The structure of rigid functions

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Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ is called *vertically rigid* if $graph(cf)$ is isometric to $graph(f)$ for all $c \neq 0$. We prove Janković's conjecture by showing that a continuous function is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$. We answer a question of Cain, Clark and Rose by showing that there exists a Borel measurable vertically rigid function which is not of the above form. We discuss the Lebesgue and Baire measurable case, consider functions bounded on some interval and functions with at least one point of continuity. We also introduce horizontally rigid functions, and show that a certain structure theorem can be proved without assuming any regularity.

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1 Introduction

An easy calculation shows that the exponential function $f(x) = e^x$ has the somewhat 'paradoxical' property that $cf(x)$ is a translate of $f(x)$ for every $c > 0$. It is also easy to see that every function of the form $a + be^{kx}$ has this property. This connection is also of interest from the point of view of functional equations. In [1] Cain, Clark and Rose introduced the notion of vertical rigidity as follows.

Definition 1.1 A function $f : \mathbb{R} \to \mathbb{R}$ is called *vertically rigid*, if $graph(cf)$ is isometric to $graph(f)$ for all $c \in (0, \infty)$. (Clearly, $c \in \mathbb{R} \setminus \{0\}$ would be the same.)

Obviously every function of the form $a + bx$ is also vertically rigid. D. Janković conjectured (see [1]) that the converse is also true for continuous functions.

Conjecture 1.2 *(D. Janković) A continuous function is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

The main result of the present paper is the proof of this conjecture.

We will need the following technical generalisations.

Definition 1.3 If C is a subset of $(0, \infty)$ and $\mathcal{G}$ is a set of isometries of the plane then we say that f is vertically rigid *for a set C via elements of $\mathcal{G}$* if for every $c \in C$ there exists a $\varphi \in \mathcal{G}$ such that $\varphi(graph(cf)) = graph(f)$.

(If we do not mention C or $\mathcal{G}$ then C is $(0, \infty)$ and $\mathcal{G}$ is the set of all isometries.)

The paper is organised as follows. In Section 2 we prove Janković's conjecture, even if we only assume that f is a continuous vertically rigid function for an uncountable set C. We show that it is sufficient to assume that f has at least one point of continuity, provided that it is vertically rigid for C *via translations*. We also show that it is sufficient to assume that f is bounded on some nondegenerate interval, provided that it is vertically rigid *via translations* and $C = (0, \infty)$. In Section 3 we show that Janković's conjecture fails for Borel measurable functions. Our example also answers a question from [1] that asks whether every vertically rigid function is of the form $a + bx$ $(a, b \in \mathbb{R})$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g. In Section 4 we prove that every Lebesgue (Baire) measurable function that is vertically rigid *via translations* is of the form $a + be^{kx}$ *almost everywhere (on a comeagre set)*. The case of general isometries remains open. We also prove that in many situations the exceptional set can be removed. In Section 5 we define the notion of a rigid set, discuss how it is connected to the notion of a rigid function, and prove an ergodic theory type result. In Section 6 we define horizontally rigid functions, and give a simple characterisation of those functions that are horizontally rigid *via translations*. Finally, in Section 7 we collect the open questions.

2 Proof of Janković's conjecture

Theorem 2.1 ***(Janković's conjecture)*** *A continuous function is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

Remark 2.2 In fact, our proof will show that it is sufficient if f is a continuous function that is vertically rigid for some uncountable set C.

It is of course very easy to see that these functions are vertically rigid and continuous. The proof of the difficult direction goes through three theorems, which are interesting in their own right. First we reduce the general case to translations, then the case of translations to horizontal translations, and finally we describe the continuous functions that are vertically rigid via horizontal translations.

Theorem 2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function vertically rigid for an uncountable set $C \subset (0, \infty)$. Then f is of the form $a + bx$ for some $a, b \in \mathbb{R}$ or f is vertically rigid for an uncountable set $D \subset (0, \infty)$ via translations.*

Proof. Let φ_c be the isometry belonging to $c \in C$. First we show that we may assume that these isometries are orientation preserving. If uncountably many of the φ_c's are orientation preserving then we are done by shrinking C. Otherwise let $C' \subset C$ be uncountable so that $\varphi_{c'}$ is orientation reversing for every $c' \in C'$. Fix $c'_0 \in C'$, then one can easily check that $c'_0 f$ is vertically rigid via orientation preserving isometries for $C'' = \left\{ \frac{c'}{c'_0} : c' \in C' \right\}$. Suppose that we have already proved the theorem in case all isometries are orientation preserving. Then either $c'_0 f$ is of the form $a + bx$, and then so is f, or $c'_0 f$ is vertically rigid for an uncountable set D via translations, but then so is f itself (for the same set D, but possibly different translations).

For a function f let S_f be the set of directions between pairs of points on the graph of f, that is,

$$S_f = \left\{ \frac{p - q}{|p - q|} : p, q \in graph(f),\ p \neq q \right\}.$$

Clearly S_f is a symmetric (about the origin) subset of the unit circle $S^1 \subset \mathbb{R}^2$. As f is a function, $(0, \pm 1) \notin S_f$. Since f is continuous, it is easy to see that S_f actually consists of two (possibly degenerate) nonempty intervals. (Indeed, if $p = (x, f(x))$ and $q = (y, f(y))$ then $x < y$ and $x > y$ define two connected sets, open half planes in $\mathbb{R}^2$, whose continuous images form S_f.)

