Level Sets of Differentiable Functions of Two Variables with Non-vanishing Gradient

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Abstract

We show that if the gradient of $f : \mathbb{R}^2 \to \mathbb{R}$ exists everywhere and is nowhere zero, then in a neighbourhood of each of its points the level set $\{x \in \mathbb{R}^2 : f(x) = c\}$ is homeomorphic either to an open interval or to the union of finitely many open segments passing through a point. The second case holds only at the points of a discrete set. We also investigate the global structure of the level sets.

Introduction

The Inverse Function Theorem is usually proved under the assumption that the mapping is continuously differentiable. In [RR] S. Radulescu and M. Radulescu generalized this theorem to mappings that are only differentiable, namely they proved that if $f : D \to \mathbb{R}^n$ is differentiable on an open set $D \subset \mathbb{R}^n$ and the derivative $f'(x)$ is non-singular for every $x \in D$, then f is a local diffeomorphism.

It is therefore natural to ask whether the Implicit Function Theorem, which is usually derived from the Inverse Function Theorem, can also be proved under these more general assumptions. (In addition, this question is also related to the Gradient Problem of C. Weil, see [Qu], and is motivated by [EKP] as well, where such a function of two variables with non-vanishing gradient is used as a tool to solve a problem of K. Ciesielski.) In [Bu] Z. Buczolich gave a negative answer to this question by constructing a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ of non-vanishing gradient such that $\{x \in \mathbb{R}^2 : f(x) = 0\} = \{(x, y) : y = x^2 \text{ or } y = 0 \text{ or } y = -x^2\}$. Indeed, this example shows that the level set is not homeomorphic to an open interval in any neighbourhood of the origin.

The goal of our paper is to show that such a level set cannot be 'much worse' than that. The main result is a kind of Implicit Function Theorem (Theorem 1.2), stating that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function of non-vanishing gradient, then in a neighbourhood of each of its points a level set is homeomorphic either to an open interval or to the union of finitely many open

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segments passing through a point. We also show, that the set of points at which the second case holds has no point of accumulation.

In order to prove this local result we have to investigate the global structure of the level set as well. In Section 2 we apply the theory of plane continua to prove that the level sets (extended by ∞) consist of Jordan curves (Corollary 2.8). From this result we show that the level set consists of arcs which have tangents at each of their points, only finitely many arcs can meet at a point and the set of points where these arcs meet has no point of accumulation. Finally, from all these the above 'Implicit Function Theorem' will follow.

In addition, we show that the notion 'arc with tangents' cannot be replaced by the more natural notion of 'differentiable curve with non-zero derivative'.

Preliminaries

The usual compactification of the plane by a single point called ∞ is denoted by $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. Throughout the paper topological notions such as closure, boundary or component, unless particularly stated, always refer to $\mathbb{R}^2$. The notations clA, $intA$ and ∂A stand for the closure, interior and boundary of a set A, respectively. The angle of the two vectors in the plane is denoted by $ang(x,y)$. The abbreviations $\{f = c\}$, $\{f < c\}$ etc. stand for $\{x \in \mathbb{R}^2 : f(x) = c\}$, $\{x \in \mathbb{R}^2 : f(x) < c\}$, etc., respectively. $B(x,\varepsilon)$ is the open disc $\{y \in \mathbb{R}^2 : |y-x| < \varepsilon\}$. The circle of center x and radius ε, is denoted by $S(x,\varepsilon)$. By an *arc* or a *Jordan curve* we mean a continuous and injective function to the plane (or to $\mathbb{S}^2$) defined on a closed interval or on a circle, respectively. (We often do not distinguish between the image of the function and the function itself.) The *contingent* of a set $H \subset \mathbb{R}^2$ at a point $x \in H$ is the union of those half-lines L that can be written as $L = lim\ L_n$, where L_n is a half-line starting from x and passing through $x_n \in H$, and x_n is converging to x ($x_n \neq x$). (By $L = lim\ L_n$ we mean that the direction of the half-lines converges.) We say that $H \subset \mathbb{R}^2$ has a *tangent* (half-tangent) at $x \in H$ if the contingent of H at x is a line (half-line). A *continuum* is a compact connected set.

1 The results

Definition 1.1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function of non-vanishing gradient and let $c \in \mathbb{R}$ be arbitrary. Then $x \in \mathbb{R}^2$ is called a *branching point* of $\{f = c\}$ if it is in the closure of at least three different components of $\{f \neq c\}$. We call $H \subset \mathbb{R}^2$ a *nice curve* if $H = \gamma \setminus \{\infty\}$ where

(i) γ is either an arc in $\mathbb{R}^2$ between two branching points or an arc in $\mathbb{S}^2$ between a branching point and ∞ or a Jordan curve in $\mathbb{S}^2$ containing ∞,

(ii) if $x \in \gamma$ and x is not an endpoint, then γ has a tangent at x, and if $x \in \gamma \setminus \{\infty\}$ and x is an endpoint, then γ has a half-tangent at x.

