f^+(x) = g_j(x)$ for every $j \in J^+$ and $f^+(x+b_i) + f(x) = g_i(x)$ for every $x \in \mathbb{R}$. Put $f' = f - f^+$. Then f' is easily seen to be anti-periodic mod b_j, hence periodic mod $2b_j$ for every $j \in J^+$, hence it is also periodic mod $G^+ = \langle \{2b_j : j \in J^+\} \rangle$. If G^+ is dense in $\mathbb{R}$, then f' must be a constant function c. But f' is anti-periodic, so $c = 0$. Therefore $f = f^+$, so f clearly solves the i^{th} equation.

On the other hand, if G^+ is discrete then so is E^+ and then we can complete the proof as in the previous case. $\square$

Question 7.5 *Is it true that* $\mathrm{sc}(\{f : f \textit{ is Baire class } 1\}) = \omega_1$*?*

8 Lebesgue and Baire measurable functions

As in Section 5, $\mathcal{I}$ shall denote the ideals $\mathcal{N}$ or $\mathcal{M}$. Thus $\mathcal{B}_\mathcal{I}$ equals the σ-algebra of Lebesgue or Baire measurable sets.

The goal of this section is to prove upper and lower estimates for $\mathrm{sc}(\{f : f \text{ is } \mathcal{B}_\mathcal{I}\text{-measurable}\})$ in terms of some cardinal invariants of the ideal $\mathcal{I}$. These estimates give the exact value of the solvability cardinal consistently.

Definition 8.1

$$\begin{aligned}
\operatorname{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I},\ \bigcup \mathcal{A} \notin \mathcal{I}\},\\
\operatorname{non}(\mathcal{I}) &= \min\{|A| : A \subset \mathbb{R},\ A \notin \mathcal{I}\},\\
\operatorname{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I},\ \forall I \in \mathcal{I}\ \exists A \in \mathcal{A},\ I \subset A\}.
\end{aligned}$$

Remark 8.2 Note that $\omega_1 \le \operatorname{add}(\mathcal{I}) \le \operatorname{non}(\mathcal{I}) \le \operatorname{cof}(\mathcal{I}) \le 2^\omega$. The last inequality follows from $\operatorname{cof}(\mathcal{I}) \le |\mathcal{A}|$, where $\mathcal{A} = \{B \in \mathcal{I} : B \text{ is Borel}\}$. It is also easy to see that $\operatorname{add}(\mathcal{I}) \le \operatorname{cf}(\operatorname{non}(\mathcal{I}))$.

Before we prove our estimates (Theorems 8.6 and 8.7) we need some preparation.

Definition 8.3 For a set $H \subset \mathbb{R}$ define

$$\mathcal{D}_H = \{D \in \mathcal{D} : D = \sum_{i=1}^n a_i T_{b_i},\ b_i \in H \text{ for every } i = 1, \dots, n\}.$$

Theorem 8.4 *Let S be a solvable system of difference equations and $H \subset \mathbb{R}$. Then S has a solution that is identically zero on H if and only if whenever (D, g) is deducible from S and $D \in \mathcal{D}_H$ then $g(0) = 0$.*

Proof. The proof is again a variation of the proof of [6, Thm. 2.2].

First suppose that f is a solution to S vanishing on H, and let (D, g) be deducible from S such that $D \in \mathcal{D}_H$; that is, $D = \sum_{i=1}^n a_i T_{b_i}$, where $b_i \in H$ for every i. Then $g(0) = (Df)(0) = \sum_{i=1}^n a_i f(b_i) = 0$, since $b_i \in H$ for every i.

Suppose now that whenever (D, g) is deducible from S and $D \in \mathcal{D}_H$ then $g(0) = 0$. Let

$$\mathcal{A} = \{D \in \mathcal{D} : \exists g\ (D, g) \text{ is deducible from } S\}.$$

Then $\mathcal{A}$ is a linear subspace of $\mathcal{D}$. Define

$$L(D) = g(0) \quad (D \in \mathcal{A}),$$

where (D, g) is deducible from S. To see that L is well defined note that S is solvable, and $Df = g$ whenever (D, g) is deducible from S and f is a solution to S. L is clearly linear, and by assumption vanishes on $\mathcal{A} \cap \mathcal{D}_H$. Define the linear space

$$\mathcal{B} = \{A + D : A \in \mathcal{A}, D \in \mathcal{D}_H\}$$

and the linear map

$$L'(A + D) = L(A) \quad (A \in \mathcal{A},\ D \in \mathcal{D}_H),$$

which is clearly well defined and extends L. Moreover, L' vanishes on $\mathcal{D}_H$. Let $L^* : \mathcal{D} \to \mathbb{R}$ be a linear extension of L', and set $f(x) = L^*(T_x)$ $(x \in \mathbb{R})$. We claim that f is a solution to S vanishing on H, which will complete the proof.

First, $L^*(T_x) = f(x) = (T_x f)(0)$ for every $x \in \mathbb{R}$, and L^* is linear, so $L^*(D) = (Df)(0)$ for every $D \in \mathcal{D}$. If (D,g) is deducible from S then so is $(T_x D, T_x g)$ for every $x \in \mathbb{R}$, hence $(Df)(x) = (T_x Df)(0) = L^*(T_x D) = (T_x g)(0) = g(x)$, so f solves S.