An orientation preserving isometry φ of the plane is either a translation or a rotation. Denote by $ang(\varphi)$ the angle of φ in case it is a rotation, and set $ang(\varphi) = 0$ if φ is a translation.

Now we define two self-maps of S^1. Denote by ϱ_α the rotation about the origin by angle α. For $c > 0$ let ψ_c be the map obtained by 'multiplying by c', that is, let

$$\psi_c((x, y)) = \frac{(x, cy)}{|(x, cy)|} \quad ((x, y) \in S^1).$$

It is easy to see that the rigidity of f implies that for every $c \in C$

$$S_f = \varrho_{ang(\varphi_c)}(\psi_c(S_f)). \tag{2.1}$$

If S_f consists of two points, then f is clearly of the form $a + bx$ and we are done.

Let now $S_f = I \cup -I$, where I is a subinterval of S^1 in the right half plane. We claim that the endpoints of I are among $(0, \pm 1)$ and $(1, 0)$. Suppose this fails, and consider the function $l(c) = arclength\,(\psi_c(I))$ $(c \in (0, \infty))$. It is easy to see that l is real analytic, and we show that it is not constant. Let us first assume that $(0, 1)$ and $(0, -1)$ are not endpoints of I, then $\lim_{c\to 0} l(c) = 0$, so l cannot be constant (as $l > 0$). Let us now suppose that either $(0, 1)$ or $(0, -1)$ is an endpoint of I, then $0 < arclength(I) < \frac{\pi}{2}$ or $\frac{\pi}{2} < arclength(I) < \pi$. In both cases $\lim_{c\to 0} l(c) = \frac{\pi}{2}$ but $l(c) \neq \frac{\pi}{2}$, so l is not constant. As l is analytic, it attains each of its values at most countably many times, so there exists a $c \in C$ so that $arclength\,(\psi_c(I)) \neq arclength(I)$, which contradicts (2.1).

(Actually, it can be shown by a somewhat lengthy calculation using the derivatives that l attains each value at most twice.)

But this easily yields $ang(\varphi_c) = 0$ or π for every $c \in C$. (Note that $(0, \pm 1) \notin S_f$ and that S_f is symmetric.) Just as above, we may assume that $ang(\varphi_c) = 0$ for all $c \in C$. (Indeed, choose C', c_0' analogously.) But then f is vertically rigid for an uncountable set via translations, so the proof is complete. □

Theorem 2.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function that is vertically rigid for a set $C \subset (0, \infty)$ via translations. Then there exists $a \in \mathbb{R}$ such that $f - a$ is vertically rigid for the same set via horizontal translations.*

Proof. We can clearly assume that $1 \notin C$. By assumption, for every $c \in C$ there exists $u_c, v_c \in \mathbb{R}$ such that

$$cf(x) = f(x + u_c) + v_c \ (\forall x \in \mathbb{R}) \tag{2.2}$$

Applying this first with $c = c_2$ then with $c = c_1$ we obtain

$$c_1c_2f(x) = c_1(f(x + u_{c_2}) + v_{c_2}) =$$

$$= c_1f(x + u_{c_2}) + c_1v_{c_2} = f(x + u_{c_1} + u_{c_2}) + v_{c_1} + c_1v_{c_2} \tag{2.3}$$

Interchanging c_1 and c_2 we get

$$c_2c_1f(x) = f(x + u_{c_2} + u_{c_1}) + v_{c_2} + c_2v_{c_1}. \tag{2.4}$$

Comparing (2.3) and (2.4) yields $v_{c_1} + c_1v_{c_2} = v_{c_2} + c_2v_{c_1}$, so

$$\frac{v_{c_1}}{c_1 - 1} = \frac{v_{c_2}}{c_2 - 1} \quad \text{for all } c_1, c_2 \in C,$$

consequently $a := \frac{v_c}{c-1}$ is the same value for all $c \in C$. Substituting this back to (2.2) gives $cf(x) = f(x + u_c) + a(c - 1)$, so $c(f(x) - a) = f(x + u_c) - a$ for all $c \in C$, hence $f - a$ is vertically rigid for C via horizontal translations. $\square$

Theorem 2.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous vertically rigid function for an uncountable set $C \subset (0, \infty)$ via horizontal translations. Then f is of the form be^{kx} $(b \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\})$.*

Before proving this theorem we need a definition and a lemma.

Definition 2.6 For a function $f : \mathbb{R} \to \mathbb{R}$ let $T_{f,C} \subset \mathbb{R}$ be the additive group generated by the set $T' = \{t \in \mathbb{R} : \exists c \in C\ \forall x \in \mathbb{R}\ f(x + t) = cf(x)\}$. (We will usually simply write T for $T_{f,C}$.)

