Can we assign the Borel hulls in a monotone way?

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Abstract

A *hull* of $A \subset \mathbb{R}$ is a set H containing A such that $\lambda(H \cap I) = \lambda(A \cap I)$ for every Lebesgue measurable set I. We investigate all four versions of the following problem. Does there exist a monotone (wrt. inclusion) map that assigns a Borel/G_δ hull to every negligible/measurable subset of $\mathbb{R}$?

Three versions turn out to be independent of *ZFC* (the usual Zermelo-Fraenkel axioms with the Axiom of Choice), while in the fourth case we only prove that the nonexistence of a monotone G_δ hull operation for all measurable sets is consistent. It remains open whether existence here is also consistent. We also answer a question of Z. Gyenes and D. Pálvölgyi which asks if monotone hulls can be defined for every chain (wrt. inclusion) of measurable sets. We also comment on the problem of hulls of all subsets of $\mathbb{R}$.

1 Introduction

Notation 1.1 *Let us denote by $\mathcal{N}, \mathcal{L}, \mathcal{B}$ and $\mathcal{G}_\delta$ the class of Lebesgue negligible, Lebesgue measurable, Borel and G_δ subsets of $\mathbb{R}$, respectively. Let λ stand for Lebesgue (outer) measure.*

Definition 1.2 A set $H \subset \mathbb{R}$ is a *hull* of $A \subset \mathbb{R}$, if $A \subset H$ and $\lambda(H \cap I) = \lambda(A \cap I)$ for every Lebesgue measurable set I.

Clearly, every set has a Borel, even a G_δ hull.

Definition 1.3 Let $\mathcal{D}$ and $\mathcal{H}$ be two subclasses of $\mathcal{P}(\mathbb{R})$ (usually $\mathcal{D}$ is $\mathcal{N}$ or $\mathcal{L}$, and $\mathcal{H}$ is $\mathcal{B}$ or $\mathcal{G}_\delta$). If there exists a map $\varphi : \mathcal{D} \to \mathcal{H}$ such that

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1. $\varphi(D)$ is a hull of D for every $D \in \mathcal{D}$
2. $D \subset D'$ implies $\varphi(D) \subset \varphi(D')$

then we say that a monotone $\mathcal{H}$ hull operation on $\mathcal{D}$ exists.

The four questions we address in this paper are the following.

Question 1.4 *Let $\mathcal{D}$ be either $\mathcal{N}$ or $\mathcal{L}$, and let $\mathcal{H}$ be either $\mathcal{B}$ or $\mathcal{G}_\delta$. Does there exist a monotone $\mathcal{H}$ hull operation on $\mathcal{D}$?*

Remark 1.5

1. The problem was originally motivated by the following question of Z. Gyenes and D. Pálvölgyi [3]. Suppose that $\mathcal{C} \subset \mathcal{L}$ is a chain of sets, i.e. for every $C, C' \in \mathcal{C}$ either $C \subset C'$ or $C' \subset C$ holds. Does there exist a monotone $\mathcal{B}/G_\delta$ hull operation on $\mathcal{C}$?

2. Another motivation for our set of problems is that it seems to be very closely related to the theory of so called *liftings*. A map $l : \mathcal{L} \to \mathcal{L}$ is called a lifting if it preserves $\emptyset$, finite unions and complement, moreover, it is constant on the equivalence classes defined in Lemma 3.5 and also it maps each equivalence class to one of its members. Note that liftings are clearly monotone. For a survey of this theory see the chapter by Strauss, Macheras and Musiał in [5], or the chapter by Fremlin in [4]. Note that the existence of Borel liftings is known to be independent of *ZFC*, but the existence of a lifting with range in a fixed Borel class is not known to be consistent.

   We also remark that liftings are usually considered as $l^* : \mathcal{L}/\mathcal{N} \to \mathcal{L}$ or $l^* : \mathcal{P}(\mathbb{R})/\mathcal{N} \to \mathcal{L}$ maps.