Finally, f vanishes on H, for if $x \in H$ then $T_x \in \mathcal{D}_H$, so $f(x) = L^*(T_x) = 0$, since L^* vanishes on $\mathcal{D}_H$. $\square$

Theorem 8.5 *Let S be a system of difference equations such that for every $(D,g) \in S$ we have $g = 0$ $\mathcal{I}$-a.e. If there exists a $\mathcal{B}_{\mathcal{I}}$-measurable solution to S, then there is also one which is zero $\mathcal{I}$-a.e.*

Proof. Let f be a $\mathcal{B}_{\mathcal{I}}$-measurable solution to S. If $\mathcal{I} = \mathcal{N}$ then let H be the set of points of approximate continuity of f, while if $\mathcal{I} = \mathcal{M}$ then let H be a residual (= comeager) set on which f is (relatively) continuous. Then $\mathbb{R} \setminus H \in \mathcal{I}$. It is sufficient to show that there exists a solution to S vanishing on H. Using the previous theorem we need to show that if (D,g) is deducible from S and $D \in \mathcal{D}_H$ then $g(0) = 0$. Let $D = \sum_{i=1}^n a_i T_{b_i}$, where $b_i \in H$ for every i. As (D,g) is deducible from S, $g = 0$ $\mathcal{I}$-a.e. and f solves (D,g). So $\sum_{i=1}^n a_i f(x + b_i) = g(x)$ for every $x \in \mathbb{R}$.

If $\mathcal{I} = \mathcal{N}$ then, using $b_i \in H$, we obtain that g is approximately continuous at 0. If $\mathcal{I} = \mathcal{M}$ then, using $b_i \in H$, we obtain that g is (relatively) continuous on the residual set $\bigcap_{i=1}^n (H - b_i)$, which contains 0. But in both cases $g = 0$ $\mathcal{I}$-a.e., so we obtain $g(0) = 0$ as required. $\square$

Theorem 8.6 $\mathrm{sc}(\{f : f$ *is $\mathcal{B}_{\mathcal{I}}$-measurable*$\}) \leq [\mathrm{cof}(\mathcal{I})]^+$.

Proof. Let S be such that each subsystem of cardinality at most $\mathrm{cof}(\mathcal{I})$ has a $\mathcal{B}_{\mathcal{I}}$-measurable solution. We have to show that S has a $\mathcal{B}_{\mathcal{I}}$-measurable solution. By Theorem 5.1 there exists a $\mathcal{B}_{\mathcal{I}}$-measurable f_0 that is an $\mathcal{I}$-a.e. solution to S. Define a new system as follows.

$$S' = \{(D, g - Df_0) : (D,g) \in S\}.$$

Then S' has a $\mathcal{B}_{\mathcal{I}}$-measurable solution if and only if S has one, and every subsystem of S' of cardinality at most $\mathrm{cof}(\mathcal{I})$ has a $\mathcal{B}_{\mathcal{I}}$-measurable solution. Moreover, each right hand side $g - Df_0$ is 0 $\mathcal{I}$-a.e. Let

$$S^* = \{(D,g) : (D,g) \text{ is deducible from } S'\}.$$

Then f solves S' if and only if it solves S^*, moreover, each right hand side of S^* is 0 $\mathcal{I}$-a.e. Also, every subsystem of S^* of cardinality at most $\mathrm{cof}(\mathcal{I})$ has a $\mathcal{B}_{\mathcal{I}}$-measurable solution since every $(D,g) \in S^*$ is deducible from a finite subsystem of S'. In addition, every equation deducible from S^* is already in S^*.

Now we prove that S^* has a $\mathcal{B}_{\mathcal{I}}$-measurable solution, which will complete the proof. By Theorem 8.5 every subsystem of S^* of cardinality at most $\mathrm{cof}(\mathcal{I})$ has an $\mathcal{I}$-a.e. zero solution. We claim that S^* itself has such a solution. Suppose on the contrary that this is not true. Let $\mathcal{A} \subset \mathcal{I}$ be such that $|\mathcal{A}| = \mathrm{cof}(\mathcal{I})$ and $\forall I \in \mathcal{I}\ \exists A \in \mathcal{A},\ I \subset A$. For any $A \in \mathcal{A}$ the system S^* has no solution vanishing outside A. By Theorem 8.4 this means that there exists a $(D_A, g_A) \in S^*$ such that $D_A \in \mathcal{D}_{\mathbb{R}\setminus A}$ and $g_A(0) \neq 0$.