Lemma 2.7 *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function for an uncountable set $C \subset (0, \infty)$ via horizontal translations such that $f(0) = 1$. Then T is dense and*

$$f(x + t) = f(x)f(t) \quad \forall x \in \mathbb{R}\ \forall t \in T.$$

Moreover, $f(t) > 0$ for every $t \in T$.

Proof. By assumption, for every $c \in C$ there exists $t_c \in \mathbb{R}$ such that $cf(x) = f(x+t_c)$ for every $x \in \mathbb{R}$. Then $t_c \in T$ for every $c \in C$. Since f is not identically zero, $t_c \neq t_{c'}$ whenever $c, c' \in C$ are distinct. Hence $\{t_c : c \in C\}$ is uncountable, so T is uncountable. As every subgroup of $\mathbb{R}$ is either discrete countable or dense, T is dense.

Every $t \in T$ can be written as $t = \sum_{i=1}^{m} n_i t_i$ $(t_i \in T', n_i \in \mathbb{Z}, i = 1, \dots, m)$ where $f(x + t_i) = c_i f(x)$ $(x \in \mathbb{R},\ i = 1, \dots, m)$.

From these we easily get

$$f(x + t) = c_t f(x), \quad \text{where } c_t = \prod_{i=1}^{m} c_i^{n_i},\ x \in \mathbb{R},\ t \in T. \tag{2.5}$$

Note that $c_t > 0$ (and also that it is not necessarily a member of C). It suffices to show that $c_t = f(t)$ for every $t \in T$, but this follows if we substitute $x = 0$ into (2.5). $\square$

Proof. (Thm. 2.5) If f is identically zero then we are done, so let us assume that this is not the case. The class of continuous vertically rigid functions for some uncountable set via horizontal translations, as well as the class of functions of the form be^{kx} $(b \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\})$ are both closed under horizontal translations and under multiplication by nonzero constants, so we may assume that $f(0) = 1$. Then the previous lemma yields that $f(t_1 + t_2) = f(t_1)f(t_2)$ $(t_1, t_2 \in T)$, and also that $f|_T > 0$. Then $g(t) = \log f(t)$ is defined for every $t \in T$, and g is clearly additive on T. But it is well-known (and an easy calculation) that an additive function on a dense subgroup is either of the form kx, or unbounded both from above and below on every nondegenerate interval.

The second alternative cannot hold, since f is continuous, so $f|_T$ is of the form e^{kx}, so by continuity f is of this form everywhere. Since C contains elements different from 1, we obtain that $f(x) = 1$ $(x \in \mathbb{R})$ is not vertically rigid for C via horizontal translations, hence $k \neq 0$. $\square$

Putting together the three above theorems completes the proof of Janković's conjecture.

We remark here that we have actually also proved the following, which applies e.g. to Baire class 1 functions.

Theorem 2.8 *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function for an uncountable set $C \subset (0, \infty)$ via translations. If f has a point of continuity then it is of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$. If f is vertically rigid via translations (i.e. $C = (0, \infty)$) and bounded on a nondegenerate interval then it is of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$, too.*

Proof. Following the proof of the last theorem we may assume in both cases that $f(0) = 1$, the translations are horizontal, and $f|_T$ is of the form e^{kx} $(k \in \mathbb{R})$.

In the first case, let x_0 be a point of continuity of f, then clearly $f(x_0) = e^{kx_0}$, since T is dense. Let now $x \in \mathbb{R}$ be arbitrary, and $t_n \in T$ $(n \in \mathbb{N})$ be such that $\lim_{n\to\infty} t_n = x_0 - x$. Using Lemma 2.7 we obtain

$$e^{kx_0} = f(x_0) = \lim_{n\to\infty} f(x + t_n) = \lim_{n\to\infty} f(x)f(t_n) = f(x) \lim_{n\to\infty} f(t_n) =$$

$$= f(x) \lim_{n\to\infty} e^{kt_n} = f(x)e^{k(x_0-x)} = f(x)e^{kx_0}/e^{kx},$$

from which $f(x) = e^{kx}$ follows.

In the second case, for every $c > 0$ there is a $t_c \in T = T_{f,(0,\infty)}$ such that $cf(x) = f(x + t_c) = f(x)f(t_c)$. By substituting $x = 0$ into the equation we get $c = f(t_c) = e^{kt_c}$ for every $c > 0$. (In particular, $k \neq 0$.) So $t_c = \frac{\log c}{k}$. If c ranges over $(0, \infty)$ then t_c ranges over $\mathbb{R}$, so we get $T = \mathbb{R}$. Hence $f|_T = f$ is of the form e^{kx}, and we are done. $\square$

Example 2.9 *There exists a function $f : \mathbb{R} \to \mathbb{R}$ that is vertically rigid for an uncountable set $C \subset \mathbb{R}$ via horizontal translations, bounded on every bounded interval, and is* not *of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

Proof. Let $P \subset \mathbb{R}$ be an uncountable linearly independent set over $\mathbb{Q}$, see e.g. [3, 19.2] or [6]. Define $\widehat{P}$ to be the generated additive subgroup. Let

$$f(x) = \begin{cases} e^x & \text{if } x \in \widehat{P} \\ 0 & \text{if } x \in \mathbb{R} \setminus \widehat{P}, \end{cases}$$

then f is clearly bounded on every bounded interval.