Theorem 1.2 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function of non-vanishing gradient and let $c \in \mathbb{R}$ be arbitrary. Then the set of branching points has no point of accumulation, and the level set $\{f = c\}$ is the disjoint union (except from the endpoints) of nice curves. Moreover, if $x \in \mathbb{R}^2$ is not a branching point, then there exists a neighbourhood U of x such that $\{f = c\} \cap U$ is homeomorphic to an open interval, while if $x \in \mathbb{R}^2$ is a branching point, then there exists a neighbourhood U of x such that $\{f = c\} \cap U$ is homeomorphic to the union of finitely many open segments passing through a point.*

In the rest of the paper we prove this theorem. As we have already mentioned in the Introduction, first we examine certain global properties of the level sets in the next section, and then apply these results in the last section to obtain Theorem 1.2.

Throughout the proof we assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function of non-vanishing gradient, D is a component of $\{f \neq c\}$ and C is a component of ∂D. We can clearly assume that $c = 0$.

2 The level set consists of Jordan curves

We start with a lemma that we shall frequently use in the sequel.

Lemma 2.1 *Let $a < b$ and $c < d$ be real numbers and $F \subset [a, b] \times [c, d]$ be a closed set that has infinitely many points on each vertical line that meets the rectangle. Then there is a point in F at which the contingent of F is not contained in a line.*

Proof Suppose, on the contrary, that at every point of F the contingent of F is contained in a line. Our assumption is that for every $a \leq x_0 \leq b$ the set $\{(x, y) \in F : x = x_0\}$ is infinite. Therefore we can choose a point of accumulation of this set for every $a \leq x_0 \leq b$, and thus we obtain a function $g : [a, b] \to [c, d]$. As F is closed, $(x_0, g(x_0)) \in F$ and because of the way the point was chosen, the contingent of F at this point must be contained in the vertical line. Hence for every $a \leq x_0 \leq b$

$$lim_{x \to x_0} \left| \frac{g(x) - g(x_0)}{x - x_0} \right| = +\infty. \tag{1}$$

But this is impossible, as [Sa, IX. 4. 4] states that for any function of a real variable the set of points at which (1) holds is of measure zero. $\square$

The proof of the next lemma is a straightforward calculation, so we omit it.

Lemma 2.2 *For every $x \in \{f = 0\}$ the contingent of $\{f = 0\}$ at x is contained in the line perpendicular to $f'(x)$.*

An easy consequence is the following (see Figure 1).

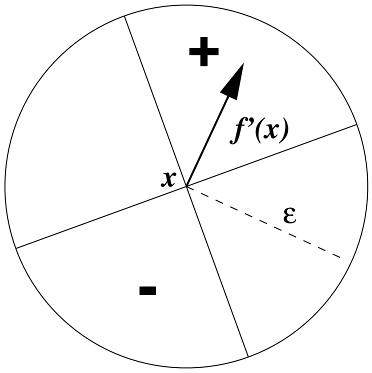

Figure 1:

Lemma 2.3 *For every $x \in \{f = 0\}$ there exists an $\varepsilon > 0$ such that if $y \neq x$ and $y \in B(x, \varepsilon)$, then $ang(f'(x), y - x) \leq \frac{\pi}{4}$ implies that $f(y) > 0$, while $ang(\text{-}f'(x), y - x) \leq \frac{\pi}{4}$ implies that $f(y) < 0$.*

The next lemma is the basic tool in the proof of the fact that $C \cup \{\infty\}$ is a Jordan curve (Corollary 2.8, the main result of the present section).

Lemma 2.4

(i) $\{f = 0\}$ is locally connected.

(ii) ∂D has no bounded component.

(iii) $\partial D \cup \{\infty\}$ is connected.

(iv) Both ∂D and $\partial D \cup \{\infty\}$ are locally connected.

(v) Both C and $C \cup \{\infty\}$ are connected and locally connected.

Proof

(i) If $\{f = 0\}$ is not locally connected at $x \in \{f = 0\}$, then for some $\varepsilon > 0$ we can find a sequence K_n $(n \in \mathbb{N})$ of distinct components of $\{f = 0\} \cap clB(x, \varepsilon)$ converging (with respect to the Hausdorff metric) to a continuum K such that $x \in K$ and $K_n \cap K = \emptyset$ for every $n \in \mathbb{N}$ ([Wh, I. 12. 1] asserts that if the compact set $M \subset \mathbb{R}^2$ is not locally connected at a point m, then for some $\varepsilon > 0$ there exists a sequence M_n of components of $M \cap clB(x, \varepsilon)$ converging to a continuum N such that $m \in N$ and $M_n \cap N = \emptyset$ for every $n \in \mathbb{N}$). We claim that $K_n \cap S(x, \varepsilon) \neq \emptyset$ for every $n \in \mathbb{N}$. Indeed, if this is the case, then there exists a Jordan curve inside $B(x, \varepsilon)$ that encloses K_n and that is disjoint from $\{f = 0\}$ ([Wh, VI. 3. 11] states that if M is a component of a compact set $N \subset \mathbb{R}^2$, then there exists a Jordan curve in the ε-neighbourhood of M that encloses M and is disjoint from N). But then the sign of f is constant on the curve and thus f attains a local extremum inside the curve, which contradicts the assumption that the gradient nowhere vanishes.

