Hausdorff measures of different dimensions are isomorphic under the Continuum Hypothesis

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June 22, 2004

Abstract

We show that the Continuum Hypothesis implies that for every $0 < d_1 \leq d_2 < n$ the measure spaces $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_1}}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_2}}, \mathcal{H}^{d_2}\right)$ are isomorphic, where $\mathcal{H}^d$ is d-dimensional Hausdorff measure and $\mathcal{M}_{\mathcal{H}^d}$ is the σ-algebra of measurable sets with respect to $\mathcal{H}^d$. This is motivated by the well-known question (circulated by D. Preiss) whether such an isomorphism exists if we replace measurable sets by Borel sets.

We also investigate the related question whether every continuous function (or the typical continuous function) is Hölder continuous (or is of bounded variation) on a set of positive Hausdorff dimension.

Introduction

The following problem is circulated by D. Preiss [Pr] (while it is unclear, who actually asked this first, see also [Cs]). Let $\mathcal{H}^d$ denote d-dimensional Hausdorff measure, see e.g. [Fa], [Fe] or [Ma], and let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}^n$.

Question 0.1 *Let $0 < d_1 < d_2 < n$. Are the measure spaces $\left(\mathbb{R}^n, \mathcal{B}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{B}, \mathcal{H}^{d_2}\right)$ isomorphic?*

An equally natural question is whether such an isomorphism exists if we replace Borel sets by measurable sets with respect to Hausdorff measures. The main result of the first part of this paper is an affirmative answer to this question assuming CH; that is, under the Continuum Hypothesis. Question 0.1 remains open.

Denote by $\mathcal{M}_{\mathcal{H}^d}$ the σ-algebra of measurable sets with respect to $\mathcal{H}^d$, in the usual sense of Carathéodory. (Again, and throughout the paper we follow standard terminology that can be found e.g. in [Fa], [Fe] or [Ma]). So using this notation our theorem will read as follows.

*Partially supported by Hungarian Scientific Foundation grant no. 37758 and F 43620.
MSC codes: Primary 28A78; Secondary 26A16, 26A45, 28A05, 03E50
Key Words: Hausdorff measure, isomorphism, CH, Hausdorff dimension, Hölder continuous, bounded variation

Theorem 0.2 *Under the Continuum Hypothesis, for every $0 < d_1 \leq d_2 < n$ the measure spaces $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_1}}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_2}}, \mathcal{H}^{d_2}\right)$ are isomorphic.*

By *isomorphism* of two measure spaces we mean a bijection f such that both f and f^{-1} are measurable set and measure preserving. We remark here that by measure preserving we could mean two different things, as we can either require that the measure of every measurable set is preserved, or that the outer measure of every set is preserved. However, as Hausdorff measures are Borel regular (and Borel sets are measurable with respect to Hausdorff measures), there is no possibility of confusion here, since measure preserving for measurable sets is equivalent to outer measure preserving for all sets.

We do not know if the assumption of the Continuum Hypothesis can be dropped in Theorem 0.2. However, it is rather unlikely as the following remarks show. The result resembles Erdős-Sierpiński Duality (see e.g. [Ox]); that is, the existence under CH of a bijection which maps Lebesgue nullsets onto sets of first category and vice versa. Denote $\mathcal{N}$ and $\mathcal{M}$ the σ-ideals of Lebesgue nullsets and sets of first category, respectively. For a σ-ideal $\mathcal{I}$, the symbol $\mathrm{non}(\mathcal{I})$ stands for the smallest cardinal κ for which there exists a set of cardinality κ that is not in $\mathcal{I}$. Now the fact that the role of CH is essential for Erdős-Sierpiński Duality follows from the well-know result that in some models of set theory $\mathrm{non}(\mathcal{N}) \neq \mathrm{non}(\mathcal{M})$ holds (see e.g. [BJ]).

In our present setting, denote $\mathcal{N}_{\mathcal{H}^d}$ the σ-ideal of negligible sets with respect to $\mathcal{H}^d$. If it were known that in some model of set theory $\mathrm{non}(\mathcal{N}_{\mathcal{H}^{d_1}}) \neq \mathrm{non}(\mathcal{N}_{\mathcal{H}^{d_2}})$ held, it would be proven that the Continuum Hypothesis cannot be dropped in Theorem 0.2. However, this is a very recent research area, and so far the above statement is only known to hold in some model when $d_2 = n$ (see [SS]).

