Measurable Envelopes, Hausdorff Measures and Sierpiński Sets

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Abstract

We show that the existence of measurable envelopes of all subsets of $\mathbb{R}^n$ with respect to the d dimensional Hausdorff measure $(0 < d < n)$ is independent of ZFC. We also investigate the consistency of the existence of $\mathcal{H}^d$-measurable Sierpiński sets.

Introduction

The following definition was motivated by the theory of analytic sets.

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Definition 0.1 Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. We call a set $H \subset X$ *small* (with respect to $\mathcal{A}$) if every subset of H belongs to $\mathcal{A}$. The σ-ideal of small subsets is denoted by $\mathcal{A}_0$. We say that $A \in \mathcal{A}$ is a *measurable envelope* of $H \subset X$ (with respect to $\mathcal{A}$) if $H \subset A$ and for every $B \in \mathcal{A}$ such that $H \subset B \subset A$ we have $A \setminus B \in \mathcal{A}_0$.

I have learnt the terminology 'every subset of X has a measurable envelope' from D. Fremlin. Another usual one is '$(X, \mathcal{A})$ admits covers' (see e.g. [Ke]), and 'measurable hull w.r.t $\mathcal{A}$' is also used.

For example it is not hard to see that if $\mathcal{A}$ is the Borel, Lebesgue or Baire σ-algebra in $\mathbb{R}^n$, then $\mathcal{A}_0$ is the σ-ideal of countable, Lebesgue negligible and first category sets, respectively. One can also prove that with respect to the Lebesgue or Baire σ-algebra, every subset of $\mathbb{R}^n$ has a measurable envelope, while in the case of the Borel sets this is not true. What makes these notions interesting is a theorem of Szpilrajn-Marczewski, asserting that if every subset of X has a measurable envelope, then $\mathcal{A}$ is closed under the Souslin operation (see [Ke, 29.13]).

This problem has been considered for various σ-algebras for a long time (see e.g. [Ma] and [Pa]).

In our paper we investigate the case $\mathcal{A} = \mathcal{A}_\mu$, where μ is an outer measure on X and $\mathcal{A}_\mu$ is the σ-algebra of μ-measurable sets (in the sense of Carathéodory). First we show, that the σ-finite case is easy. Therefore we turn to the Hausdorff measures, which are probably the most natural examples of non-σ-finite measures. We prove, that this question cannot be answered in ZFC.

As an application we give a short proof of the known statement that the existence of an $\mathcal{H}^1$-measurable Sierpiński set (see the definition below) is consistent with ZFC (this is proven for the so called 'one dimensional measures' in [DP, 3.11]).

Finally, we investigate the existence of two kinds of Sierpiński sets measurable with respect to Hausdorff measures:

Definition 0.2 A set $S \subset \mathbb{R}^2$ is a *Sierpiński set in the sense of measure* if S is (one dimensional) Lebesgue negligible on each vertical line, but co-negligible (that is, the complement of S is negligible) on each horizontal line. A set $S \subset \mathbb{R}^2$ is a *Sierpiński set in the sense of cardinality* if S is countable on each vertical line, but co-countable on each horizontal line.

On one hand, we prove that for $0 < d < 2$ the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of measure is independent of ZFC. (In the remaining cases the answer is trivial.) On the other hand, we show in ZFC that in the non-trivial cases ($0 < d \leq 2$) there exists no $\mathcal{H}^d$-measurable Sierpiński set in the sense of cardinality.

1 Measurable envelopes with respect to outer measures

If μ is a σ-finite outer measure on a set X, then it is easy to check that every subset of X has a measurable envelope (with respect to $\mathcal{A}_\mu$). It is also known

([Fr]), as the following example shows, that σ-finiteness is essential.

Example 1.1 *Put $X = \omega_2 \times \omega_2$ and let ν be the outer measure on X that is 0 for countable subsets and 1 otherwise. Define*

$$\mu(H) = \sum_{\alpha \in \omega_2} [\nu(H \cap (\{\alpha\} \times \omega_2)) + \nu(H \cap (\omega_2 \times \{\alpha\}))];$$

that is, let $\mu(H)$ be the number of uncountable horizontal and vertical sections. We claim that $\omega_1 \times \omega_2$ has no measurable envelope.

