On a converse to Banach's Fixed Point Theorem

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Abstract

We say that a metric space (X, d) possesses the *Banach Fixed Point Property (BFPP)* if every contraction $f : X \to X$ has a fixed point. The Banach Fixed Point Theorem says that every complete metric space has the BFPP. However, E. Behrends pointed out [2] that the converse implication does not hold; that is, the BFPP does not imply completeness, in particular, there is a non-closed subset of $\mathbb{R}^2$ possessing the BFPP. He also asked [3] if there is even an open example in $\mathbb{R}^n$, and whether there is a 'nice' example in $\mathbb{R}$. In this note we answer the first question in the negative, the second one in the affirmative, and determine the simplest such examples in the sense of descriptive set theoretic complexity.

Specifically, first we prove that if $X \subset \mathbb{R}^n$ is open or $X \subset \mathbb{R}$ is simultaneously F_σ and G_δ and X has the BFPP then X is closed. Then we show that these results are optimal, as we give an F_σ and also a G_δ non-closed example in $\mathbb{R}$ with the BFPP.

We also show that a nonmeasurable set can have the BFPP. Our non-G_δ examples provide metric spaces with the BFPP that cannot be remetrised by any compatible complete metric. All examples are in addition bounded.

1 Introduction

Converses to the Banach Fixed Point Theorem have a very long history. The earliest such result seems to be that of Bessaga [4], but see also [1], [5], [8], [9], [10], [12], [14], [15], [16] and [18]. There are also numerous result of this kind in linear spaces as well.

The version we consider in this note is the following.

Definition 1.1 We say that a metric space (X, d) possesses the *Banach Fixed Point Property (BFPP)* if every contraction $f : X \to X$ has a fixed point.

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Note that the empty set does not possess the BFPP as the empty function is a contraction with no fixed point, so this would cause no problem, but for the sake of simplicity we simply assume that all sets and metric spaces considered are nonempty.

At the Problem Session of the 34th Winter School in Abstract Analysis E. Behrends presented the following example, which he referred to as 'folklore'.

Theorem 1.2 *Let $X = graph\left(sin(1/x)\restriction_{(0,1]}\right)$. Then $X \subset \mathbb{R}^2$ is a non-closed set possessing the Banach Fixed Point Property.*

Proof. X is clearly not closed. Let $f : X \to X$ be a contraction of Lipschitz constant $q < 1$. For $H \subset (0,1]$ define $X\restriction_H = graph\,(sin(1/x)\restriction_H)$. Choose $\varepsilon > 0$ so that $diam\left(X\restriction_{(0,\varepsilon)}\right) < \frac{2}{q}$, then $diam\left(f\left(X\restriction_{(0,\varepsilon)}\right)\right) < 2$. Hence $f\left(X\restriction_{(0,\varepsilon)}\right)$ cannot contain both a local minimum and a local maximum on the graph. But this set is clearly connected, which easily implies that it is contained in at most two monotone parts of the graph. Therefore there exists $\delta_1 > 0$ such that $f\left(X\restriction_{(0,\varepsilon)}\right) \subset X\restriction_{[\delta_1,1]}$. By compactness $f\left(X\restriction_{[\varepsilon,1]}\right) \subset X\restriction_{[\delta_2,1]}$ for some $\delta_2 > 0$, and hence setting $\delta = min\{\delta_1, \delta_2\}$ gives $f(X) \subset X\restriction_{[\delta,1]}$. But then the Banach Fixed Point Theorem applied to $X\restriction_{[\delta,1]}$ provides a fixed point. $\square$

E. Behrends asked the following two questions.

Question 1.3 ([3]) *Is there an open non-closed subset of $\mathbb{R}^n$ with the Banach Fixed Point Property for some $n \in \mathbb{N}$?*

Question 1.4 ([3]) *Is there a 'simple' non-closed subset of $\mathbb{R}$ with the Banach Fixed Point Property?*

2 When Banach's Fixed Point Theorem implies completeness

First we answer Question 1.3.