We also prove in this paper that under CH for every $0 < d_1 \leq d_2 < n$ the measure spaces $(\mathbb{R}^n, \mathcal{B}^{\sigma f}_{d_1}, \mathcal{H}^{d_1})$ and $(\mathbb{R}^n, \mathcal{B}^{\sigma f}_{d_2}, \mathcal{H}^{d_2})$ are isomorphic, where $\mathcal{B}^{\sigma f}_{d}$ stands for the σ-algebra of Borel subsets of $\mathbb{R}^n$ that are of σ-finite or co-σ-finite $\mathcal{H}^d$-measure. This result shows that Question 0.1 cannot be solved by the natural approach, as it is not enough to consider only d_1-sets and d_2-sets; that is, sets of positive and finite measure with respect to $\mathcal{H}^{d_1}$ and $\mathcal{H}^{d_2}$.

In the second part of the paper, motivated by Question 0.1, we consider the following set of problems. Can we find for every $f : [0,1] \to \mathbb{R}$ continuous/Borel/typical continuous (in the Baire category sense, see e.g. [Br]) function a set of positive Hausdorff dimension on which the function agrees with a function of bounded variation/Lipschitz/Hölder continuous function? For example it is clear that showing that every Borel function is Hölder continuous of some suitable exponent on a set of sufficiently large Hausdorff dimension would answer Question 0.1 in the negative. The other versions are less closely related to our problem, however, they are of independent interest. We prove the following two results.

Theorem 0.3 *Fix $0 < \alpha \leq 1$. A typical continuous function is not Hölder continuous of exponent α on any set of Hausdorff dimension larger than $1 - \alpha$.*

We do not know whether $1-\alpha$ is sharp.

Theorem 0.4 *A typical continuous function does not agree with any function of bounded variation on any set of Hausdorff dimension larger than* $\frac{1}{2}$.

This theorem is motivated by an analogous result of Humke and Laczkovich [HL], who proved that a typical continuous function is not monotonic on any set of positive Hausdorff dimension. So one would expect the same for functions of bounded variation, and it is a very natural problem whether this dimension is indeed 0, or $\frac{1}{2}$ or perhaps something in between.

Question 0.5 *Is it true that a typical continuous function agrees with a function of bounded variation on a set of Hausdorff dimension* $\frac{1}{2}$*?*

1 Preliminaries

In this section we present the lemmas we use in the sequel. Probably most of them are well-known, however, we could not find suitable references, so we could not avoid including them. Let λ denote one-dimensional Lebesgue measure, then $\mathcal{M}_\lambda$ is the class of Lebesgue measurable sets. A *Borel isomorphism* is a bijection such that images and preimages of Borel sets are Borel.

Lemma 1.1 *Let* $B \subset \mathbb{R}^n$ *be Borel such that* $0 < \mathcal{H}^d(B) < \infty$ *and let* $I = [0, \mathcal{H}^d(B)]$. *Then there exists an isomorphism* f *between the measure spaces* $(B, \mathcal{M}_{\mathcal{H}^d}, \mathcal{H}^d)$ *and* $(I, \mathcal{M}_\lambda, \lambda)$ *that is also a Borel isomorphism.*

Proof. We may clearly assume $\mathcal{H}^d(B) = 1$. It is stated in [Ke, 12.B] that every Borel set, hence B is a standard Borel space (that is, B is Borel isomorphic to some Polish space). Hence we can apply [Ke, 17.41] which states that every continuous (that is, singletons are of measure zero) probability measure on a standard Borel space is isomorphic by a Borel isomorphism to Lebesgue measure on the unit interval. $\square$

Lemma 1.2 *Let* $B \subset \mathbb{R}^n$ *be a Borel set of infinite but* σ*-finite* $\mathcal{H}^d$*-measure. Then there exists an isomorphism* f *between the measure spaces* $(B, \mathcal{M}_{\mathcal{H}^d}, \mathcal{H}^d)$ *and* $([0,\infty), \mathcal{M}_\lambda, \lambda)$ *that is also a Borel isomorphism.*