3. In light of the theory of liftings it is natural to ask if a monotone Borel$/G_\delta$ hull operation on $\mathcal{P}(\mathbb{R})$ (i.e. all subsets of $\mathbb{R}$) can be defined. We will see in Section 3 that this is actually equivalent to the existence of a monotone Borel$/G_\delta$ hull operation on $\mathcal{L}$.

4. We remark here that throughout this paper $\mathbb{R}$ could be replaced by $\mathbb{R}^n$, or more generally, by an uncountable Polish space endowed with a nonzero continuous σ-finite Borel measure. (The arguments using the density topology can be got around using that for such measures there exists a measure preserving Borel isomorphism with a subinterval of $\mathbb{R}$ [6].)

The paper is organised as follows. First, in the next section we settle the independence of the existence of a monotone Borel$/G_\delta$ hull on $\mathcal{N}$. The consistency of the nonexistence immediately yields the consistency of the nonexistence of a monotone Borel$/G_\delta$ hull on $\mathcal{L}$. Then, in Section 3, we prove that under *CH* there is a monotone Borel hull on $\mathcal{L}$, and prove partial results concerning G_δ hulls. We conclude the paper by collecting the open questions in Section 4.

2 Monotone hulls for nullsets

Recall that $\mathrm{non}(\mathcal{N}) = \min\{|H| : H \subset \mathbb{R},\ H \notin \mathcal{N}\}$, where $|H|$ denotes cardinality. In the sequel the cardinal κ is identified with its initial ordinal, i.e. with the smallest ordinal of cardinality κ, and also every ordinal is identified with the set of smaller ordinals. For the standard set theory notation and techniques we use here see e.g. [8] and [1].

Theorem 2.1 *In a model obtained by adding ω_2 Cohen reals to a model satisfying the Continuum Hypothesis (CH) there is no monotone Borel hull operation on $\mathcal{N}$.*

Proof. We need two well-known facts. Firstly, $\mathrm{non}(\mathcal{N}) = \omega_2$ in this model [1]. Secondly, in this model there is no strictly increasing (wrt. inclusion) sequence of Borel sets of length ω_2 (this is proved in [7], see also [2]).

Assume that $\varphi : \mathcal{N} \to \mathcal{B}$ is a monotone hull operation. Choose $H = \{x_\alpha : \alpha < \mathrm{non}(\mathcal{N})\} \notin \mathcal{N}$, and consider $\varphi(\{x_\beta : \beta < \alpha\})$ for $\alpha < \mathrm{non}(\mathcal{N})$. This is an increasing ω_2 long sequence of Borel sets, which cannot stabilise, since then H would be contained in a nullset. But then we can select a strictly increasing subsequence of length ω_2, a contradiction. $\square$

The following is immediate.

Corollary 2.2 *Under the same assumption there exists no monotone G_δ hull operation on $\mathcal{N}$.*

Remark 2.3 We will see in Remark 3.14 that the length ω_2 is optimal in the sense that all shorter chains have monotone G_δ hulls.

Recall that $\mathrm{add}(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{N},\ \bigcup \mathcal{F} \notin \mathcal{N}\}$ and $\mathrm{cof}(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{N},\ \forall N \in \mathcal{N}\ \exists F \in \mathcal{F}$ such that $N \subset F\}$, and also that $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$ is consistent [1] (note that e.g. *CH* implies this equality).