Let us now divide $S(x, \varepsilon)$ into three sub-arcs of equal length. At least one of these pieces must intersect infinitely many of the sets K_n $(n \in \mathbb{N})$. Let us call this subsequence K_{n_i} $(i \in \mathbb{N})$. This chosen sub-arc can be separated from x by a narrow rectangle (see Figure 2).

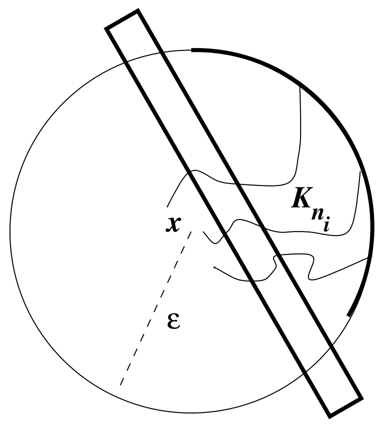

Figure 2:

As $K_{n_i} \to K$, $x \in K$, $K_{n_i} \cap S(x, \varepsilon) \neq \emptyset$ and K_{n_i} is connected $(i \in \mathbb{N})$, for all $i \in \mathbb{N}$ large enough, K_{n_i} must 'cross' the rectangle from the sub-arc to a point close to x. As the sets K_{n_i} $(i \in \mathbb{N})$ are disjoint, we obtain infinitely many different points of intersection on the lines considered in Lemma 2.1, so we can apply this lemma (to a rotated copy of our rectangle) with $F = \{f = 0\}$. But then we get that the contingent of $\{f = 0\}$ is not contained in a line, which contradicts Lemma 2.2.

(ii) Suppose that a component C of ∂D is contained in $B((0,0), R)$. Applying [Wh, VI. 3. 11] (see (i)) to $\partial D \cap clB((0,0), R)$ we can again find a Jordan curve γ around the component C that is disjoint from ∂D. So γ is either contained in D or disjoint from D. In the first case the sign of f is constant on the curve, so f attains a local extremum, which is impossible. In the second case D must be inside γ, as D is connected and at least one point of it is inside γ, since there are points of ∂D inside γ. But no component of $\{f = 0\}$ can be bounded, since it would result in a local extremum of f as f vanishes on the boundary of the components.

(iii) Easily follows from (ii).

(iv) If $x \in \mathbb{R}^2$, then we can repeat the argument of (i). The only difference is that we prove $K_n \cap S(x, \varepsilon) \neq \emptyset$ $(n \in \mathbb{N})$ by applying (ii).

In the case of ∞ we argue as follows. Let $\pi : \mathbb{S}^2 \setminus \{s\} \to \mathbb{R}^2$ be the usual stereo-graphic projection from a point $s \in \mathbb{S}^2$ which is not in $\partial D \cup \{\infty\}$. As this is a local homeomorphism in a neighbourhood of ∞, it is sufficient to prove that $\pi(\partial D \cup \{\infty\})$ is locally connected at the point $x = \pi(\infty) \in \mathbb{R}^2$. Suppose, on the contrary, that this is not true. It is enough to find a point $y \neq x$ at which the contingent of $\pi(\partial D \cup \{\infty\})$ is not contained in a line, since π is a local diffeomorphism at $p = \pi^{-1}(y)$, and so in this case the contingent of $\partial D \cup \{\infty\}$ at p is also not contained in a line, which contradicts Lemma 2.2. But we can again obtain this by the argument of (i), once we show that $K_n \cap S(x, \varepsilon) \neq \emptyset$ $(n \in \mathbb{N})$. So let $K_n \subset B(x, \varepsilon)$. As $x \notin K_n$, its inverse image $\pi^{-1}(K_n)$ is bounded, and moreover $K_n \cap S(x, \varepsilon) \neq \emptyset$, therefore $\pi^{-1}(K_n)$ is a component of $\partial D \cup \{\infty\}$. (Note that K_n was originally a component of $\pi(\partial D \cup \{\infty\}) \cap clB(x, \varepsilon)$.) But this is impossible by (ii).

(v) C is clearly connected and by (ii) unbounded. Thus $C \cup \{\infty\}$ is connected as well. What remains to prove is that these sets are locally connected. If $x \in \mathbb{R}^2$, then we can repeat the argument of (i). The only difference is that $K_n \cap S(x, \varepsilon) \neq \emptyset$ simply follows from the connectedness of C. For the case of ∞ we repeat the proof of (iv). We again simply use the connectedness of C to verify that $K_n \cap S(x, \varepsilon) \neq \emptyset$.

□

Corollary 2.5 *Every $x \in \partial D \cup \{\infty\}$ can be accessed from any point of D by an arc in $D \cup \{x\}$.*

Proof [Ku, §61, II. 11] states that if E is a component of the complement of a locally connected and closed set F, then every point of ∂E can be accessed from E by a connected set. A remark at the end of the proof adds that it can also be accessed by an arc.

Suppose first that $x \neq \infty$. Let us apply this theorem to $E = D$ and $F = \partial D$. Indeed, D is a component of of the complement of ∂D, which is closed and by (iv) of Lemma 2.4 locally connected. Thus the statement follows.