Proof. Define a system $\mathcal{A}$ of pairwise disjoint Borel subsets of B by transfinite recursion, such that $\mathcal{H}^d(A) = 1$ for every $A \in \mathcal{A}$. By σ-finiteness this procedure stops at some countable ordinal, so $\mathcal{A}$ is countable. Put $m = \mathcal{H}^d(B \setminus \cup\mathcal{A})$. As we can always find a Borel subset of $\mathcal{H}^d$-measure 1 inside a Borel set of $\mathcal{H}^d$-measure at least 1 (see e.g. [Ma, 8.20]), we obtain $m < 1$. In particular, $\mathcal{A}$ is infinite, say $\mathcal{A} = \{A_0, A_1, \ldots\}$. By the previous lemma $B \setminus \cup_{i=0}^{\infty} A_i$ is isomorphic to the interval $[-m, 0)$ (or more precisely to [-m,0], but these two intervals are isomorphic again by the previous lemma), and A_i is isomorphic to $[i, i+1)$ for every i. Therefore B is clearly isomorphic to $[-m, \infty)$ which is obviously isomorphic to $[0, \infty)$. $\square$

Lemma 1.3 *Let $n \in \mathbb{N}$, $d \geq 0$ and $H \subset \mathbb{R}^n$ be arbitrary. Then the following statements are equivalent:*

(i) H is $\mathcal{H}^d$-measurable,

(ii) $\mathcal{H}^d(B) = \mathcal{H}^d(B \cap H) + \mathcal{H}^d(B \cap H^C)$ for every Borel set $B \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(B) < \infty$,

(iii) $H \cap B$ is $\mathcal{H}^d$-measurable for every Borel set $B \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(B) < \infty$,

(iv) For every Borel set $B \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(B) < \infty$, we have $H \cap B = A \cup N$, where A is Borel and N is $\mathcal{H}^d$-negligible.

Proof. (i) $\iff$ (ii) By definition, the set H is $\mathcal{H}^d$-measurable in the sense of Carathéodory if and only if $\mathcal{H}^d(X) = \mathcal{H}^d(X \cap H) + \mathcal{H}^d(X \cap H^C)$ for every $X \subset \mathbb{R}^n$. As outer measures are subadditive, this is equivalent to $\mathcal{H}^d(X) \geq \mathcal{H}^d(X \cap H) + \mathcal{H}^d(X \cap H^C)$ for every $X \subset \mathbb{R}^n$. Once this inequality fails to hold, using the Borel regularity of Hausdorff measures (see e.g. [Ma, 4.5]), there is a Borel set $B \supset X$ such that $\mathcal{H}^d(B) < \mathcal{H}^d(X \cap H) + \mathcal{H}^d(X \cap H^C)$. Therefore B is of finite measure, moreover, $\mathcal{H}^d(B) < \mathcal{H}^d(B \cap H) + \mathcal{H}^d(B \cap H^C)$, in particular, B is clearly of positive measure.

(ii) $\iff$ (iii) The third condition obviously implies the second one as $\mathcal{H}^d$ is additive on measurable sets. So suppose (ii). It is enough to show that $\mathcal{H}^d(A) \geq \mathcal{H}^d(A \cap (H \cap B)) + \mathcal{H}^d(A \cap (H \cap B)^C)$ for every Borel set $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^d(A) < \infty$. We can assume that $\mathcal{H}^d(A \cap B) > 0$, otherwise the above inequality clearly holds, as the first term on the right hand side is 0, while the second term is not greater than the left hand side. Now

$$\mathcal{H}^d(A) \geq \mathcal{H}^d(A \cap B) + \mathcal{H}^d(A \cap B^C) \geq$$
$$\mathcal{H}^d((A \cap B) \cap H) + \mathcal{H}^d((A \cap B) \cap H^C) + \mathcal{H}^d(A \cap B^C) \geq$$
$$\mathcal{H}^d(A \cap (H \cap B)) + \mathcal{H}^d(A \cap (H \cap B)^C),$$

where the first inequality holds as B is measurable, the second one is (ii) applied to $A \cap B$, and the third one follows from the subadditivity of $\mathcal{H}^d$ since $A \cap (H \cap B)^C$ is the (disjoint) union of $(A \cap B) \cap H^C$ and $A \cap B^C$.