Lemma 2.1 *Let $n \in \mathbb{N}$ and $X \subset \mathbb{R}^n$ such that there exist $y, z \in \mathbb{R}^n$ so that $y \notin X$ but the nondegenerate segment $(y,z) \subset X$. Then X does not possess the Banach Fixed Point Property.*

Proof. We can clearly assume $y = (0,\dots,0)$ and $z = (1,\dots,0)$. Then

$$f(x) = \left(\frac{1}{2}arctan|x|, 0, \dots, 0\right) \quad (x \in \mathbb{R}^n)$$

is a contraction, since the absolute value of vectors and $arctan$ are both Lipschitz functions of constant 1. By our assumptions $f(X) \subset X$. As no contraction can have more than one fixed point, and the origin is clearly a fixed point, we obtain that $f\restriction_X$ has no fixed point. $\square$

Corollary 2.2 *For every $n \in \mathbb{N}$ every open subset of $\mathbb{R}^n$ possessing the Banach Fixed Point Property coincides with $\mathbb{R}^n$, hence it is closed.*

Proof. Let $U \subset \mathbb{R}^n$ be open but not closed, then there exists $z \in U$ and $x \notin U$. Let y be the closest point of $[x, z) \setminus U$ to z. $\square$

Now we turn to Question 1.4, the case of $X \subset \mathbb{R}$. In this section we show that there is no example that is simultaneously F_σ and G_δ.

Lemma 2.3 *Let $X \subset \mathbb{R}$ such that $0 \in \overline{X} \setminus X$ and 0 is a bilateral accumulation point of $int(X^c)$. Then X does not possess the Banach Fixed Point Property.*

Proof. Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of intervals in $X^c \cap (0, \infty)$ so that $b_{n+1} < a_n$ for every n and $a_n, b_n \to 0$. Fix a monotone decreasing sequence $z_n \in X$ such that $|z_n| < \frac{b_n - a_n}{2}$. Now, for $x \in X$, $x > 0$ let n_x be the minimal number for which $b_{n_x} < x$, and define $f(x) = z_{n_x}$. Define f on $X \cap (-\infty, 0)$ in a a similar manner. We claim that f is a contraction. First let $0 < x < y$ be two points in X. If $n_x = n_y$ then $f(x) = f(y)$, while if $n_x > n_y$ then $|f(x) - f(y)| < |z_{n_y}| < \frac{b_{n_y} - a_{n_y}}{2} < \frac{|x-y|}{2}$, hence f is a contraction on $X \cap (0, \infty)$. Similarly, f is a contraction on $X \cap (-\infty, 0)$. Moreover, $|f(x)| = |z_{n_x}| < \frac{b_{n_x} - a_{n_x}}{2} < \frac{|x|}{2}$, which shows that for every $x < 0 < y$ in X we have $|f(x) - f(y)| < \frac{|x-y|}{2}$, hence f is a contraction on X.

Since $0 \notin X$, the above inequality $|f(x)| < \frac{|x|}{2}$ also shows that f has no fixed point. This finishes the proof. $\square$

A *portion* of a set is a relatively open nonempty subset. A set that is simultaneously F_σ and G_δ is called *ambiguous* (or Δ^0_2 in descriptive set theory). A set X is ambiguous iff for every nonempty closed set F either X or X^c contains a portion of F [13].

Theorem 2.4 *Every simultaneously F_σ and G_δ subset of $\mathbb{R}$ with the Banach Fixed Point Property is closed.*

Proof. Suppose that $X \subset \mathbb{R}$ is a non-closed ambiguous set with the BFPP. By applying a translation we can assume that $0 \in \overline{X} \setminus X$. By the previous lemma 0 is not a bilateral accumulation point of $int(X^c)$, so without loss of generality there exists $\varepsilon > 0$ such that X is dense in $[0, \varepsilon]$. Let I be an arbitrary closed nondegenerate subinterval of $[0, \varepsilon]$. As X is ambiguous, either X or X^c contains a portion of I, but as X is dense in I, the second alternative cannot hold. Hence X contains a subinterval of I, and as I was arbitrary, $int(X)$ is dense in $[0, \varepsilon]$.