Proof One can easily see that μ is an outer measure, $H \in \mathcal{A}_\mu$ iff H is either countable or co-countable on each section, and $H \in (\mathcal{A}_\mu)_0$ iff H is countable on each section.

Suppose now that M is a measurable envelope of $\omega_1 \times \omega_2$. Clearly $M \cap (\{\alpha\} \times \omega_2)$ is countable for every $\alpha \in \omega_2 \setminus \omega_1$ and M is co-countable on each horizontal section. But this gives a contradiction, as for any $A \subset \omega_2 \setminus \omega_1$ of cardinal ω_1 we have that $M \cap (A \times \omega_2)$ is also of cardinal ω_1, but the projection to the second coordinate of this set is the whole ω_2. $\square$

2 Measurable envelopes with respect to Hausdorff measures

Let $\mathcal{H}^d$ denote the d-dimensional Hausdorff measure in $\mathbb{R}^n$. If μ is an outer measure, then instead of 'measurable envelope with respect to $\mathcal{A}_\mu$' we shall sometimes say 'measurable envelope with respect to μ'.

If $d = 0$ or $d > n$ then every subset of $\mathbb{R}^n$ is $\mathcal{H}^d$-measurable, hence every subset has a measurable envelope. If $d = n$ then $\mathcal{H}^d$ is σ-finite, therefore we get the same conclusion. In the remaining cases we have

Theorem 2.1 *The following statement is independent of ZFC: for all $n \in \mathbb{N}$ and $0 < d < n$ every subset of $\mathbb{R}^n$ has a measurable envelope with respect to $\mathcal{H}^d$.*

Before the proof we need two lemmas. (λ denotes the one-dimensional Lebesgue measure here.)

Lemma 2.2 *Let $B \subset \mathbb{R}^n$ be Borel such that $0 < \mathcal{H}^d(B) < \infty$. Then there exists a bijection f between B and the interval $I = [0, \mathcal{H}^d(B)]$ such that both f and its inverse preserve the Borel sets, measurable sets and zero sets, and which is measure preserving between the measure spaces $(B, \mathcal{H}^d)$ and (I, λ).*

Proof [Ke, 12.B] and [Ke, 17.41]. $\square$

Now we turn to the second lemma.

Definition 2.3 *add* $\mathcal{N}$ is the minimal cardinal κ for which there are κ Lebesgue negligible sets A_α ($\alpha < \kappa$) such that $\cup_{\alpha<\kappa} A_\alpha$ is of positive outer measure.

Remark Note that if the sets A_α ($\alpha < \lambda$) are Lebesgue measurable for some $\lambda <$ *add* $\mathcal{N}$, then so is their union $\cup_{\alpha<\lambda} A_\alpha$, as it can be shown by well-ordering the sets and noting that all but countably many of them must be almost covered by the preceding sets.

The following lemma is essentially contained in [Fe, 2.5.10].

Lemma 2.4 *Let $0 < d < n$ and suppose* add $\mathcal{N} = 2^\omega$. *Then there exists a disjoint family $\{M_\alpha : \alpha < 2^\omega\}$ of $\mathcal{H}^d$-measurable subsets of $\mathbb{R}^n$ of finite $\mathcal{H}^d$-measure, such that a set $H \subset \mathbb{R}^n$ is $\mathcal{H}^d$-measurable iff $H \cap M_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$.*

Proof Let $\{B_\alpha : \alpha < 2^\omega\}$ be be an enumeration of the Borel subsets of $\mathbb{R}^n$ of finite $\mathcal{H}^d$-measure, and put $M_\alpha = B_\alpha \setminus (\cup_{\beta<\alpha} B_\beta)$. These are clearly pairwise disjoint sets of finite $\mathcal{H}^d$-measure. Moreover *add* $\mathcal{N} = 2^\omega$ together with the above remark and Lemma 2.2 applied to B_α gives that M_α is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$. The other direction being trivial we only have to verify that if $H \subset \mathbb{R}^n$ is such that $H \cap M_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$, then H is itself $\mathcal{H}^d$-measurable. Let $A \subset \mathbb{R}^n$ arbitrary. We show that