(iii) $\iff$ (iv) As negligible sets are measurable, the fourth condition implies the third, while we immediately obtain the other direction if we apply Lemma 1.1 to B. $\square$

Remark 1.4 We could also assume that the above B is compact, but we will not need this fact.

The following lemma is essentially [Fe, 2.5.10].

Lemma 1.5 *Let $0 < d < n$ and suppose CH holds. Then there exists a disjoint family $\{B_\alpha : \alpha < \omega_1\}$ of Borel subsets of $\mathbb{R}^n$ of finite $\mathcal{H}^d$-measure, such that a set $H \subset \mathbb{R}^n$ is $\mathcal{H}^d$-measurable iff $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$.*

Proof. Let $\{A_\alpha : \alpha < \omega_1\}$ be an enumeration of the Borel subsets of $\mathbb{R}^n$ of positive finite $\mathcal{H}^d$-measure, and put $B_\alpha = A_\alpha \setminus (\cup_{\beta<\alpha} A_\beta)$. These are clearly pairwise disjoint Borel sets of finite $\mathcal{H}^d$-measure. The other direction being trivial we only have to verify that if $H \subset \mathbb{R}^n$ is such that $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$, then H itself is $\mathcal{H}^d$-measurable. By Lemma 1.3 we only have to show that $H \cap A_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$. But $A_\alpha = \cup_{\beta \leq \alpha} B_\beta$, therefore $H \cap A_\alpha = \cup_{\beta \leq \alpha}(H \cap B_\alpha)$, which is clearly $\mathcal{H}^d$-measurable, which completes the proof. □

Lemma 1.6 *Let $0 < d < n$ and suppose CH holds. Then there exists a disjoint family $\{S_\alpha : \alpha < \omega_1\}$ of Borel subsets of $\mathbb{R}^n$ of infinite but σ-finite $\mathcal{H}^d$-measure, such that a set $H \subset \mathbb{R}^n$ is $\mathcal{H}^d$-measurable iff $H \cap S_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$.*

Proof. First we check that uncountably many B_α of Lemma 1.5 are of positive $\mathcal{H}^d$-measure. Otherwise, as $\mathbb{R}^n$ is not σ-finite and as by [Ma, 8.20] every Borel set of infinite $\mathcal{H}^d$-measure contains a Borel set of $\mathcal{H}^d$-measure 1, we could find a Borel set of positive and finite measure that is disjoint from these B_α. But this set was enumerated as A_α for some α, moreover $A_\alpha = \cup_{\beta \leq \alpha} B_\beta$. Since A_α is disjoint from all B_β of positive measure, by CH, countably many zerosets cover A_α, so it is a zeroset, a contradiction.

This obviously implies that for some integer N uncountably many B_α are of measure at least $\frac{1}{N}$. Now we can recursively define a partition of the set of countable ordinals into countable intervals $\{I_\alpha : \alpha < \omega_1\}$ such that for every α the $\mathcal{H}^d$-measure of $\cup_{\xi \in I_\alpha} B_\xi$ is infinite. On the other hand, this measure is clearly σ-finite. Now we check that $S_\alpha = \cup_{\xi \in I_\alpha} B_\xi$ works. Every S_α is clearly a Borel subsets of $\mathbb{R}^n$ of infinite but σ-finite $\mathcal{H}^d$-measure. Now we have to show that the last statement of the lemma holds, namely that $\mathcal{H}^d$-measurability "reflects" on the sets S_α. One direction is trivial, so in order to prove the other one let us assume that $H \cap S_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < \omega_1$. We have to show that H is $\mathcal{H}^d$-measurable. But this is obvious by the previous lemma, as every B_α can be covered by some S_β, hence $H \cap B_\alpha = B_\alpha \cap (H \cap S_\beta)$ which is measurable. □

Lemma 1.7 *Assume CH. A set $H \subset \mathbb{R}^n$ is of σ-finite $\mathcal{H}^d$-measure iff it can be covered by countably many of the above S_α.*

Proof. One direction is trivial. For the other one we can assume that $H \subset \mathbb{R}^n$ is of finite $\mathcal{H}^d$-measure. There exists a Borel set B of positive finite $\mathcal{H}^d$-measure containing H. This set was enumerated in Lemma 1.5 as A_α for some α, so by CH it can be covered by countably many B_α hence also by countably many S_α. □

2 Isomorphic measure spaces

In this section we use the above lemmas to prove the existence of isomorphisms under CH.