Theorem 2.4 *Assume* $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$. *Then there exists a monotone G_δ hull operation on $\mathcal{N}$.*

Proof. Let $\{N_\alpha : \alpha < \mathrm{cof}(\mathcal{N})\}$ be a cofinal family in $\mathcal{N}$, that is, $\forall N \in \mathcal{N}\ \exists \alpha < \mathrm{cof}(\mathcal{N})$ such that $N \subset N_\alpha$. For every $\alpha < \mathrm{cof}(\mathcal{N})$ define, using transfinite recursion, $A_\alpha =$ a G_δ hull of $(\cup_{\beta<\alpha} A_\beta \cup N_\alpha)$. Clearly, $\{A_\alpha : \alpha < \mathrm{cof}(\mathcal{N})\}$ is a cofinal increasing sequence of G_δ sets. Now, for every $N \in \mathcal{N}$ define $\varphi(N) = A_{\alpha_N}$, where α_N is the minimal index so that $H \subset A_{\alpha_N}$. It is easy to see that $\varphi : \mathcal{N} \to \mathcal{G}_\delta$ is a monotone hull operation. $\square$

The following is again immediate.

Corollary 2.5 *Under the same assumption there exists a monotone Borel hull operation on $\mathcal{N}$.*

3 Monotone hulls for all sets

First we note (Statement 3.2 below) that the title of this section is justified, as there is no difference between working with measurable sets or arbitrary sets.

We need a well-known lemma first. Recall that the *density topology* consists of those measurable sets that have Lebesgue density 1 at each of their points (see e.g. [9]). Closure in this topology is denoted by $\overline{H}^d$.

Lemma 3.1 *$\overline{H}^d$ is a hull of H for every $H \subset \mathbb{R}$.*

Proof. Assume that $\lambda(H \cap I) < \lambda(\overline{H}^d \cap I)$ for some $I \in \mathcal{L}$. As $\overline{H}^d \in \mathcal{L}$, this implies that there exists $L \in \mathcal{L}$ with $\lambda(L) > 0$ such that $L \subset \overline{H}^d \setminus H$. Set $L_0 = \{x \in L : x \text{ is a density point of } L\}$. By the Lebesgue Density Theorem $L \setminus L_0 \in \mathcal{N}$, which easily implies that $L_0 \neq \emptyset$ is open in the density topology. But $L_0 \subset \overline{H}^d$ is disjoint from H, a contradiction. $\square$

Statement 3.2 *The existence of a monotone Borel/G_δ hull operation on $\mathcal{P}(\mathbb{R})$ is equivalent to the existence of a monotone Borel/G_δ hull operation on $\mathcal{L}$.*

Proof. On the one hand, the restriction to $\mathcal{L}$ of a monotone hull operation on $\mathcal{P}(\mathbb{R})$ is itself a monotone hull operation.

On the other hand, by the previous lemma there exists a monotone hull operation $\psi : \mathcal{P}(\mathbb{R}) \to \mathcal{L}$. Hence if φ is a monotone hull operation on $\mathcal{L}$ then $\varphi \circ \psi$ is a monotone hull operation on $\mathcal{P}(\mathbb{R})$. $\square$

Theorem 2.1 immediately implies the following.

Corollary 3.3 *In a model obtained by adding ω_2 Cohen reals to a model satisfying CH there is no monotone Borel or G_δ hull operation on $\mathcal{L}$.*

Now we turn to the positive results.

Theorem 3.4 *Assume CH. Then there is a monotone Borel hull operation on $\mathcal{L}$.*

Before we prove this theorem we need a few lemmas. In case $\mathcal{H} = \mathcal{B}$ the first one is a special case of a well-known result about Borel liftings, but there are no such results in case of $\mathcal{G}_\delta$.

Let us denote by $A \Delta B$ the symmetric difference of A and B.

Lemma 3.5 *(CH) There exists a monotone map $\psi : \mathcal{L} \to \mathcal{G}_\delta$ so that $\lambda(M \Delta \psi(M)) = 0$ for every $M \in \mathcal{L}$ and so that $\lambda(M \Delta M') = 0$ implies $\psi(M) = \psi(M')$ for every $M, M' \in \mathcal{L}$.*

Proof. Let us say that $M, M' \in \mathcal{L}$ are *equivalent*, if $\lambda(M \Delta M') = 0$. Denote by $[M]$ the equivalence class of M and by $\mathcal{L}/\mathcal{N}$ the set of classes. We say that $[M_1] \leq [M_2]$ if there are $M_1' \in [M_1]$ and $M_2' \in [M_2]$ such that $M_1' \subset M_2'$.