To see that the case $x = \infty$ is similar, note first that $\partial D \cup \{\infty\}$ is also locally connected by (iv) of Lemma 2.4. Choose a point $s \in \partial D$ and let $\pi : \mathbb{S}^2 \setminus \{s\} \to \mathbb{R}^2$ be the usual stereo-graphic projection from the point s. We can now apply the above result to the open set $\pi(D) \subset \mathbb{R}^2$ and its boundary $\partial\pi(D) = \pi(\partial D \cup \{\infty\} \setminus \{s\})$, since this latter set is clearly locally connected. Therefore we obtain an arc $\gamma \subset \pi(D)$ accessing $\pi(\infty)$, and then $\pi^{-1}(\gamma)$ is the required arc. □

Statement 2.6 *For every $x \in \partial D$ there exists a Jordan curve in $\partial D \cup \{\infty\}$ containing both x and ∞.*

Before the proof of this statement we need a technical lemma. It is surely well known, but we could not find it in the literature, so we include a proof here.

Lemma 2.7 *Let γ_1 and γ_2 be arcs between a point $x \in \mathbb{R}^2$ and $\{\infty\}$ such that they are disjoint except from the endpoints, and let G be one of the two components of the complement of the Jordan curve formed by the two arcs. In addition, let φ be a Jordan curve in $\mathbb{R}^2$ which contains x in its interior. Then there exists a sub-arc of φ that joins γ_1 and γ_2 such that $\varphi \subset G$ except from the endpoints.*

Proof $\varphi : [0,1] \to \mathbb{R}^2 \setminus B(x,\varepsilon)$ for some $\varepsilon > 0$, and we may assume that $\varphi(0) = \varphi(1)$ is on one of the arcs. Let $c_0 = 0$ and let us construct inductively a sequence of triples in the following way (see Figure 3).

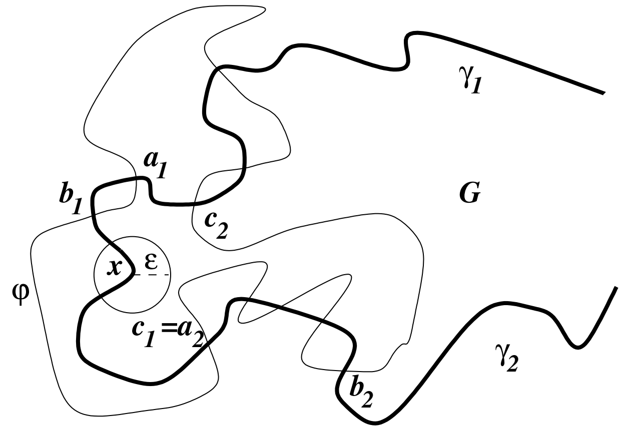

Figure 3:

For every $n = 1, 2, \ldots$ put $a_n = c_{n-1}$ and define c_n as the smallest number (greater than a_n) for which $\varphi(c_n)$ is already on the other arc than $\varphi(a_n)$. Finally, let b_n be the largest number (smaller than c_n) for which $\varphi(b_n)$ is still on the same arc as $\varphi(a_n)$. We claim that for some $n \in \mathbb{N}$ $c_n = 1$. Otherwise, a_n $(n \in \mathbb{N})$ is a strictly increasing sequence converging to a number $a \in [0,1]$. So the sequence $[b_n, c_n]$ $(n \in \mathbb{N})$ of disjoint intervals also converges to a, thus $\varphi([b_n, c_n])$ converges to the point $\varphi(a)$. But these images are arcs between γ_1 and γ_2, and they are disjoint from $B(x,\varepsilon)$, hence they cannot converge to a point.

If we replace the sub-arcs of φ between $\varphi(a_i)$ and $\varphi(b_i)$ $(i = 1, \ldots, n)$ by the corresponding sub-arcs of γ_1 or γ_2, we obtain a continuous closed curve ψ of the same rotation number around x as φ, but the rotation number around x of φ is 1 (or -1), thus ψ must intersect G. But then for the corresponding $[b_i, c_i]$, $\varphi|_{[b_i,c_i]}$ is the required sub-arc. $\square$

Now we prove Statement 2.6.

Proof By Corollary 2.5 there exists an arc γ_1 from x to ∞ in $D \cup \{x, \infty\}$. As f has no local extremum, there must be another component E of $\{f \neq 0\}$ (of different sign than D) such that $x \in \partial E$. Let γ_2 be a similar arc in $E \cup \{x, \infty\}$ from x to ∞.