It is easy to see that $\frac{p}{2} \in \mathbb{R} \setminus \widehat{P}$ for every $p \in P$, so $\widehat{P} \neq \mathbb{R}$, hence f is not continuous, so it is not of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.

For every $p \in P$ and $x \in \mathbb{R}$ we have $x \in \widehat{P} \iff x + p \in \widehat{P}$, which easily implies $f(x+p) = e^p f(x)$. Hence f is vertically rigid for the uncountable set $C = \{e^p : p \in P\}$. $\square$

Janković's conjecture has the following curious corollary.

Corollary 2.10 *There are continuous functions f and g with isometric graphs so that f is vertically rigid but g is not.*

Proof. If we rotate the graph of $f(x) = e^x$ clockwise by $\frac{\pi}{4}$ then we obtain the graph of a continuous function. By Theorem 2.1 it is not vertically rigid. $\square$

3 A Borel measurable counterexample

In this section we show that Janković's conjecture fails for Borel measurable functions. Our example also answers Question 1 in [1] of Cain, Clark and Rose, which asks whether every vertically rigid function is of the form $a+bx$ $(a, b \in \mathbb{R})$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g. By Thm. 2 of [1] $a + be^g$ is vertically rigid if and only if $b = 0$ or g is surjective.

Theorem 3.1 *There exists a Borel measurable vertically rigid function $f : \mathbb{R} \to [0, \infty)$ (via horizontal translations) that is not of the form $a + bx$ $(a, b \in \mathbb{R})$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g.*

For definitions and basic results on Baire measurable sets (= sets with the property of Baire), meagre (= first category) and comeagre (= residual) sets consult e.g. [5] or [3]. For Polish spaces and Borel isomorphisms see e.g. [3].

Proof. Let P be a Cantor set (nonempty nowhere dense compact set with no isolated points) that is linearly independent over $\mathbb{Q}$, see e.g. [3, 19.2]. (One can also derive the existence of such a set from [6] using the well-known fact that every uncountable Borel or analytic set contains a Cantor set.) It is easy to see that for all $n_1, \dots, n_k \in \mathbb{Z}$ the set $P_{n_1,\dots,n_k} = \{n_1p_1 + \dots + n_kp_k : p_1, \dots, p_k \in P\}$ is compact, hence the group $\widehat{P}$ generated by P (that is, the union of the $P_{n_1,\dots,n_k}$'s) is a Borel, actually F_σ set. As P is linearly independent, each element of $\widehat{P}$ can be uniquely written in the form $n_1p_1 + \dots + n_kp_k$.

Since P and $(0, \infty)$ are uncountable Polish spaces, we can choose a Borel isomorphism $g : P \to (0, \infty)$. Let $f : \mathbb{R} \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \widehat{P} \\ \prod_{i=1}^{k} g(p_i)^{n_i} & \text{if } x = \sum_{i=1}^{k} n_i p_i \in \widehat{P},\ n_i \in \mathbb{Z},\ p_i \in P,\ i = 1, \dots, k. \end{cases}$$

This function is Borel, as it is Borel on the countably many Borel sets $P_{n_1,\dots,n_k}$, and zero on the rest. However, f is not continuous, as it is unbounded on the compact set P. Therefore f is not of the form $a + bx$. Suppose now that f is of the form $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g. Clearly $b \neq 0$, since f is not constant, therefore $\frac{f-a}{b} = e^g$ is Borel measurable, and then so is g by taking logarithm. But it is well-known that every Borel (or even Lebesgue) measurable additive function is of form kx ($k \in \mathbb{R}$), hence f is continuous, a contradiction.

What remains to show is that f is vertically rigid via horizontal translations. For every $c > 0$ there exists a $p \in P$ such that $g(p) = c$. Now we check that $cf(x) = f(x+p)$ for all $x \in \mathbb{R}$. Clearly $x \in \widehat{P}$ if and only if $x + p \in \widehat{P}$. Therefore $cf(x) = f(x+p) = 0$ if $x \notin \widehat{P}$. Let now $x = n_1p_1 + \dots + n_kp_k \in \widehat{P}$, and assume without loss of generality that $p = p_1$ ($n_1 = 0$ is also allowed). Then $cf(x) = g(p)f(x) = g(p)g(p)^{n_1}g(p_2)^{n_2} \cdots g(p_k)^{n_k} = g(p)^{n_1+1}g(p_2)^{n_2} \cdots g(p_k)^{n_k} = f((n_1+1)p + n_2p_2 + \dots + n_kp_k) = f(x+p)$, which finishes the proof. □

4 Lebesgue and Baire measurable functions

It is easy to see that the example in the previous section is zero almost everywhere (on a comeagre set). Indeed, it can be shown that every $P_{n_1,\dots,n_k}$ has uncountably many pairwise disjoint translates.