It is sufficient to define $\Psi : \mathcal{L}/\mathcal{N} \to \mathcal{G}_\delta$ so that $[M] \le [M']$ implies $\Psi([M]) \subset \Psi([M'])$ for every $M, M' \in \mathcal{L}$, and so that $\Psi([M]) \in [M]$ for every $M \in \mathcal{L}$.

Enumerate $\mathcal{L}/\mathcal{N}$ as $\{[M_\alpha] : \alpha < \omega_1\}$. For every $\alpha < \omega_1$ define

$$\Psi([M_\alpha]) = \bigcap_{\substack{\beta<\alpha \\ [M_\beta]\ge[M_\alpha]}} \Psi([M_\beta]) \ \cap \ \Big(\text{a } G_\delta \text{ hull of } \bigcup_{\substack{\gamma<\alpha \\ [M_\gamma]\le[M_\alpha]}} \Psi([M_\gamma]) \cup M_\alpha\Big).$$

It is not hard to check that this is a G_δ set so that $[M_\gamma] \le [M_\alpha] \le [M_\beta]$ implies $\Psi([M_\gamma]) \subset \Psi([M_\alpha]) \subset \Psi([M_\beta])$, and so that $\Psi([M_\alpha]) \in [M_\alpha]$, hence the construction works. $\square$

Remark 3.6 1. Actually we will not use the fact that ψ is constant on the equivalence classes.

2. We do not know if *CH* is needed in this lemma.

The following lemma is the only result we can prove for $\mathcal{B}$ but not for $\mathcal{G}_\delta$.

Lemma 3.7 *(CH) There exists a monotone hull operation $\varphi : \mathcal{N} \to \mathcal{B}$ so that*

1. *$\varphi(N \cup N') \subset \varphi(N) \cup \varphi(N')$ for every $N, N' \in \mathcal{N}$ (subadditivity),*
2. *$\bigcup\{\varphi(N) : N \subset B, N \in \mathcal{N}\} \setminus B \in \mathcal{N}$ for every $B \in \mathcal{B}$.*

Proof. Let $\{A_\alpha : \alpha < \omega_1\}$ and α_N be as in Theorem 2.4 (note that $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \omega_1$ under *CH*). Set $A^*_\alpha = A_\alpha \setminus \cup_{\beta<\alpha} A_\beta$. Enumerate $\mathcal{B}$ as $\{B_\alpha : \alpha < \omega_1\}$ and for every $\alpha < \omega_1$ define the countable set

$$\mathcal{B}_\alpha = \{\cup_{i=0}^n B_{\beta_i} : n \in \mathbb{N},\ \beta_i < \alpha\ (0 \le i \le n)\}.$$

Note that every $\mathcal{B}_\alpha$ is closed under finite unions.

Now define

$$\varphi(N) = \bigcup_{\alpha \le \alpha_N} \Big(A^*_\alpha \cap \bigcap_{\substack{B\in\mathcal{B}_\alpha \\ N\cap A^*_\alpha \subset B}} B\Big).$$

This is clearly a disjoint union. It is easy to see that φ is a monotone Borel hull operation (note that $\varphi(N) \subset A_{\alpha_N}$).