Since $\partial D \cup \{\infty\} \subset \mathbb{S}^2$ is closed and locally connected, it is locally arc-wise connected (this is [Ku, §50, II. 1]). Moreover, it is connected by (iii) of Lemma 2.4, hence arc-wise connected. Therefore there exists an arc ψ_1 in $\partial D \cup \{\infty\}$ from x to ∞. γ_1 and γ_2 joined together is a Jordan curve in $\mathbb{S}^2$, thus it splits its complement into two components, and ψ_1 must be contained in one of them (except from the endpoints). Let us denote the other component by G. It is sufficient to show that there exists an arc ψ_2 between x and ∞ in $(G \cap \partial D) \cup \{x, \infty\}$. First we check that this set is locally connected. We only have to consider x and ∞. But at these points the locally connected set $\partial D \cup \{\infty\}$ is cut into two pieces by a Jordan curve that intersects the set in no additional point (in a neighbourhood), hence the set must be locally connected on 'both sides' of the Jordan curve. So $(G \cap \partial D) \cup \{x, \infty\}$ is locally connected. Now we show that x and ∞ are in the same component of $(G \cap \partial D) \cup \{x, \infty\}$. Otherwise, the component of x must be bounded, thus it can be enclosed by a Jordan curve φ that is disjoint from $(G \cap \partial D) \cup \{x, \infty\}$. But then by the previous lemma we obtain an arc from γ_1 (which is in D) to γ_2 (which is not in D) such that this arc is in G except from its endpoints. This is a contradiction.

The common component of x and ∞ in $(G \cap \partial D) \cup \{x, \infty\}$ is locally connected, as it is a component of a locally connected set. By the same argument as at the beginning of the previous paragraph we obtain that it is arc-wise connected. Hence we can construct an arc ψ_2 in $(G \cap \partial D) \cup \{x, \infty\}$ between x and ∞, and then ψ_1 and ψ_2 together form the required Jordan curve. □

Now we are able prove the main result of this section.

Corollary 2.8 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function of non-vanishing gradient, D be a component of $\{f \neq 0\}$ and C be a component of ∂D. Then $C \cup \{\infty\}$ is a Jordan curve.*

Proof We apply [Ku, §52, VI. 1], which asserts that if a locally connected continuum (of at least two points) contains no θ-curve and no separating point, then it is a Jordan curve. (A θ-curve is the union of three arcs between two points such that the arcs are disjoint except from the endpoints. A point is a separating point if its complement is disconnected.)

By (ii) of Lemma 2.4 we know that $C \cup \{\infty\}$ is a locally connected continuum. First we check that it contains no θ-curve. The complement (in $\mathbb{S}^2$) of a θ-curve consists of three Jordan domains, and D must be contained in one of them. But the boundary of this domain is the union of two arcs of the θ-curve, therefore the third arc cannot be in ∂D, which is a contradiction. We still have to check that $C \cup \{\infty\}$ contains no separating point, which easily follows from Statement 2.6. □

3 The rest of the proof of Theorem 1.2

Statement 3.1 *For every $x \in \mathbb{R}^2$ and $R > 0$ there are only finitely many components of $\{f \neq 0\}$ intersecting the disc $B(x, R)$.*

Proof Suppose that the converse is true. As every component is unbounded (otherwise we could find a local extremum), there exists an arc $\gamma_D \subset D$ for every component D such that γ_D joins $S(x, R)$ and $S(x, 2R)$. We may assume that $\gamma_D \subset clB(x, 2R)$. Indeed, denote by x_D the first point along the arc γ_D that is on $S(x, 2R)$ and cut off the rest of γ_D. Our aim is to get a contradiction by a similar argument as in the proof of Lemma 2.4. So choose a small sub-arc of $S(x, 2R)$ (e.g. one tenth of the circle) that contains infinitely many of the points x_D, and separate this sub-arc from $S(x, R)$ by a narrow rectangle as in Figure 4.

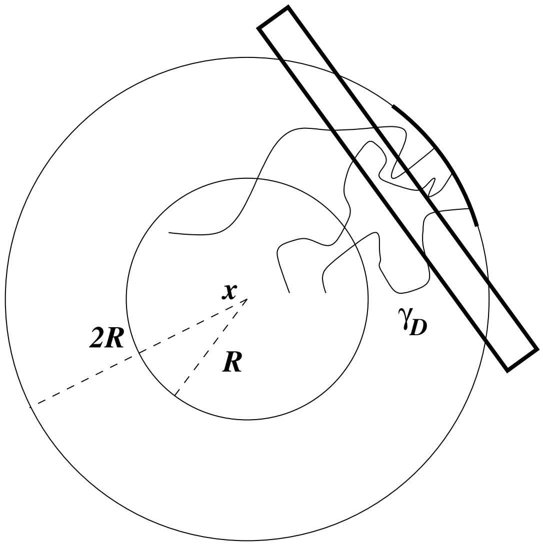

Figure 4:

Let us consider a segment in our rectangle as in Lemma 2.1. It is intersected by all of the disjoint arcs γ_D. Hence we obtain infinitely many points on our segment that are in distinct components of $\{f \neq 0\}$, therefore there must also be infinitely many points of the set $\{f = 0\}$ on the segment. Thus Lemma 2.1 can be applied, and we get a contradiction by Lemma 2.2. $\square$

Corollary 3.2 *The set of branching points has no point of accumulation in $\mathbb{R}^2$.*

Proof By the previous statement there are only finitely many components of $\{f \neq 0\}$ intersecting $B(x, 1)$. Therefore the proof is complete once we show that for two distinct components D_1 and D_2 there are at most two branching points in $\partial D_1 \cap \partial D_2$. Suppose, on the contrary, that $x_1, x_2, x_3 \in \mathbb{R}^2$ are three such points, and let $d_1 \in D_1,\ d_2 \in D_2$ be arbitrary. By Lemma 2.5 we can join x_1, x_2 and x_3 by six disjoint arcs (except from the endpoints) to d_1 and d_2, and so we obtain three disjoint arcs (except from the endpoints) from d_1 to d_2 through the points x_1, x_2 and x_3. Let us denote them by γ_1, γ_2 and γ_3. One of these arc, say γ_2, must be surrounded by the Jordan curve formed by the other two arcs. But then x_2 cannot be a branching point, since if it were on the boundary of a third component of $\{f \neq 0\}$, then this component would be inside the Jordan curve, hence bounded, which is impossible. $\square$

We need one more lemma to prove the main result, our version of the Implicit Function Theorem (Statement 3.4).