Set $F = [0, \varepsilon] \setminus int(X)$. As $0 \in F$, we have $F \neq \emptyset$, so either X or X^c contains a portion of F, but the first alternative clearly cannot hold, so there exists an open interval $J \subset [0, \varepsilon]$ so that the nonempty set $F \cap J$ is disjoint from X. Fix $f \in J \setminus X$ and by the denseness of $int(X)$ also an $x \in J \cap int(X)$. Let y be the closest point to x of $(int(X))^c$ between x and f. As $y \in F \cap J$, we obtain $y \notin X$, hence by Lemma 2.1 X does not possess the BFPP. $\square$

3 When Banach's Fixed Point Theorem holds for strange sets

In this section we give the examples of non-closed sets with the BFPP of lowest possible Borel classes. For every $n \geq 2$ Theorem 1.2 clearly provides an ambiguous example in $\mathbb{R}^n$, Corollary 2.2 shows that no open example is possible, and obviously there is no closed example. In the language of descriptive set theory, Δ^0_2 is best possible, as there are no Σ^0_1 and Π^0_1 examples. In $\mathbb{R}$ Theorem 2.4 shows that there is no ambiguous example, and this will be shown to be optimal when we prove below that there are F_σ and also G_δ examples. That is, Σ^0_2 and Π^0_2 are possible, but Δ^0_2 is not.

The space of compact subsets of $\mathbb{R}$ endowed with Hausdorff metric is a complete metric space (see e.g. [11] for definitions and basic facts). We say that *a typical compact set has a property* if the compact sets not having the property form a first category (in the sense of Baire) set in the above space.

The following lemma is interesting in its own right. For simplicity we only prove it in $\mathbb{R}$, but it easily generalises to higher dimensions.

Lemma 3.1 *A contractive image of a typical compact $K \subset \mathbb{R}$ cannot contain a portion of K.*

Proof. Recall that if each of a countable set of properties hold for a typical compact set, then they also hold simultaneously, as first category sets are closed under countable unions. Therefore it is enough to show that for a fixed pair of rationals $p < q$, for a typical compact set K either $K \cap (p,q) = \emptyset$ or $K \cap (p,q)$ cannot be covered by a contractive image of K. Similarly, it suffices to check that for a fixed $r < 1$ if f is a contraction of ratio at most r then either $K \cap (p,q) = \emptyset$ or $K \cap (p,q) \not\subset f(K)$. As (in fact, in every dimension) every contraction can be extended to $\mathbb{R}$ with the same Lipschitz constant [6, 2.10.43.] we may assume that $f : \mathbb{R} \to \mathbb{R}$.

Therefore it suffices to prove that for a fixed $r < 1$ and for a fixed pair of rationals $p < q$

$$N = \{K \subset \mathbb{R} \text{ cpt } : \exists f : \mathbb{R} \to \mathbb{R} \text{ contr. of ratio} \leq r, \quad \emptyset \neq K \cap (p,q) \subset f(K)\}$$

is a nowhere dense subset of the space of compact sets. Let $B(K_0, \varepsilon_0)$ be the open ball of center K_0 and radius $\varepsilon_0 > 0$. We need to find a ball inside this one that is disjoint from N. It is well known and easy to see that the finite sets form a dense subset of our space, so we may assume that K_0 is finite; $K_0 = \{x_1, \dots, x_n\}$.

Suppose first that $K_0 \cap [p,q] = \emptyset$. Define $\varepsilon_1 = min\{dist(K_0, (p,q)), \varepsilon_0\} > 0$. Then for every $K \in B(K_0, \varepsilon_1)$ we have $K \cap (p,q) = \emptyset$, hence $B(K_0, \varepsilon_1) \cap N = \emptyset$.

So we can assume that $K_0 \cap [p,q] \neq \emptyset$, e.g. $x_{i_0} \in [p,q]$. Let (a,b) be a subinterval of $(p,q) \cap (x_{i_0} - \varepsilon_0, x_{i_0} + \varepsilon_0)$. Choose an integer

$$k > \frac{n+2}{1-r}, \tag{1}$$

and choose two arithmetic progressions $\{y_1, \dots, y_k\}$ and $\{z_1, \dots, z_k\}$ in (a, b), each of length k and of some difference $d > 0$ so that

$$dist(\{y_1, \dots, y_k\}, \{z_1, \dots, z_k\}) \geq kd. \tag{2}$$

Define

$$K_1 = K_0 \cup \{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\},$$

then $K_1 \in B(K_0, \varepsilon_0)$. Choose

$$\varepsilon_1 = min \left\{ dist\left(K_1, B(K_0, \varepsilon_0)^c\right), \frac{d}{4} \right\},$$

then clearly $B(K_1, \varepsilon_1) \subset B(K_0, \varepsilon_0)$. It is also easy to see that the intervals $Y_i = (y_j - \varepsilon_1, y_j + \varepsilon_1)$, $Z_i = (z_j - \varepsilon_1, z_j + \varepsilon_1)$ for $1 \leq j \leq k$ are all disjoint. Also put $X_i = (x_i - \varepsilon_1, x_i + \varepsilon_1)$ for every $1 \leq i \leq n$.