For every $\alpha < \omega_1$ define $\varphi_\alpha(N) = A^*_\alpha \cap \varphi(N)$ $(N \in \mathcal{N})$. In order to check subadditivity, let $N, N' \in \mathcal{N}$. We may assume $\alpha_N \le \alpha_{N'}$, so clearly $\alpha_{N \cup N'} = \alpha_{N'}$. It suffices to check that each φ_α is subadditive. If $\alpha > \alpha_N$ then actually $\varphi_\alpha(N \cup N') = \varphi_\alpha(N')$, so we are done. Suppose now $\alpha \le \alpha_N$. Let $x \in A^*_\alpha$ so that $x \notin \varphi(N) \cup \varphi(N')$. Then there exist $B \supset N \cap A^*_\alpha$ and $B' \supset N' \cap A^*_\alpha$ in $\mathcal{B}_\alpha$ so that $x \notin B, B'$. But then $B \cup B' \in \mathcal{B}_\alpha$ witnesses that $x \notin \varphi(N \cup N')$ since $x \notin B \cup B' \supset (N \cup N') \cap A^*_\alpha$.

Finally, to prove *2* it is sufficient to show that $N \subset B_\alpha$ implies $\varphi(N) \setminus B_\alpha \subset A_\alpha$ for every $N \in \mathcal{N}$ and $\alpha < \omega_1$. So let $x \in \varphi_\beta(N)$ for some $\beta > \alpha$. We have to show $x \in B_\alpha$. But this simply follows from the definition of φ since $B_\alpha \in \mathcal{B}_\beta$. $\square$

Lemma 3.8 *Let $\mathcal{H}$ be either $\mathcal{B}$ or $\mathcal{G}_\delta$. Assume that there exists a monotone map $\psi : \mathcal{L} \to \mathcal{H}$ so that $\lambda(M \Delta \psi(M)) = 0$ for every $M \in \mathcal{L}$ and also that there exists a monotone hull operation $\varphi : \mathcal{N} \to \mathcal{H}$ so that*

1. *$\varphi(N \cup N') \subset \varphi(N) \cup \varphi(N')$ for every $N, N' \in \mathcal{N}$,*
2. *$\bigcup\{\varphi(N) : N \subset H, N \in \mathcal{N}\} \setminus H \in \mathcal{N}$ for every $H \in \mathcal{H}$.*

Then φ can be extended to a monotone hull operation $\varphi^* : \mathcal{L} \to \mathcal{H}$.

Proof. We may assume that $\psi(N) = \emptyset$ for every $N \in \mathcal{N}$ (by redefining ψ on $\mathcal{N}$ to be constant $\emptyset$, if necessary).

Define

$$\varphi^*(M) = \psi\big(M\big) \cup \varphi\big(M \setminus \psi(M)\big) \cup \varphi\Big(\bigcup_{\substack{N \subset \psi(M) \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M)\Big).$$

Clearly $\varphi^*(M) \in \mathcal{H}$. As the union of first two terms contains M, we obtain $M \subset \varphi^*(M)$. Moreover, $\varphi^*(M)$ is a hull of M, since the first term is equivalent to M and the last two terms are nullsets. It is also easy to see that φ^* extends φ.

We still have to check monotonicity of φ^*. First we prove

$$N' \in \mathcal{N},\ M' \in \mathcal{L},\ N' \subset \psi(M') \Rightarrow \varphi(N') \subset \varphi^*(M'). \tag{1}$$

Indeed, the case $N' = \emptyset$ is trivial to check, otherwise

$$\varphi(N') \subset \bigcup_{\substack{N \subset \psi(M') \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \subset \Big(\bigcup_{\substack{N \subset \psi(M') \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M')\Big) \cup \psi(M') \subset$$
$$\subset \varphi\Big(\bigcup_{\substack{N \subset \psi(M') \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M')\Big) \cup \psi(M') \subset \varphi^*(M'),$$

which proves (1).

Let now $M \subset M'$ be arbitrary elements of $\mathcal{L}$. We need to show that all three terms of $\varphi^*(M)$ are in $\varphi^*(M')$.

Firstly, $\psi(M) \subset \psi(M')$.