Lemma 3.3 *Let D_1 and D_2 be two components of $\{f \neq 0\}$. Then $\partial D_1 \cap \partial D_2 = C_1 \cap C_2$, where C_1 and C_2 are components of ∂D_1 and ∂D_2, respectively. Moreover, this intersection is a sub-arc (possibly empty) of both of the curves C_1 and C_2. An endpoint of such a sub-arc is either a branching point or ∞, but the other points of the sub-arc are not branching points.*

Proof In order to justify the first equality it is sufficient to show that ∂D_1 cannot intersect two components of ∂D_2. Suppose, on the contrary, that C_2 and C_2' are two such components. By Corollary 2.8 they are Jordan curves in $\mathbb{S}^2$ (apart from ∞), hence they split $\mathbb{R}^2$ into three domains G_1, G_2 and G_3 such that the boundaries of these domains are C_2, C_2' and $C_2 \cup C_2'$, respectively. Consequently, $D_1 \subset G_3$ and $D_2 \subset G_3$ must hold. As ∂D_1 intersects both C_2 and C_2', by Lemma 2.5 there exists an arc γ in D_2 (apart from the endpoints) which joins C_2 and C_2'. But γ splits G_3 into two parts and D_1 must be contained in one of them, which contradicts e.g. $C_2 \subset \partial D_1$.

To prove that the intersection is a sub-arc of e.g. C_1, let us denote by γ_{D_1} and γ_{D_2} two Jordan curves in $\mathbb{S}^2$ such that $\gamma_{D_1} \setminus \{\infty\} = C_1$ and $\gamma_{D_2} \setminus \{\infty\} = C_2$. It is sufficient to show that there are no three points x, y and z on $\gamma_{D_1} \setminus \{\infty\}$ in this order such that $x, z \in \gamma_{D_2}$ and $y \notin \gamma_{D_2}$. Suppose, on the contrary, that there are three such points. On the sub-arc of $\gamma_{D_1} \setminus \{\infty\}$ from y to x there exists a first point x' in $\gamma_{D_1} \cap \gamma_{D_2}$. Similarly, the first point of this intersection on $\gamma_{D_1} \setminus \{\infty\}$ in the other direction is denoted by z'. But these two points, x' and z' are connected by a sub-arc of $\gamma_{D_1} \setminus \{\infty\}$ and a sub-arc of $\gamma_{D_2} \setminus \{\infty\}$, which form a Jordan curve in $\{f = 0\}$, which results in a local extremum of f, a contradiction.

Now we have to check that if an endpoint of a sub-arc is not ∞, then it is a branching point. If $x \in \mathbb{R}^2$ is such an endpoint, then x and ∞ splits γ_{D_1} and γ_{D_2} into sub-arcs between x and ∞ such that three of these arcs are disjoint except from their endpoints. But these three arcs divide the plane into three domains, therefore at least one of them is disjoint from both D_1 and D_2. As x is on the boundary of all three domains, it must meet the closure of a component different from D_1 and D_2. Thus x is a branching point.

We now show that (except for the endpoints) no point of the sub-arc can be a branching point. Suppose, on the contrary, that there exists such a point y. By Lemma 2.5 we can join two other points of the sub-arc to two points $d_1 \in D_1$ and $d_2 \in D_2$ such that the obtained Jordan curve contains y in its interior. As y is a branching point, a third component D_3 must intersect the interior of this Jordan curve, moreover, it cannot intersect the curve itself, therefore it must be enclosed by the Jordan curve. But then D_3 is bounded, and f attains a local extremum inside it, which is impossible. □

Now we can prove the most important statement of Theorem 1.2.

Statement 3.4 *If $x \in \mathbb{R}^2$ is not a branching point, then there exists a neighbourhood U of x such that $\{f = 0\} \cap U$ is homeomorphic to an open interval, while if $x \in \mathbb{R}^2$ is a branching point, then there exists a neighbourhood U of x such that $\{f = 0\} \cap U$ is homeomorphic to the union of finitely many open segments passing through a point.*

Proof Let $x \in \mathbb{R}^2$ be arbitrary. By Statement 3.1 there are only finitely many components of $\{f \neq 0\}$ intersecting $B(x, 1)$, therefore for some $\varepsilon > 0$ the point x is on the boundary of every component that intersects $B(x, \varepsilon)$. We can also assume by Corollary 3.2 that $B(x, \varepsilon)$ contains no branching point, with the possible exception of x itself.