Theorem 2.1 *Under CH for every $0 < d_1 \leq d_2 < n$ the measure spaces $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_1}}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{M}_{\mathcal{H}^{d_2}}, \mathcal{H}^{d_2}\right)$ are isomorphic.*

Proof. Find two partitions $\{S_\alpha^{d_1} : \alpha < \omega_1\}$ and $\{S_\alpha^{d_2} : \alpha < \omega_1\}$ of $\mathbb{R}^n$ as in Lemma 1.6. By Lemma 1.2 find isomorphisms $f_\alpha : S_\alpha^{d_1} \to S_\alpha^{d_2}$ for every α. Define $f = \cup_{\alpha<\omega_1} f_\alpha$. We have to check that f is an isomorphism. First we prove that f preserves measurable sets. Suppose $H \in \mathcal{M}_{\mathcal{H}^{d_1}}$. By Lemma 1.6 it is sufficient to show that $f(H) \cap S_\alpha^{d_2} \in \mathcal{M}_{\mathcal{H}^{d_2}}$ for every α. But $f(H) \cap S_\alpha^{d_2} = f_\alpha(H \cap S_\alpha^{d_1})$ which is $\mathcal{H}^{d_2}$-measurable as f_α is an isomorphism. Similarly, f^{-1} also preserves measurable sets. Now we show that f preserves measure. (Again, the same argument works for f^{-1}.) As mentioned in the Introduction it is enough to show this for measurable sets. First, f preserves non-σ-finiteness since it clearly preserves the property that characterizes σ-finiteness in Lemma 1.7, so we can restrict ourselves to σ-finite sets. But such a set is partitioned by the countably many $S_\alpha^{d_1}$ that cover it, and the countably many isomorphisms f_α preserve measure, so the proof is complete. $\square$

Remark 2.2 If $d = 0$ then all subsets of $\mathbb{R}^n$ are measurable, while if $d = n$ then $\mathcal{H}^d$ is σ-finite, therefore the theorem cannot be extended to these cases.

As we already mentioned in the Introduction, it is unknown whether CH can be dropped from the theorem, but the paper [SS] is a huge step in this direction.

Now we prove the existence of another type of isomorphism. Let $\mathcal{B}_d^{\sigma f}$ denote the σ-algebra of Borel subsets of $\mathbb{R}^n$ that are of σ-finite or co-σ-finite $\mathcal{H}^d$-measure.

Theorem 2.3 *Under CH for every $0 < d_1 \leq d_2 < n$ the measure spaces $\left(\mathbb{R}^n, \mathcal{B}_{d_1}^{\sigma f}, \mathcal{H}^{d_1}\right)$ and $\left(\mathbb{R}^n, \mathcal{B}_{d_2}^{\sigma f}, \mathcal{H}^{d_2}\right)$ are isomorphic.*

Proof. Let f be as above. It is sufficient to show that $B \in \mathcal{B}_{d_1}^{\sigma f}$ implies $f(B) \in \mathcal{B}_{d_2}^{\sigma f}$. It is also sufficient to show this for σ-finite sets, as then for B co-σ-finite we obtain $f(\mathbb{R}^n \setminus B) \in \mathcal{B}_{d_2}^{\sigma f}$ which implies $f(B) = \mathbb{R}^n \setminus f(\mathbb{R}^n \setminus B) \in \mathcal{B}_{d_2}^{\sigma f}$. So let B be a σ-finite Borel set. By Lemma 1.7 the set B is partitioned by the countably many Borel sets $S_\alpha^{d_1}$ that cover it, and the countably many Borel isomorphisms f_α produce a σ-finite Borel image. $\square$

This result shows that Question 0.1 cannot be solved by the natural approach, as it is not enough to consider only d_1-sets and d_2-sets; that is, sets of positive and finite measure with respect to $\mathcal{H}^{d_1}$ and $\mathcal{H}^{d_2}$.