Secondly, define $N' = \big(M \setminus \psi(M)\big) \cap \psi(M')$. Using the subadditivity of φ and then (1) we obtain

$$\varphi\big(M \setminus \psi(M)\big) \subset \varphi\Big(\big(M \setminus \psi(M)\big) \cap \psi(M')\Big) \cup \varphi\Big(\big(M \setminus \psi(M)\big) \setminus \psi(M')\Big) \subset$$
$$\subset \varphi\big(N'\big) \cup \varphi\big(M' \setminus \psi(M')\big) \subset \varphi^*(M').$$

Thirdly, let

$$N' = \Big(\bigcup_{\substack{N \subset \psi(M) \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M)\Big) \cap \psi(M').$$

Using the subadditivity of φ and then (1) we obtain

$$\varphi\Big(\bigcup_{\substack{N \subset \psi(M) \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M) \Big) \subset$$

$$\subset \varphi\Big(\big(\bigcup_{\substack{N \subset \psi(M) \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M) \big) \cap \psi(M') \Big) \cup \varphi\Big(\big(\bigcup_{\substack{N \subset \psi(M) \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M) \big) \setminus \psi(M') \Big) \subset$$

$$\subset \varphi(N') \cup \varphi\Big(\bigcup_{\substack{N \subset \psi(M') \\ \emptyset \neq N \in \mathcal{N}}} \varphi(N) \setminus \psi(M') \Big) \subset \varphi^*(M').$$

This concludes the proof. □

Now we prove Theorem 3.4.

Proof. Lemma 3.5 and Lemma 3.7 show that in case of $\mathcal{H} = \mathcal{B}$ the requirements of Lemma 3.8 can be satisfied, so the proof of Theorem 3.4 is complete. □

Remark 3.9

1. We remark that subadditive monotone maps are actually additive.
2. The proof actually gives a monotone $F_{\sigma\delta\sigma}$ hull. However, we do not know whether a monotone G_δ hull operation on $\mathcal{L}$ exists (Question 4.5). Of course, in light of the previous theorem, under *CH*, this is equivalent to assigning G_δ hulls only to the Borel (or $F_{\sigma\delta\sigma}$) sets in a monotone way.

Question 3.10 *Is there a monotone G_δ hull operation on $\mathcal{B}$? Or on $F_{\sigma\delta\sigma}$? Or on any other fixed Borel class e.g. $\mathcal{F}_\sigma$? (Of course $\mathcal{G}_\delta$ and the simpler ones are not interesting.)*

Our next goal is to prove Theorem 3.11, the partial result we have concerning monotone G_δ hull operations on $\mathcal{L}$. It shows that it is not possible to prove in *ZFC* the nonexistence of G_δ hulls on $\mathcal{L}$ along the lines of Theorem 2.1, that is, only by considering long chains of sets.

Theorem 3.11 *Assume that there exists a monotone G_δ hull operation ψ on $\mathcal{N}$ (which follows e.g. from $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$). Let $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ be a chain of sets, that is, for every $C, C' \in \mathcal{C}$ either $C \subset C'$ or $C' \subset C$ holds. Then there exists a monotone G_δ hull operation on $\mathcal{C}$.*

Proof. By Lemma 3.1 we may assume that $\mathcal{C} \subset \mathcal{L}$.

We may also assume that $C \subset [0,1]$ for every $C \in \mathcal{C}$, since it is sufficient to construct the hulls separately in every $[n, n+1]$. Partition $\mathcal{C}$ into the intervals $\mathcal{I}_r = \{C \in \mathcal{C} : \lambda(C) = r\}$. Let $R = \{r \in [0,1] : \mathcal{I}_r \neq \emptyset\}$, and fix an element $C_r \in \mathcal{I}_r$ for every $r \in R$. Well-order R as $\{r_\alpha : \alpha < |R|\}$, and set $R_\alpha = \{r_\beta : \beta < \alpha\}$.