Let us now first suppose that x is not a branching point, that is only two components of $\{f \neq 0\}$ intersect $B(x, \varepsilon)$. As f has no local extremum, every point of $\{f = 0\}$ is on the boundary of both components. Thus by Lemma 3.3 we obtain that $\{f = 0\} \cap B(x, \varepsilon)$ is the intersection of an arc with $B(x, \varepsilon)$ such that the endpoints of the arc are outside the disc. If we now choose an open neighbourhood U of x inside $B(x, \varepsilon)$ by (i) of Lemma 2.4 such that $U \cap \{f = 0\}$ is connected, then this intersection must be homeomorphic to an open interval.

Let us now consider the case when x is a branching point. As f has no local extrema, every point in $\{f = 0\} \cap B(x, \varepsilon)$ is on the common boundary of at least two components of $f \neq 0$. Since x is the only branching point in the disc, we obtain that $\{f = 0\} \cap B(x, \varepsilon)$ is the intersection of $B(x, \varepsilon)$ and the disjoint union (except from the point x) of finitely many arcs starting from x and running to branching points outside the disc (indeed, we apply Lemma 3.3 to every pair of components intersecting the disc). If we now choose an open neighbourhood U of x inside $B(x, \varepsilon)$ such that $U \cap \{f = 0\}$ is connected, then this intersection must be homeomorphic to a half-open interval on each of the above arcs starting from x. So the only thing that remains to show is that there is an even number of these arcs, which easily follows from the fact that the components of $\{f \neq 0\}$ surrounding x must be of alternating signs. □

Our next goal is to complete the proof of Theorem 1.2.

Definition 3.5 Let D be a component of $\{f \neq 0\}$, C be a component of ∂D and γ be a Jordan curve such that $\gamma = C \cup \{\infty\}$. Then γ is separated into sub-arcs by the branching points and ∞. We call these sub-arcs (together with their endpoints) *edges*.

Statement 3.6 *The edges are disjoint except from their endpoints. If φ is an edge, then $\varphi \setminus \{\infty\}$ is a nice curve (see Definition 1.1).*

Proof If two edges correspond to the same component of $\{f \neq 0\}$, then they are clearly disjoint except from their endpoints. Let now $\varphi_1 \subset \gamma_1 \subset \partial D_1$ and $\varphi_2 \subset \gamma_2 \subset \partial D_2$ be two edges corresponding to the distinct components D_1 and D_2 such that they have a point x in common which is not an endpoint of at least one of them. As an edge is an arc between branching points and ∞, no point inside an edge can be a branching point, thus x is not a branching point. By Lemma 3.3 $(\gamma_1 \setminus \{\infty\}) \cap (\gamma_2 \setminus \{\infty\})$ is a sub-arc of γ_1 such that its endpoints are branching points or ∞ and the other points of the sub-arc are not branching points, therefore this sub-arc must agree with $\varphi_1 \setminus \{\infty\}$. Similarly $(\gamma_1 \setminus \{\infty\}) \cap (\gamma_2 \setminus \{\infty\})$ must agree with $\varphi_2 \setminus \{\infty\}$, therefore the two edges coincide.

To show that $\varphi \setminus \{\infty\}$ is a nice curve note that it clearly satisfies (i) of Definition 1.1 (by Corollary 2.8). To see that (ii) is also satisfied, let $x \in \varphi \setminus \{\infty\}$. By Lemma 2.2 the contingent of φ at x is at most the line perpendicular to $f'(x)$.

Suppose first that x is an endpoint. An easy compactness argument shows that the contingent of φ at x cannot be empty, thus it is sufficient to show that it does not contain both possible half-lines. If we apply Lemma 2.3 to x we obtain two opposite sectors of $B(x, \varepsilon)$ containing the directions of the two possible half-lines, and moreover $\varphi \cap B(x, \varepsilon)$ must also be contained in these two sectors. But φ is an arc which starts from x and never returns there, from which easily follows that it cannot approach x arbitrarily close inside both sectors.

Suppose next that x is not an endpoint of φ, thus it is not a branching point. We have to show that the contingent contains both possible half-lines. By Statement 3.4 we can find a neighbourhood U of x such that $\{f = 0\} \cap U$ is a sub-arc of φ. Then Lemma 2.3 provides a small disc $B(x, \varepsilon)$ inside U which consists of four sectors, two opposite sectors S_1 and S_3 containing the directions of the two possible half-lines and two other sectors S_2 and S_4, in which f is positive and negative, respectively. Because of the different signs, these latter sectors cannot be connected by an arc in $\{f \neq 0\}$, therefore for every $\delta < \varepsilon$ there must be points of $\{f = 0\}$ on both semi-circles of center x and of radius δ running from S_2 to S_4 and crossing through S_1 or S_3. Consequently, the contingent contains both half-lines by an easy compactness argument. $\square$

Therefore the proof of Theorem 1.2 is complete.