Now we claim that $B(K_1, \varepsilon_1) \cap N = \emptyset$, which will finish the proof. Let $K \in B(K_1, \varepsilon_1)$ be arbitrary. Clearly $K \subset \bigcup_{i=1}^n X_i \cup \bigcup_{j=1}^k Y_j \cup \bigcup_{j=1}^k Z_j$, and K intersects all these intervals. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction of ratio at most r. Denote by m_Y (resp. m_Z) the number of intervals Y_j (resp. Z_j) met by some $f(I)$, where I ranges over the X_i's, Y_j's and Z_j's. We will be done once we show that $m_Y < k$ or $m_Z < k$.

Using $\varepsilon_1 \leq \frac{d}{4}$ and (2) we obtain

$$diam\left(f(\bigcup_{j=1}^k Y_j)\right) < diam\left(\bigcup_{j=1}^k Y_j\right) = (k-1)d + 2\varepsilon_1 \leq kd - 2\varepsilon_1 \leq \tag{3}$$

$$\leq dist(\{y_1, \dots, y_k\}, \{z_1, \dots, z_k\}) - 2\varepsilon_1 = dist(\bigcup_{j=1}^k Y_j, \bigcup_{j=1}^k Z_j),$$

so $f\left(\bigcup_{j=1}^k Y_j\right)$ cannot intersect both $\bigcup_{j=1}^k Y_j$ and $\bigcup_{j=1}^k Z_j$. Of course, the same holds for $f\left(\bigcup_{j=1}^k Z_j\right)$, so without loss of generality we may assume that

$$\left(\bigcup_{j=1}^k Y_j\right) \cap f\left(\bigcup_{j=1}^k Y_j\right) = \emptyset \text{ or } \left(\bigcup_{j=1}^k Y_j\right) \cap f\left(\bigcup_{j=1}^k Z_j\right) = \emptyset. \tag{4}$$

For $1 \leq j_1 < j_2 \leq k$ we have $dist(Y_{j_1}, Y_{j_2}) \geq d - 2\varepsilon_1 \geq 2\varepsilon_1$, so if I is an interval of length $2\varepsilon_1$ then $f(I)$ cannot intersect both Y_{j_1} and Y_{j_2}. Moreover, if $H \subset \mathbb{R}$ intersects t many distinct Y_j intervals, then clearly $diam(H) > d(t-1) - 2\varepsilon_1 > d(t-1) - d = d(t-2)$, hence

$$t < \frac{diam(H)}{d} + 2. \tag{5}$$

We would like to apply this to $f\left(\bigcup_{j=1}^k Y_j\right)$ and $f\left(\bigcup_{j=1}^k Z_j\right)$. Clearly

$$diam\left(f(\bigcup_{j=1}^{k} Y_j)\right) \leq r diam\left(\bigcup_{j=1}^{k} Y_j\right) = r[(k-1)d + 2\varepsilon_1] \leq rkd,$$

so by (5) $f\left(\bigcup_{j=1}^k Y_j\right)$ can only meet at most $rk+2$ many Y_j's, and similarly for $f\left(\bigcup_{j=1}^k Z_j\right)$. In fact, by (4) we only need to calculate with one of these two amounts, and altogether we obtain

$$m_Y < rk + 2 + n,$$

where n comes from the X_i's. But by (1) $rk + 2 + n < k$, which finishes the proof. □

Remark 3.2 Note that if every contraction $f : X \to X$ is constant, then X clearly has the Banach Fixed Point Property.

Theorem 3.3 *There exists a non-closed G_δ set $X \subset \mathbb{R}$ with the Banach Fixed Point Property. Moreover, $X \subset [0,1]$ and every contraction mapping X into itself is constant.*

Proof. Let $K \subset \mathbb{R}$ be a nonempty compact set such that no portion of K can be covered by a contractive image of K. Then K is clearly nowhere dense. Let

$$X = (K + \mathbb{Q})^c \cap [0,1],$$

then X is G_δ. As $K + \mathbb{Q}$ is a nonempty set of the first category, it is not open in $[0,1]$, hence X is not closed.