3 Typical continuous functions

In this section we consider a set of problems related to the previous section. The reason is that a possible way to settle the isomorphism problem might be answering the following question. For the sake of simplicity, in this section we restrict ourselves to functions defined on $[0,1]$ instead of $\mathbb{R}^n$.

Question 3.1 *For which pair of reals $0 < \alpha, \beta \leq 1$ is it true that for every Borel function $f : [0,1] \to \mathbb{R}$ there exists a set $H \subset [0,1]$ of Hausdorff dimension at least β such that f restricted to H is Hölder continuous of exponent α?*

(Again, see [Fa], [Fe] or [Ma] for definitions). As every Borel function is continuous on a set of positive measure, we can ask the same question for continuous functions as well. Here typical continuous functions (in the Baire category sense, see e.g. [Br]) are natural candidates for badly behaving examples. Indeed, the following holds.

Theorem 3.2 *Fix $0 < \alpha \leq 1$. A typical continuous function is not Hölder continuous of exponent α on any set of Hausdorff dimension larger than $1 - \alpha$.*

Proof. Denote $C[0,1]$ the Banach space of continuous real-valued functions defined on $[0,1]$. We say that $g \in C[0,1]$ is Hölder continuous of exponent α and constant K if $|g(x) - g(y)| \leq K|x - y|^\alpha$ for every $x, y \in [0,1]$. Length of an interval I is denoted by $|I|$. For $f, g \in C[0,1]$, the set $\{x \in [0,1] : f(x) = g(x)\}$ is abbreviated by $\{f = g\}$. For $H \subset [0,1]$ denote

$$\mathcal{H}^d_\infty(H) = \inf \left\{ \sum_{n=1}^{\infty} |I_n|^d : \{I_n\}_{n=1}^\infty \text{ is a sequence of intervals, } H \subset \cup_{n=1}^\infty I_n \right\}.$$

So this is the usual Hausdorff premeasure with no restriction on the lengths of the covering intervals. It is well-known and easy to check that $\mathcal{H}^d(H) = 0$ if and only if $\mathcal{H}^d_\infty(H) = 0$.

First we show that it is sufficient to prove that for all integers $N, M > 0$ the set

$$D(N,M) = \{f \in C[0,1] : \mathcal{H}^{1-\alpha+\frac{1}{N}}_\infty(\{f = g\}) < \frac{1}{M}$$
$$\text{for every Hölder function } g \text{ of exponent } \alpha \text{ and constant } 1\}$$

contains a dense open subset of $C[0,1]$. Indeed, this implies that the set $D = \cap_{N=1}^\infty \cap_{M=1}^\infty D(N,M)$ contains a dense G_δ set, hence is residual, which in turn implies that the typical continuous function does not agree with a Hölder continuous function of exponent α and constant 1 on any set of Hausdorff dimension larger than $1 - \alpha$. Moreover, the map $f \mapsto Kf$ is a homeomorphism of $C[0,1]$, hence $KD = \{Kf : f \in D\}$ is residual for every K. Therefore $\cap_{K=1}^\infty KD$ is also residual, and so the typical continuous function does not agree with a Hölder continuous function of exponent α on any set of Hausdorff dimension larger than $1 - \alpha$.

Now we show that we can assume $0 < \alpha < 1$. Indeed, if the statement of the theorem holds for $1 - \frac{1}{L}$ for every L, then intersecting the corresponding sequence of residual subsets of $C[0,1]$ we obtain the case $\alpha = 1$.