Therefore it is still possible that the complete analogue of Janković's conjecture holds: every vertically rigid Lebesgue (Baire) measurable function is of the form $a+bx$ or $a+be^{kx}$ *almost everywhere (on a comeagre set).* In this section we prove this in case of translations. The general case remains open, see Section 7. We also prove that in many situations the exceptional set can be removed.

Theorem 4.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function for an uncountable set $C \subset (0, \infty)$* via translations. *If f is Lebesgue (Baire) measurable then it is of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$ almost everywhere (on a comeagre set).*

Proof. By Theorem 2.4 we can assume that f is vertically rigid for C via horizontal translations. As in the proof of Theorem 2.5 we can also assume that $f(0) = 1$. Then Lemma 2.7 implies that

$$f(x+t) = f(x)f(t) \quad \forall x \in \mathbb{R}\ \forall t \in T \tag{4.1}$$

and $f(t) > 0$ for every $t \in T$.

First we show that the sign of f is constant almost everywhere (on a comeagre set). It is easy to see from (4.1) that the sets $\{f > 0\}$, $\{f = 0\}$, and $\{f < 0\}$ are all Lebesgue (Baire) measurable sets periodic modulo every $t \in T$. It is a well-known and easy consequence of the Lebesgue density theorem (the fact that every set with the Baire property is open modulo meagre) that if a measurable set H has a dense set of periods then either H or $\mathbb{R} \setminus H$ is of measure zero (meagre). But the above three sets cover $\mathbb{R}$, hence at least one of them is of positive measure (nonmeagre), and then that one is of full measure (comeagre). If $f = 0$ almost everywhere (on a comeagre set) then we are done, otherwise we may assume that $f > 0$ almost everywhere (on a comeagre set). (Indeed,

$-f$ is also rigid via horizontal translations, and then we can apply a horizontal translation and a positive multiplication to achieve $f(0) = 1$.)

Set $D = \{f > 0\}$ and define the measurable function $g = \log f$ on D. Recall that $D + t = D$ ($\forall t \in T$) and note that $T \subset D$. Clearly

$$g(x+t) = g(x) + g(t) \quad \forall x \in D \ \forall t \in T,$$

so $g|_T$ is additive. Now we show that $g|_T$ is of the form kx. Let us suppose that this is not the case. As we have mentioned above, if an additive function is not of the form kx then it is unbounded on every interval from above (and also below). For every Lebesgue (Baire) measurable function there is a measurable set of positive measure (nonmeagre) on which the function is bounded. So let $M \subset D$ be a measurable set of positive measure (nonmeagre) such that $|g|_M| \leq K$ for some $K \in \mathbb{R}$. By the Lebesgue density theorem (the fact that every Baire measurable set is open modulo meagre) there exists $\varepsilon > 0$ so that $(M+s) \cap M \neq \emptyset$ for every $s \in (-\varepsilon, \varepsilon)$. Choose $t_0 \in T$ in $(-\varepsilon, \varepsilon)$ so that $g(t_0) > 2K$. Fix an arbitrary $m_0 \in M \cap (M - t_0)$, then $g(m_0 + t_0) = g(m_0) + g(t_0) > g(m_0) + 2K$, which is absurd, since $m_0 + t_0, m_0 \in M$ and $|g|_M| \leq K$.

Now define $h(x) = g(x) - kx$ ($x \in D$). This is a measurable function that is periodic modulo every $t \in T$. Indeed,

$$h(x+t) = g(x+t) - k(x+t) = g(x) - kx + g(t) - kt = h(x) + 0 = h(x).$$

It is a well-known consequence of the Lebesgue density theorem (the fact that every Baire measurable set is open modulo meagre) that if the periods of a measurable function form a dense set then the function is constant almost everywhere (on a comeagre set). Hence $g(x) = kx + c$ almost everywhere (on a comeagre set), so $f(x) = e^c e^{kx}$ almost everywhere (on a comeagre set), so we are done. $\square$

Our next theorem shows that the measure zero (meagre) set can be removed, unless f is constant almost everywhere (on a comeagre set). Theorem 3.1 provides an almost everywhere (on a comeagre set) constant but nonconstant function that is vertically rigid via horizontal translations.

Theorem 4.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function that is of the form $a + bx$ ($b \neq 0$) or $a + be^{kx}$ ($bk \neq 0$) almost everywhere (on a comeagre set). Then f is of this form everywhere.*

Let us denote the 1-dimensional Hausdorff measure by $\mathcal{H}^1$. For the definition and properties see [2] or [4]. First we prove the following lemma.

Lemma 4.3 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be arbitrary functions, and let φ be an isometry such that $\varphi(graph(f)) = graph(g)$. Let $f', g' : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f' = f$ almost everywhere (on a comeagre set) and $g' = g$ almost everywhere (on a comeagre set). Let us also assume that $graph(f')$, $\varphi(graph(f'))$, $graph(g')$ and $\varphi^{-1}(graph(g'))$ are coverable by the graphs of countably many Lipschitz (continuity suffices for the category case) functions. Then $\varphi(graph(f')) = graph(g')$.*

Proof. By symmetry of f' and g' (with φ^{-1}), it suffices to show that $graph(g') \subset \varphi(graph(f'))$. Since the latter set is closed, it also suffices to show that $\varphi(graph(f'))$ covers a dense subset of $graph(g')$. We will actually show that $\varphi(graph(f'))$ covers $\mathcal{H}^1$ a.e. (relatively comeagre many) points of $graph(g')$, which will finish the proof.