Now, let $f : X \to X$ be a non-constant contraction. As above, let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a contraction extending f. As X is dense in $[0,1]$, we have $\tilde{f}([0,1]) \subset [0,1]$. We can clearly assume that $\tilde{f}$ is constant on $(-\infty, 0]$ and $[1, \infty)$, hence $ran(\tilde{f}) \subset [0,1]$. Then $ran(\tilde{f})$ is a nondegenerate interval $I \subset [0,1]$. Pick $q_0 \in \mathbb{Q}$ so that $(K + q_0) \cap int(I) \neq \emptyset$. As $\tilde{f}(X) \subset X$, we have $X^c \cap I \subset \tilde{f}(X^c)$, so $(K + q_0) \cap I \subset (K + \mathbb{Q}) \cap I \subset \tilde{f}(K + \mathbb{Q}) \cup \{\tilde{f}(0), \tilde{f}(1)\} = \bigcup_{q \in \mathbb{Q}} \tilde{f}(K + q) \cup \{\tilde{f}(0), \tilde{f}(1)\}$. Since K is nowhere dense, there is a nondegenerate interval $[a,b] \subset int(I)$ intersecting $K + q_0$ such that $a, b \notin K + q_0$. The closed set $[a,b] \cap (K + q_0) \subset \bigcup_{q \in \mathbb{Q}} \tilde{f}(K+q) \cup \{\tilde{f}(0), \tilde{f}(1)\}$, which is a covering by countably many closed sets, hence by the Baire Category Theorem one of them covers a portion of $K + q_0$, which contradicts the choice of K. □

Theorem 3.4 *There exists a non-closed F_σ subset of $[0,1]$ with the Banach Fixed Point Property.*

Proof. Again, let $K \subset \mathbb{R}$ be a nonempty nowhere dense compact set such that no portion of K can be covered by a contractive image of K. Then clearly K has no isolated points, so K is homeomorphic to the Cantor set [11, 7.4]. We can clearly assume that $min(K) = 0$ and $max(K) = 1$. Let $\{I_n\}_{n\in\mathbb{N}}$ be the set of contiguous open intervals of K. Set

$$X = \cup_{n\in\mathbb{N}}\overline{I_n}.$$

That is, X is '$[0,1] \setminus K$ plus the endpoints'. This set is clearly F_σ, and it is not closed, as it is dense in $[0,1]$ but only contains countably many points of K.

In order to show that it has the BFPP let $f : X \to X$ be a contraction, and as above, let $\tilde{f} : \mathbb{R} \to [0,1]$ be a contraction extending f (here we use again that X is dense in $[0,1]$) that is constant on $(-\infty, 0]$ and $[1,\infty)$. If $\tilde{f}$ is constant then we are done, otherwise $ran(\tilde{f})$ is a nondegenerate interval $I \subset [0,1]$. If $I \subset X$ then (by connectedness) we have $I \subset \overline{I_{n_0}}$ for some $n_0 \in \mathbb{N}$, and therefore $f\restriction_{\overline{I_{n_0}}}$ has a fixed point.

So we can assume $X^c \cap I \neq \emptyset$. Then using again that X is a union of closed intervals we obtain that $X^c \cap int(I) \neq \emptyset$. Choose a nondegenerate interval $[a,b] \subset int(I)$ intersecting K so that $a, b \notin K$. Similarly as above, $X^c \cap I \subset \tilde{f}(X^c) \subset \tilde{f}(K)$. As this last set is closed, $\overline{X^c \cap I} \subset \tilde{f}(K)$. Set $E = \bigcup_{n\in\mathbb{N}} \left(\overline{I_n} \setminus I_n\right)$; that is, the set of endpoints. Then $K \cap [a,b] = \overline{(K \setminus E) \cap [a,b]} = \overline{X^c \cap [a,b]} \subset \overline{X^c \cap I} \subset \tilde{f}(K)$, which is impossible by the choice of K. $\square$

It is well known [11, 3.11] that there is a complete metric equivalent to the usual one on a set $X \subset \mathbb{R}^n$ iff X is G_δ. Combining this fact with the above theorem and Theorem 2.4 we obtain the following.