Remark As we have already mentioned in the Introduction, it is not true that the level sets consist of differentiable arcs with non-zero derivative:

We can argue as follows. For every $n \in \mathbb{N}$ let γ_n be a smooth curve in the (closed) region bounded by $y = 0,\ y = x^2,\ S(0, \frac{1}{2^{3n}})$ and $S(0, \frac{1}{2^{3n+2}})$ such that its endpoints are $(0, \frac{1}{2^{3n+1}})$ and $(0, \frac{1}{2^{3n}})$, it meets $S(0, \frac{1}{2^{3n+2}})$ and such that if we continue γ_n by two horizontal segments to the left and to right, then we get a smooth curve (see Figure 5). Let $H \subset \mathbb{R}^2$ be the union of the curves γ_n $(n \in \mathbb{N})$, the horizontal segments connecting them and the two half-lines $\{(x, y) : \ x \leq 0,\ y = 0\}$ and $\{(x, y) : \ x \geq 1,\ y = 0\}$. One can show that

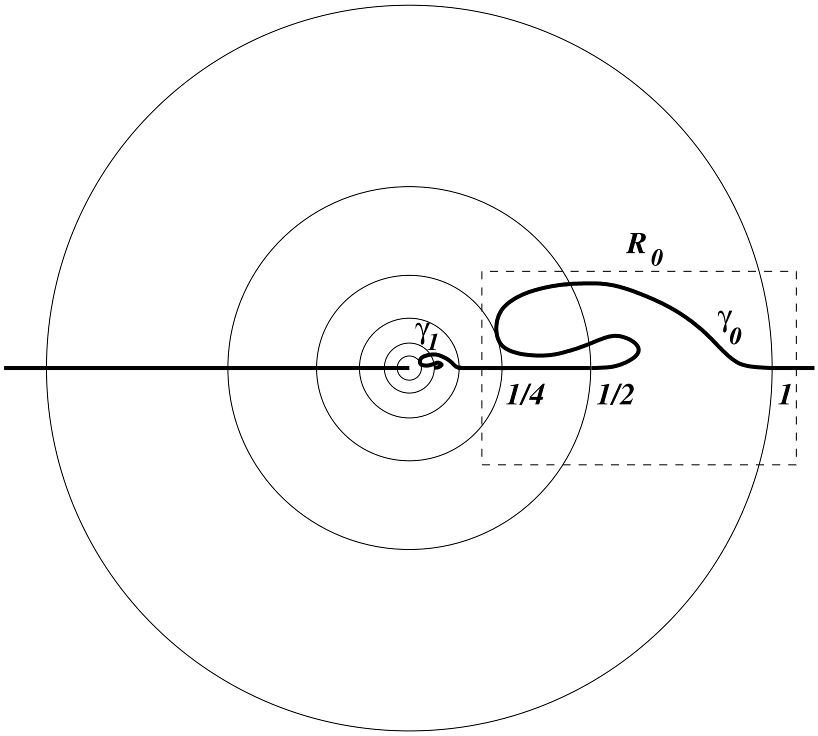

Figure 5:

the Jordan curve $H \cup \{\infty\}$ is the level set $\{f = 0\}$ of a suitable differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ of non-vanishing gradient, but it is not a differentiable curve of non-zero derivative. We only sketch the proofs here.

To construct the function f let us fix for every $n \in \mathbb{N}$ a rectangle R_n (as in Figure 5) containing γ_n in its interior, such that R_n is between the parabolas $y = 2x^2$ and $y = \text{-}2x^2$. It is not hard to see that there exists a diffeomorphism $\Phi^{(n)} : R_n \to R_n$ for which $\Phi^{(n)}(x, y) \in \{y = 0\}$ iff $(x, y) \in \gamma \cap R_n$ (this latter set is the union of γ_n and the two horizontal segments). We can also clearly assume that $\Phi^{(n)}$ coincides with the identity function close to the edges of R_n. Now define

$$f(x,y) = \begin{cases} \Phi_2^{(n)}(x,y) & \text{if } (x,y) \in R_n, \\ y & \text{otherwise,} \end{cases}$$

where $\Phi_2^{(n)}$ is the second coordinate function of $\Phi^{(n)} = (\Phi_1^{(n)}, \Phi_2^{(n)})$. It is easy to check that $f'(0,0)$ exists and is equal to $(0,1)$, and one can also see that the function f satisfies all the other requirements.

To see that the Jordan curve $H \cup \{\infty\}$ cannot have a non-zero derivative at the origin, we parametrize it by a function φ such that $\varphi(0) = (0,0)$, and consider

$$\left|\frac{\varphi(t)-\varphi(0)}{t-0}\right| = \left|\frac{\varphi(t)}{t}\right|.$$

Note that there exists a sequence $t_n \to 0$ ($t_n > 0$) such that $\varphi(t_n) = (0, \frac{1}{2^{3n+1}})$ and an other sequence $t'_n \to 0$ ($t'_n > 0$) for which we have $\varphi(t'_n) \in S(0, \frac{1}{2^{3n+2}})$, while $t'_n \geq t_n$. This shows that the right hand side derivative of φ at 0 cannot be a finite, non-zero value.

Finally we pose the following problem.

Problem 3.7 *Characterize the level sets* $\{f = c\}$ *of differentiable functions* $f : \mathbb{R}^2 \to \mathbb{R}$ *of non-vanishing gradient.*

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