If an element of $graph(g')$ fails to be covered by $\varphi(graph(f'))$ then it is either in $graph(g') \setminus graph(g)$ or in $\varphi(graph(f) \setminus graph(f')) \cap graph(g')$. The first set is clearly of $\mathcal{H}^1$ measure zero (relatively meagre in $graph(g')$), so it suffices to show that this is also true for the second. Equivalently, we need that $graph(f) \setminus graph(f')$ only covers a $\mathcal{H}^1$ measure zero (relatively meagre) subset of $\varphi^{-1}(graph(g'))$. Suppose that $\varphi^{-1}(graph(g')) \subset \bigcup_{n=1}^{\infty} graph(h_n)$, where the h_n's are Lipschitz (continuous) functions. As $graph(h_n) \cap (graph(f) \setminus graph(f'))$ is clearly of $\mathcal{H}^1$ measure zero for every n, we are done in the measure case.

Let us now write $\{x \in \mathbb{R} : f'(x) \neq f(x)\} = \bigcup_{m=1}^{\infty} N_m$, where each N_m is nowhere dense. It is enough to show that each $graph(f|_{N_m})$ only covers a relatively nowhere dense subset of $\varphi^{-1}(graph(g'))$. Fix an m, and suppose that $graph(f|_{N_m})$ is dense in an open subarc $U \subset \varphi^{-1}(graph(g'))$. By the Baire Category Theorem there exists a relatively open subarc $V \subset U$ that is covered by one of the $graph(h_n)$'s. But this is impossible, as the arc V is in $graph(h_n)$, and the set $N_m \subset \mathbb{R}$ is nowhere dense, so even $N_m \times \mathbb{R}$ covers at most a relatively nowhere dense subset of V, hence $graph(f|_{N_m})$ cannot be dense in V. $\square$

Proof. (Thm. 4.2) Using the notation of the above lemma, let first f be a vertically rigid function such that $f = f'$ almost everywhere (on a comeagre set), where f' is of the form $a + be^{kx}$ ($bk \neq 0$). The above lemma implies that f' is also vertically rigid with the same isometries φ_c. By considering the unique asymptote and the limit at $\pm\infty$ of f' we obtain that every φ_c is a translation. By Theorem 2.4 we may assume that every φ_c is actually horizontal, hence f' is of the form be^{kx}. Hence $cf'(x) = f'\left(x + \frac{\log(c)}{k}\right)$ for every $x \in \mathbb{R}$, $c > 0$ and the same holds for f. Assume now that there is an x_0 so that $f(x_0) \neq f'(x_0)$, then $cf(x_0) \neq cf'(x_0)$ for every $c > 0$, therefore $f\left(x_0 + \frac{\log(c)}{k}\right) \neq f'\left(x_0 + \frac{\log(c)}{k}\right)$ for every $c > 0$, which is a contradiction as $f = f'$ almost everywhere (on a comeagre set).

Assume now that f' is of the form $a + bx$ ($b \neq 0$). First we show that f' is vertically rigid by the same isometries as f. For every $c > 0$ set $g = cf$, $g' = cf'$, and let φ_c be the isometry mapping $graph(f)$ onto $graph(g)$. As $graph(f) \cap graph(f')$ contains at least two points and $\varphi_c(graph(f) \cap graph(f'))$ is the graph of a function we obtain that the line $\varphi_c(graph(f'))$ is not vertical, and similarly for $\varphi_c^{-1}(graph(g'))$. Therefore they are coverable by the graphs of countably many, actually a single, Lipschitz (continuous) function, hence the previous lemma applies. Hence f' is vertically rigid by the same isometries as f.

Similarly to Theorem 2.3 we can assume that f is vertically rigid via orientation preserving isometries for a set C of positive outer measure (nonmeagre). So φ_c is a rotation or translation for every $c \in C$, and by splitting C into two parts and keeping one with positive outer measure (nonmeagre), we can assume that $A = \{ang(\varphi_c) : c \in C\}$ is a subset of the left or the right half of the unit circle. We could calculate $ang(\varphi_c)$ explicitly, but we only need that it is a nonconstant real analytic function. From this it is easy to see that the set A is of positive outer measure (nonmeagre). Assume now that there is an x_0 so that $f(x_0) \neq f'(x_0)$. We prove that this contradicts the fact that $\varphi_c(graph(f))$ is the graph of a function for every $c \in C$. For this it suffices to show that S_f (see Theorem 2.3) is of full measure (comeagre). But this clearly follows simply by looking at the pairs (p_0, q) and (q, p_0) where $p_0 = (x_0, f(x_0))$ and q ranges over $graph(f) \cap graph(f')$. □

5 Rigid sets

The starting point is the proof of Theorem 2.3. So far we are only able to prove this result for continuous functions, and consequently we can only handle translations in the Borel/Lebesgue/Baire measurable case. But generalisations of the ideas concerning the sets S_f could tackle this difficulty. For a Borel function f the set S_f is analytic (see e.g. [3]), and every analytic set has the Baire property, so the result of this section can be considered as the first step towards handling Borel functions with general isometries. See Equation (2.1) for the following notations.