The system $\{(D_A, g_A) : A \in \mathcal{A}\}$ is of cardinality $\mathrm{cof}(\mathcal{I})$, hence it has a solution f vanishing $\mathcal{I}$-a.e. Let $A_0 \in \mathcal{A}$ be such that f vanishes outside A_0. Then $D_{A_0} f = g_{A_0}$, thus $(D_{A_0} f)(0) = g_{A_0}(0) \neq 0$, but on the other hand $D_{A_0} \in \mathcal{D}_{\mathbb{R}\setminus A_0}$, so $(D_{A_0} f)(0) = 0$. This contradiction finishes the proof. $\square$

Theorem 8.7 $\mathrm{sc}(\{f : f$ *is* $\mathcal{B}_\mathcal{I}$*-measurable*$\}) \geq [\mathrm{cf}(\mathrm{non}(\mathcal{I}))]^+ \geq [\mathrm{add}(\mathcal{I})]^+ \geq \omega_2$.

Proof. We have to construct an S with no $\mathcal{B}_\mathcal{I}$-measurable solutions such that each subsystem of cardinality less than $\mathrm{cf}(\mathrm{non}(\mathcal{I}))$ has a $\mathcal{B}_\mathcal{I}$-measurable solution.

First we construct a set $B \subset \mathbb{R}$ such that (i) $B \notin \mathcal{I}$, $\mathbb{R} \setminus B \notin \mathcal{I}$, (ii) $|(B+b)\Delta B| < \mathrm{non}(\mathcal{I})$ for every $b \in B$, and (iii) $B \cap (-B) = \emptyset$.

Let $V \subset \mathbb{R}$ be such that $V \notin \mathcal{I}$ and $|V| = \mathrm{non}(\mathcal{I})$. We may assume that V is a linear space over the rationals. Let $\{v_\alpha : \alpha < \mathrm{non}(\mathcal{I})\}$ be a basis of V. Represent the nonzero elements of V as $v = \sum_{i=1}^n q_i v_{\alpha_i}$, where $q_i \in \mathbb{Q} \setminus \{0\}$ and $\alpha_1 < \ldots < \alpha_n$. Define $\varphi(v) = q_n$, and

$$B = \{v \in V : \varphi(v) > 0\}.$$

Clearly, (iii) holds. Note that $V = B \cup (-B) \cup \{0\}$, hence (i) is satisfied. Let $b \in B \setminus \{0\}$ be arbitrary. Suppose $b = \sum_{i=1}^n q_i v_{\alpha_i}$, where $q_i \in \mathbb{Q} \setminus \{0\}$ and $\alpha_1 < \ldots < \alpha_n$. Then $(B+b)\Delta B$ is included in the linear space generated by $\{v_\alpha : \alpha \leq \alpha_n\}$, which is of cardinality less than $\mathrm{non}(\mathcal{I})$. So (ii) holds as well.

We claim that the system

$$S = \{(\Delta_b, \Delta_b \chi_B) : b \in B\}$$

satisfies the requirements. First we check that each right hand side is zero $\mathcal{I}$-a.e. Indeed, if $b \in B$ then $\{x \in \mathbb{R} : (\Delta_b \chi_B)(x) \neq 0\} \subset (B+b)\Delta B \in \mathcal{I}$, since $|(B+b)\Delta B| < \mathrm{non}(\mathcal{I})$.

Suppose that S has a $\mathcal{B}_\mathcal{I}$-measurable solution. Then, by Theorem 8.5, S has an $\mathcal{I}$-a.e. zero solution f_0 as well. Then $\Delta_b f_0 = \Delta_b \chi_B$ for every $b \in B$, so $f_0 - \chi_B$ is periodic mod every $b \in B$. Then it is also periodic mod each $b \in -B$. In particular, $f_0 - \chi_B$ is constant on $B \cup (-B)$. But $f_0 = 0$ $\mathcal{I}$-a.e., $B \notin \mathcal{I}$, and $B \cap (-B) = \emptyset$ which is impossible.

What remains to show is that each subsystem S' of S of cardinality less than $\mathrm{cf}(\mathrm{non}(\mathcal{I}))$ has a $\mathcal{B}_\mathcal{I}$-measurable solution. Let B' be the corresponding subset of B, where $|B'| < \mathrm{cf}(\mathrm{non}(\mathcal{I}))$. Now we put

$$A = \langle B' \rangle + \bigcup_{b' \in B'} [(B+b')\Delta B].$$

Then $|A| < \mathrm{non}(\mathcal{I})$, hence $A \in \mathcal{I}$. It is easy to see, by checking the cases $x \in A$ and $x \notin A$, that $f = \chi_{B \cap A}$ is a $\mathcal{B}_\mathcal{I}$-measurable solution to S'. $\square$

Corollary 8.8 *The Continuum Hypothesis implies*

$$\mathrm{sc}(\{f : f \textit{ is measurable}\}) = \mathrm{sc}(\{f : f \textit{ has the Baire property}\}) = \omega_2 = (2^\omega)^+.$$

Question 8.9 *Is* $\mathrm{sc}(\{f : f \textit{ is } \mathcal{B}_\mathcal{I}\textit{-measurable}\})$ *equal to* $[\mathrm{cof}(\mathcal{I})]^+$*? Is* $\mathrm{sc}(\{f : f \textit{ is } \mathcal{B}_\mathcal{I}\textit{-measurable}\})$ *equal to* $[\mathrm{cf}(\mathrm{non}(\mathcal{I}))]^+$*?*

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