Now we define $\varphi(C_{r_\alpha})$ by transfinite recursion as follows. Fix two countable sets $R_\alpha^- \subset \{r \in R_\alpha : r < r_\alpha\}$ and $R_\alpha^+ \subset \{r \in R_\alpha : r > r_\alpha\}$ so that $\forall r \in R_\alpha$,

$r < r_\alpha$ $\exists r' \in R_\alpha^-$ such that $r \le r' < r_\alpha$, and similarly, $\forall r \in R_\alpha$, $r > r_\alpha$ $\exists r' \in R_\alpha^+$ such that $r_\alpha < r' \le r$. (Note that R_α^- and R_α^+ may be singletons or even empty.) Set

$$\varphi(C_{r_\alpha}) = \Big[\text{a } G_\delta \text{ hull of } \Big(C_{r_\alpha} \cup \cup_{r \in R_\alpha^-} \varphi(C_r)\Big)\Big] \cap \cap_{r \in R_\alpha^+} \varphi(C_r).$$

It is easy to see that this is a monotone G_δ hull operation on $\{C_r : r \in R\}$.

We may assume that for the hull operation ψ we have $\psi(\emptyset) = \emptyset$. Then we can define a monotone G_δ hull operation φ_t on $\mathcal{I}_t$ for each $t \in R$ as follows. Let

$$\varphi_t(C) = \varphi(C_t) \cup \psi(C \setminus C_t) \quad (C \in \mathcal{I}_t).$$

For each $t \in R$ fix a countable set $R_t^{++} \subset \{r \in R : r > t\}$ so that $\forall\, r \in R$, $r > t$ $\exists r' \in R_t^{++}$ such that $t < r' \le r$. Set

$$\varphi(C) = \varphi_t(C) \cap \cap_{r \in R_t^{++}} \varphi(C_r)$$

for every $C \in \mathcal{I}_t$ and every $t \in R$. This is a proper definition since for $C = C_t$ this is just an equality. It is easy to check that $\varphi(C)$ is a G_δ hull of C and that φ is monotone. $\square$

Finally, we prove in *ZFC* that rather long well-ordered chains have monotone G_δ hulls.

Lemma 3.12 *Let $\xi \le \mathrm{add}(\mathcal{N})$ and $\mathcal{C} = \{M_\alpha : \alpha < \xi\} \subset \mathcal{P}(\mathbb{R})$ be such that $M_\alpha \subset M_\beta$ for every $\alpha \le \beta < \xi$. Then there exists a monotone G_δ hull operation on $\mathcal{C}$.*

Proof. By Lemma 3.1 we may assume that $\mathcal{C} \subset \mathcal{L}$.

By transfinite recursion define A_α to be a G_δ hull of the set $M_\alpha \cup \cup_{\beta<\alpha}(A_\beta \setminus M_\alpha)$. Clearly every $A_\beta \setminus M_\alpha$ is a nullset, moreover there are $|\alpha| < \mathrm{add}(\mathcal{N})$ many of them, hence A_α is a hull of M_α, too. $\square$

Recall that κ^+ is the successor cardinal of κ and also that every $\xi < \kappa^+$ has a cofinal (i.e. unbounded) subset of order type at most κ.

Theorem 3.13 *Let $\eta < \mathrm{add}(\mathcal{N})^+$ and $\mathcal{C} = \{M_\alpha : \alpha < \eta\} \subset \mathcal{P}(\mathbb{R})$ be such that $M_\alpha \subset M_\beta$ for every $\alpha \le \beta < \eta$. Then there exists a monotone G_δ hull operation on $\mathcal{C}$.*

Proof. By Lemma 3.1 we may assume that $\mathcal{C} \subset \mathcal{L}$.