Definition 5.1 We call a symmetric (about the origin) set $H \subset S^1$ *rigid* for a set $C \subset (0, \infty)$ if for every $c \in C$ there is an α such that

$$H = \varrho_\alpha(\psi_c(H)). \tag{5.1}$$

Lemma 5.2 *Let U be a regular open set (i.e. $int(cl(U)) = U$) that is rigid for an uncountable set C. Then $U = \emptyset$, or $U = S^1$, or every connected component of U is an interval whose endpoints are among $(0, \pm 1)$ and $(\pm 1, 0)$.*

Proof. Let A be the set of arclengths of the connected components of U, then A is countable. Let I be a connected component of U showing that U is not of the desired form, then $0 < arclength(I) < \pi$ since U is symmetric and regular. As in the proof of Theorem 2.3 let us prove that the real analytic function $l(c) = arclength(\psi_c(I))$ ($c \in (0, \infty)$) is not constant. If I is in the left or right half of S^1 then we already showed this there, so we may assume that $(0, 1)$ or $(0, -1)$ is in I. Since $\lim_{c \to \infty} \psi_c(x) \in \{(0, \pm 1), (\pm 1, 0)\}$ for every $x \in S^1$, we obtain that $\lim_{c \to \infty} l(c) \in \mathbb{Z}\frac{\pi}{2}$. Hence we are done using $0 < arclength(I) < \pi$ unless $arclength(I) = \frac{\pi}{2}$. But if $arclength(I) = \frac{\pi}{2}$ then $\lim_{c \to \infty} l(c) = 0$ since $(0, 1)$ or $(0, -1)$ is in I, and therefore l cannot be constant.

Hence l attains each of its values at most countably many times, so there is a $c \in C$ such that $arclength(\psi_c(I)) \notin A$, contradicting (5.1). □

One can also show, using an argument similar to the above one (by considering the possible distances of pairs in H), that the rigid sets (for $C = (0, \infty)$)

of cardinality smaller than the continuum are the following: the empty set, the symmetric sets of two elements and the set $\{(0, \pm 1), (\pm 1, 0)\}$.

The next statement is somewhat of ergodic theoretic flavour.

Theorem 5.3 *Let H be a Baire measurable set that is rigid for an uncountable set C. Then in each of the four quarters of S^1 determined by $(0, \pm 1)$ and $(\pm 1, 0)$ either H or $S^1 \setminus H$ is meagre.*

Proof. H can be written as $H = U \Delta F$ in a unique way, where U is regular open, F is meagre and Δ stands for symmetric difference, see [5, 4.6]. Then it is easy to see by the uniqueness of U that U is rigid for C, so we are done by the previous lemma. $\square$

6 Horizontally rigid functions

In this section we characterise the functions that are horizontally rigid via translations. This answers Question 3 of [1] in the case of translations.

Definition 6.1 A function $f : \mathbb{R} \to \mathbb{R}$ is *horizontally rigid*, if $graph(f(cx))$ is isometric to $graph(f(x))$ for all $c \in (0, \infty)$.

Theorem 6.2 *A function $f : \mathbb{R} \to \mathbb{R}$ is horizontally rigid via translations if and only if there exists $r \in \mathbb{R}$ such that f is constant on $(-\infty, r)$ and (r, ∞).*

Proof. These functions are trivially horizontally rigid via translations. As the proof of the other direction resembles that of Theorem 2.4, we only sketch it.

For every $c > 0$ there exist $u_c, v_c \in \mathbb{R}$ such that $f(cx) = f(x + u_c) + v_c$ $(x \in \mathbb{R})$. We may assume $u_1 = v_1 = 0$. If $c \in (0, \infty) \setminus \{1\}$ then there is an $x_c \in \mathbb{R}$ such that $cx_c = x_c + u_c$, and substituting this back to the above equation we get $v_c = 0$. Hence $f(cx) = f(x + u_c)$ $(x \in \mathbb{R})$ for every $c \in (0, \infty)$.

First we show that if f has a period $p > 0$ then f is constant. Using the last equation twice we obtain

$$f(cx) = f(x+u_c) = f(x+u_c+p) = f\left((x+p)+u_c\right) = f\left(c(x+p)\right) = f(cx+cp).$$

If x ranges over $\mathbb{R}$ then so does cx, hence cp is also a period. If c ranges over $(0, \infty)$, then so does cp, hence every positive number is a period, so f is constant.