Corollary 3.5 *There is a bounded Borel (even F_σ) subset of $\mathbb{R}$ with the Banach Fixed Point Property that is not complete with respect to any equivalent metric.*

Finally we show that even a nonmeasurable set can have the BFPP. A set $B \subset [0,1]^n$ is called a *Bernstein set* if $B \cap F \neq \emptyset$ and $B^c \cap F \neq \emptyset$ for every uncountable closed set $F \subset [0,1]^n$. It is well known that every Bernstein set is nonmeasurable [17, 5.3] (which works for $[0,1]^n$ instead of $\mathbb{R}$).

Theorem 3.6 *For every integer $n > 0$ there exists a nonmeasurable set in $\mathbb{R}^n$ with the Banach Fixed Point Property. Moreover, there exists a Bernstein set in $[0,1]^n$ with the BFPP, such that every contraction mapping this set into itself is constant.*

Proof. It suffices to prove the second statement. Enumerate the uncountable closed sets $F \subset [0,1]^n$ as $\{F_\alpha : \alpha < 2^\omega\}$, and also the non-constant contractions $f : \mathbb{R}^n \to \mathbb{R}^n$ as $\{f_\alpha : \alpha < 2^\omega\}$. We define a characteristic function $\varphi : [0,1]^n \to \{0,1\}$, and the Bernstein set with the required properties will be $X = \{x \in [0,1]^n : \varphi(x) = 1\}$.

Suppose we have already defined φ on a set $D_\alpha \subset [0,1]^n$ of cardinality $< 2^\omega$. We define it for four more points. As every uncountable closed set is of cardinality 2^ω, we can pick two distinct points $x_\alpha, y_\alpha \in F \setminus D_\alpha$ and define $\varphi(x_\alpha) = 0$, $\varphi(y_\alpha) = 1$. This will make sure that X will be a Bernstein set in $[0,1]^n$.

As $ran(f_\alpha)$ is a nondegenerate connected set, its projection on every line is an interval, and for a suitable line this interval is nondegenerate. Hence $|ran(f_\alpha)| = 2^\omega$. Therefore $|ran(f_\alpha) \setminus (D_\alpha \cup \{x_\alpha, y_\alpha, Fix(f_\alpha)\})| = 2^\omega$, where $Fix(f_\alpha)$ is the (unique) fixed point of f_α. As the inverse images of the points of this set form a disjoint family of size 2^ω of nonempty sets, and $|D_\alpha \cup \{x_\alpha, y_\alpha, Fix(f_\alpha)\}| < 2^\omega$, there exists $u_\alpha \in ran(f_\alpha) \setminus (D_\alpha \cup \{x_\alpha, y_\alpha, Fix(f_\alpha)\})$ such that $f_\alpha^{-1}(u_\alpha) \cap (D_\alpha \cup \{x_\alpha, y_\alpha, Fix(f_\alpha)\}) = \emptyset$. Pick an arbitrary $v_\alpha \in f_\alpha^{-1}(u_\alpha)$, then $v_\alpha \neq u_\alpha$. Finally, define $\varphi(u_\alpha) = 0$, $\varphi(v_\alpha) = 1$.

After finishing the induction define φ to be 0 outside $\bigcup_{\alpha<2^\omega} D_\alpha$. As we mentioned above, X is easily seen to be a Bernstein set in $[0,1]^n$. In order to get a contradiction, let $f : X \to X$ be a non-constant contraction. Then it can be extended to $\mathbb{R}^n$, so $f = f_\alpha$ for some α. But then $v_\alpha \in X$ and $f(v_\alpha) = f_\alpha(v_\alpha) = u_\alpha \notin X$, a contradiction. $\square$

Remark 3.7 It is not hard to see that if $X = sin(1/x)\restriction_{(0,1]}$ then there exists a function $f : X \to X$ with no fixed points such that $|f(x) - f(y)| < |x - y|$ for every $x, y \in X$. (Just 'map each wave horizontally to the next one'.) It would be interesting to know what happens if we replace the class of contractions with this larger class of strictly distance-decreasing functions.

Question 3.8 *Is there for some $n \in \mathbb{N}$ a non-closed F_σ subset $X \subset \mathbb{R}^n$ with the Banach Fixed Point Property such that every contraction $f : X \to X$ is constant? Is there such a simultaneously F_σ and G_δ set?*

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