Now what remains to be proven is that $D(N, M)$ contains a dense open set. Let $f_0 \in C[0,1]$ and $r_0 > 0$ be given. The ball centered at f_0 and of radius r_0 is denoted by $B(f_0, r_0)$. We have to find $f_1 \in C[0,1]$ and $r_1 > 0$ such that $B(f_1, r_1) \subset B(f_0, r_0) \cap D(N, M)$. By uniform continuity of f_0, for a large enough integer m, the inequality $|x - y| < \frac{2}{m}$ implies $|f_0(x) - f_0(y)| < \frac{r_0}{5}$. The exact value of m will be chosen later. Let k be another positive integer to be fixed later. We define the piecewise constant function $\bar{f}_1$ as follows. For a pair of integers $0 \le i < m$ and $0 \le j < k$ set

$$\bar{f}_1(x) = f_0(\frac{i}{m}) + j\frac{r_0}{5k} \qquad \text{for} \qquad \frac{i}{m} + \frac{j}{mk} \le x < \frac{i}{m} + \frac{j+1}{mk}.$$

Put $\bar{f}_1(1) = f_0(1)$. Now choose a finite system $\mathcal{I}$ of pairwise disjoint open intervals covering 1 and all numbers of the form $\frac{i}{m} + \frac{j}{mk}$ such that

$$\sum_{I \in \mathcal{I}} |I|^{1-\alpha+\frac{1}{N}} < \frac{1}{2M}. \tag{1}$$

We can clearly choose a continuous $f_1 \in C[0,1]$ which is linear on each $I \in \mathcal{I}$, agrees with $\bar{f}_1$ outside $\cup\mathcal{I}$ and satisfies $f_1(0) = f_0(0)$ and $f_1(1) = f_0(1)$. Denote the supremum-norm of a not necessarily continuous function f by $||f||$. One can easily check that $||f_0 - \bar{f}_1|| \le \frac{2}{5}r_0$ and that $||\bar{f}_1 - f_1|| \le \frac{2}{5}r_0$, hence $||f_0 - f_1|| \le \frac{4}{5}r_0$. So if we put $r_1 = \frac{r_0}{20k} \le \frac{r_0}{5}$ then $B(f_1, r_1) \subset B(f_0, r_0)$. Now we claim that the inequality

$$\left(\frac{2}{mk}\right)^{\alpha} < \frac{r_0}{10k} \tag{2}$$

implies that for every $f \in B(f_1, r_1)$, every Hölder continuous function g of exponent α and constant 1 and every fixed $0 \le i < m$, there is at most one $0 \le j < k$ such that $\left(\{f = g\} \cap \left(\frac{i}{m} + \frac{j}{mk}, \frac{i}{m} + \frac{j+1}{mk}\right)\right) \setminus \cup\mathcal{I} \neq \emptyset$. Indeed, by the concavity of the function x^α it is enough to check that for this fixed i the graph of f and g cannot meet over two consecutive intervals of the k intervals of length $\frac{1}{mk}$, but this is clear from the fact that the value of $\bar{f}_1$ 'jumps' by $\frac{r_0}{5k}$, moreover $\frac{r_0}{5k} - 2r_1 = \frac{r_0}{10k}$, and from (2).

This means that $\{f = g\}$ can be covered by the elements of $\mathcal{I}$ and by m intervals of length $\frac{1}{mk}$. Before we use this fact to estimate $\mathcal{H}_\infty^{1-\alpha+\frac{1}{N}}(\{f = g\})$, we need some more preparations.

By rearranging (2) we obtain

$$k < \left(\frac{r_0}{10 \cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}} m^{\frac{\alpha}{1-\alpha}},$$

and we will also make sure that even the following holds.

$$\frac{1}{2}\left(\frac{r_0}{10 \cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}} m^{\frac{\alpha}{1-\alpha}} < k < \left(\frac{r_0}{10 \cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}} m^{\frac{\alpha}{1-\alpha}} \tag{3}$$

The last condition we need for the estimation of the Hausdorff premeasure is that

$$m\left(\frac{1}{mk}\right)^{1-\alpha+\frac{1}{N}} < \frac{1}{2M}. \tag{4}$$

Indeed, together with (1) this implies that $\mathcal{H}_\infty^{1-\alpha+\frac{1}{N}}(\{f=g\}) < \frac{1}{M}$ for every $f \in B(f_1, r_1)$ and every Hölder continuous function g of exponent α and constant 1. Using the left hand side of (3) we obtain

$$m\left(\frac{1}{mk}\right)^{1-\alpha+\frac{1}{N}} < \left(\frac{1}{2}\left(\frac{r_0}{10\cdot 2^\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{-(1-\alpha+\frac{1}{N})} m^{1-(1-\alpha+\frac{1}{N})-\frac{\alpha(1-\alpha+\frac{1}{N})}{1-\alpha}}, \tag{5}$$

and as the exponent

$$1-(1-\alpha+\frac{1}{N})-\frac{\alpha(1-\alpha+\frac{1}{N})}{1-\alpha} = -\frac{1}{(1-\alpha)N} < 0,$$

we can choose a large enough m such that (4) holds, and then we can fix k according to (3), so the proof is complete. $\square$