We prove this lemma by induction on η. Fix a cofinal subset $X \subset \eta$ of order type $\xi \le \mathrm{add}(\mathcal{N})$ and also a monotone G_δ hull operation $\varphi_X : \{M_\alpha : \alpha \in X\} \to \mathcal{G}_\delta$ by the previous lemma. Every complementary interval $[\beta, \gamma)$ of X (i.e. every interval that is maximal disjoint from X) is of order type $< \eta$, hence by the inductive assumption there exists a monotone G_δ hull operation $\varphi_{[\beta,\gamma)} : \{M_\alpha : \alpha \in [\beta,\gamma)\} \to \mathcal{G}_\delta$. Also fix a measure zero G_δ hull $H_{[\beta,\gamma)}$ of $\cup_{\delta<\beta,\, \delta\in X}\big(\varphi_X(M_\delta) \setminus M_\beta\big)$. Now for every $[\beta,\gamma)$ and every $\alpha \in [\beta,\gamma)$ define

$$\varphi(M_\alpha) = \big(H_{[\beta,\gamma)} \cup \varphi_{[\beta,\gamma)}(M_\alpha)\big) \cap \varphi_X(M_\gamma),$$

and also define $\varphi(M_\alpha) = \varphi_X(M_\alpha)$ for every $\alpha \in X$. It is then easy to see that this is a monotone G_δ hull operation on $\mathcal{C}$. $\square$

Remark 3.14 As $\mathrm{add}(\mathcal{N}) \geq \omega_1$, we obtain that length ω_2 of the chain in the proof of Theorem 2.1 was optimal.

4 Concluding remarks and open problems

Now we pose a few somewhat vague problems, some of which may turn out to be very easy.

Question 4.1 *It would be interesting to know what happens*

1. *if we look at the category analogue of Question 1.4, that is, when $\mathcal{N}$ and $\mathcal{L}$ are replaced by the first Baire category (=meager) sets and the sets with the property of Baire.*
2. *if we require that our monotone hulls be translation or isometry invariant.*
3. *if we replace $\subset$ by $\subsetneqq$ in Question 1.4, that is, we require that strict containment is preserved.*

As for $\subsetneqq$-preserving hulls, let us note that the case of $\mathcal{L}$ is easy.

Statement 4.2 *There is no $\subsetneqq$-preserving monotone Borel hull on $\mathcal{L}$.*

Proof. Let $C \subset \mathbb{R}$ be the Cantor set and let $B \subset C$ be a Bernstein subset [9], that is, a set such that $B \cap F \neq \emptyset$ and $B \cap (C \setminus F) \neq \emptyset$ for every uncountable closed set $F \subset C$. Then $C \setminus A$ is countable for every Borel set A containing B, as uncountable Borel sets contain uncountable closed sets [6].

Clearly, $C \setminus B$ is uncountable, so let $\{x_\alpha : \alpha < \omega_1\}$ be distinct points of this set, then the strictly increasing chain $C_\alpha = (\mathbb{R} \setminus C) \cup B \cup \{x_\beta : \beta < \alpha\}$ cannot have a strictly monotone Borel hull φ, as already $\varphi(C_0)$ is of countable complement in $\mathbb{R}$. $\square$

But we do not know the answer to the case of $\mathcal{N}$.

We now repeat the open questions of the paper for the sake of completeness.

Question 4.3 *Is there (in ZFC) a monotone map $\psi : \mathcal{L} \to \mathcal{G}_\delta$ so that $\lambda(M \Delta \psi(M)) = 0$ for every $M \in \mathcal{L}$? If yes, is there one such that $\lambda(M \Delta M') = 0$ implies $\psi(M) = \psi(M')$ for every $M, M' \in \mathcal{L}$?*

Question 4.4 *Is there a monotone G_δ hull operation on $\mathcal{B}$? Or on $F_{\sigma\delta\sigma}$? Or on any other fixed Borel class e.g. $\mathcal{F}_\sigma$? (Of course $\mathcal{G}_\delta$ and the simpler ones are not interesting.)*

Let us conclude with the most important open question.

Question 4.5 *Is it possible to assign G_δ hulls to all (measurable) subsets of $\mathbb{R}$ in a monotone way?*

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