Using $f(cx) = f(x + u_c)$ again twice we obtain

$$f(c_1(c_2x)) = f(c_2x + u_{c_1}) = f\left(c_2\left(x + \frac{u_{c_1}}{c_2}\right)\right) = f\left(x + \frac{u_{c_1}}{c_2} + u_{c_2}\right).$$

Interchanging c_1 and c_2 and comparing the two equations we get

$$f\left(x + \frac{u_{c_1}}{c_2} + u_{c_2}\right) = f\left(x + \frac{u_{c_2}}{c_1} + u_{c_1}\right),$$

so $\pm\left[\left(\frac{u_{c_1}}{c_2}+u_{c_2}\right)-\left(\frac{u_{c_2}}{c_1}+u_{c_1}\right)\right]$ is a period, and hence it is zero. Therefore

$$\frac{u_{c_1}}{1-\frac{1}{c_1}}=\frac{u_{c_2}}{1-\frac{1}{c_2}} \text{ for every } c_1,c_2\in(0,\infty)\setminus\{1\}.$$

Set $r=\frac{u_c}{1-\frac{1}{c}}$, then $u_c=r\left(1-\frac{1}{c}\right)$ for every $c\in(0,\infty)$. Substituting this back to $f(cx)=f(x+u_c)$ gives $f(cx)=f\left(x+r\left(1-\frac{1}{c}\right)\right)$. Writing $\frac{x}{c}$ in place of x yields $f(x)=f\left(\frac{1}{c}(x-r)+r\right)$ for every $c\in(0,\infty)$.

Let $x_0<r$ be fixed and let c range over $(0,\infty)$, then $\frac{1}{c}(x_0-r)+r$ ranges over $(-\infty,r)$, so $f(x)$ is constant for $x<r$. Similarly, $f(x)$ is also constant for $x>r$. $\square$

7 Open questions

The most important open question is the following. By Theorem 4.1 the difficulty is to handle rotations.

Question 7.1 *Is every vertically rigid Lebesgue (Baire) measurable function of the form $a+bx$ or $a+be^{kx}$ $(a,b,k\in\mathbb{R})$ almost everywhere (on a comeagre set)? Or is this conclusion true at least for Borel measurable functions, or Baire class 1 functions, or functions with at least one point of continuity?*

Remark 7.2 It would be more natural to replace vertical rigidity by *almost* vertical rigidity. However, it is not clear how this should be defined, as a set can have a measure zero projection on one line and positive measure projection on another.

Question 7.3 *Let f be a vertically rigid function and $c>0$ such that there exists an isometry between graph(f) and graph(cf) that is not a translation (or also not a reflection). Is then f of the form $a+bx$? Or is this true for Borel, Lebesgue, or Baire measurable functions? And if we assume the same for* every *isometry between graph(f) and graph(cf)?*

Perhaps the following question can be answered in the negative by an easy transfinite recursion. A positive answer to the analytic (see e.g. [3] for the definition of analytic sets) version would answer Question 7.1 for Borel functions.

Question 7.4 *Let $I\subset S^1$ be the open subarc of arclength $\frac{\pi}{2}$ connecting $(0,1)$ and $(1,0)$. For a rigid set H can $H\cap I$ be anything else but $\emptyset$, a point, I minus a point, or I? How about analytic, Borel, or Lebesgue (Baire) measurable rigid sets?*

Question 7.5 *What is the role of the uncountable set $C\subset(0,\infty)$ in the results of this paper? When is it sufficient to assume that it is infinite, dense, sufficiently large finite, or contains a $c\neq 1$?*

Remark 7.6 Let $c_0 \neq 1$. It is easy to see that there exists a continuous f satisfying $c_0 f(x) = f(x+1)$ for every x. Indeed, if we define f to be an arbitrary continuous function on $[0,1]$ satisfying $c_0 f(0) = f(1)$ then f extends to $\mathbb{R}$ in a unique manner. Then f is vertically rigid via horizontal translations for the set $C = \{c_0^n : n \in \mathbb{Z}\}$. Hence it is not sufficient to assume for Janković's conjecture that C is infinite.

There also exists a continuous nonlinear function f whose graph consists of two half lines starting from the origin so that $graph(2f)$ is a rotated copy of $graph(f)$.

Question 7.7 *Is every horizontally rigid function of the form $a + bx$ or of the form described in Theorem 6.2? Or is this true if we assume Borel, Lebesgue, or Baire measurability? Is every continuous horizontally rigid function of the form $a + bx$?*

Question 7.8 *What can we say in higher dimensions?*

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References

[1] B. Cain, J. Clark, D. Rose, Vertically rigid functions, *Real Anal. Exchange* **31**, no. 2, (2005/2006), 515–518.

[2] K. J. Falconer, *The geometry of fractal sets.* Cambridge Tracts in Mathematics No. 85, Cambridge University Press, 1986.

[3] A. S. Kechris, *Classical Descriptive Set Theory.* Springer-Verlag, 1995.

[4] P. Mattila: *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge Studies in Advanced Mathematics No. 44, Cambridge University Press, 1995.

[5] J. C. Oxtoby: *Measure and Category. A survey of the analogies between topological and measure spaces.* Second edition. Graduate Texts in Mathematics No. 2, Springer-Verlag, 1980.

[6] J. von Neumann, Ein System algebraisch unabhängiger Zahlen, *Math. Ann.* **99**, (1928), 134-141.