However, Theorem 3.2 is an upper estimate of the dimension, while in view of the question of isomorphisms we are more interested in lower estimates. Unfortunately we cannot prove any.

Question 3.3 *Fix $0 < \alpha < 1$. Is the typical continuous function Hölder continuous of exponent α on a set of positive Hausdorff dimension? Or on a set of Hausdorff dimension $1-\alpha$?*

Motivated by the previous problem we can formulate some other natural and interesting questions. See [BH] and [HL] for similar results. E.g. in [HL] it is shown that a typical continuous function agrees with a monotone function only on a set of dimension 0. It is therefore natural to guess that the same holds with functions of bounded variation, as they are the differences of monotone functions. However, we were able to prove only the following.

Theorem 3.4 *A typical continuous function does not agree with any function of bounded variation on any set of Hausdorff dimension larger than $\frac{1}{2}$.*

Proof. The proof is similar to that of Theorem 3.2. Total variation of a function g is denoted by $Var(g)$. It is sufficient to prove that for all integers $N, M > 0$ the set

$$\left\{f \in C[0,1] : \mathcal{H}_\infty^{\frac{1}{2}+\frac{1}{N}}(\{f=g\}) < \frac{1}{M} \text{ for every } g \text{ with } Var(g) \le 1\right\} \tag{6}$$

contains a dense open set. Given f_0 and r_0 define $\bar{f}_1$ as in Theorem 3.2 (with m and k yet unspecified). Choose $\mathcal{I}$ such that

$$\sum_{I\in\mathcal{I}} |I|^{\frac{1}{2}+\frac{1}{N}} < \frac{1}{2M}, \tag{7}$$

and define f_1 and r_1 as before.

Now for every $f \in B(f_1, r_1)$, g as in (6), and fixed $0 \le i < m$, denote l_i the number of intervals over which the graph of f and g meet outside $\cup\mathcal{I}$; that is,

$$l_i = \#\left\{j : \left(\{f = g\} \cap \left(\frac{i}{m} + \frac{j}{mk}, \frac{i}{m} + \frac{j+1}{mk}\right)\right) \setminus \cup\mathcal{I} \neq \emptyset\right\}.$$

As the 'jumps' of $\bar{f}_1$ are of height $\frac{r_0}{5k}$, moreover $\frac{r_0}{5k} - 2r_1 = \frac{r_0}{10k}$, it is easily seen that $Var(g) \ge \sum_{i=1}^{m-1}\left(\frac{r_0}{10k}(l_i - 1)\right) = \frac{r_0}{10k}\left(\sum_{i=1}^{m-1} l_i - m\right)$. Thus $Var(g) \le 1$ implies that $\sum_{i=1}^{m-1} l_i$, that is, the number intervals of length $\frac{1}{mk}$ needed to cover $\{f = g\} \setminus \cup\mathcal{I}$ is at most $m + \frac{10k}{r_0}$.

Choose k=m. Then $\mathcal{H}_\infty^{\frac{1}{2}+\frac{1}{N}}(\{f = g\})$ can be estimated by (7) and

$$\left(m + \frac{10m}{r_0}\right)\left(\frac{1}{m^2}\right)^{\frac{1}{2}+\frac{1}{N}} = \left(\frac{10}{r_0} + 1\right) m^{-\frac{2}{N}},$$

which is smaller than $\frac{1}{2M}$ if m is large enough. □

Question 3.5 *Does a typical continuous function agree with some function of bounded variation on some set of Hausdorff dimension $\frac{1}{2}$? Or on a set of positive dimension?*

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