$$\mathcal{H}^d(A) \geq \mathcal{H}^d(H \cap A) + \mathcal{H}^d(H^C \cap A).$$

We can obviously assume that $\mathcal{H}^d(A) < \infty$ and thus we can find a Borel set B such that $A \subset B$ and $\mathcal{H}^d(A) = \mathcal{H}^d(B)$, therefore $B = B_\alpha$ for some $\alpha < 2^\omega$. Thus it is sufficient to prove that $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable, as that would imply

$$\mathcal{H}^d(A) = \mathcal{H}^d((H \cap B_\alpha) \cap A) + \mathcal{H}^d((H \cap B_\alpha)^C \cap A),$$

but $(H \cap B_\alpha) \cap A = H \cap A$ and $(H \cap B_\alpha)^C \cap A = H^C \cap A$. In order to show that $H \cap B_\alpha$ is $\mathcal{H}^d$-measurable, note that $B_\alpha = \cup_{\beta \leq \alpha} M_\beta$, therefore $H \cap B_\alpha = \cup_{\beta \leq \alpha}(H \cap M_\alpha)$, which is again easily seen to be $\mathcal{H}^d$-measurable. $\square$

Definition 2.5 *non*$^*\mathcal{N}$ is the minimal cardinal κ such that in every subset of the reals of positive outer Lebesgue measure we can find a subset of positive outer Lebesgue measure and of cardinal $\leq \kappa$.

$cov\,\mathcal{N}$ is the minimal cardinal κ such that $\mathbb{R}$ can be covered by κ Lebesgue negligible sets.

Remark $non^*\mathcal{N} < cov\,\mathcal{N}$ is consistent with ZFC as it holds in the so called 'random real model', see [LM, Lemma 8].

Now we can turn to the proof of Theorem 2.1.

Proof First we show that $add\,\mathcal{N} = 2^\omega$ implies that for every $n \in \mathbb{N}$ and $0 < d < n$ every subset of $\mathbb{R}^n$ has a measurable envelope with respect to $\mathcal{H}^d$. (This proves that this statement is consistent, as $add\,\mathcal{N} = 2^\omega$ follows e.g. from CH or MA.) Fix n, d and $H \subset \mathbb{R}^n$. Let $\{M_\alpha : \alpha < 2^\omega\}$ be as in Lemma 2.4. As M_α is of finite measure for every $\alpha < 2^\omega$, we can find a $\mathcal{H}^d$-measurable set H_α such that $H \cap M_\alpha \subset H_\alpha \subset M_\alpha$ and $\mathcal{H}^d(H \cap M_\alpha) = \mathcal{H}^d(H_\alpha)$. We claim that $A = \cup_{\alpha<2^\omega} H_\alpha$ is a measurable envelope of H. Clearly A is $\mathcal{H}^d$-measurable by Lemma 2.4. Suppose $H \subset B \subset A$, B is $\mathcal{H}^d$-measurable and $C \subset A \setminus B$. We want to show that C is measurable, therefore it is sufficient to check that $C \cap M_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$, which is obvious, as it is of $\mathcal{H}^d$-measure zero.

Next we prove that for $n = 2$ and $d = 1$ it is consistent that there exists a subset of the plane without a $\mathcal{H}^1$-measurable envelope. We assume $non^*\mathcal{N} < cov\,\mathcal{N}$. One can easily find a set $A \subset \mathbb{R}$ of full outer measure and of cardinal $non^*\mathcal{N}$, and we claim that $A \times \mathbb{R}$ has no $\mathcal{H}^1$-measurable envelope. Otherwise, if M is such an envelope, then it is (one dimensional) Lebesgue measurable on each vertical and horizontal line, therefore it is Lebesgue negligible on all vertical lines over $\mathbb{R} \setminus A$ and co-negligible on all horizontal lines. As $non^*\mathcal{N} < cov\,\mathcal{N}$, $\mathbb{R} \setminus A$ is not negligible, hence we can choose a set $B \subset \mathbb{R} \setminus A$ of positive outer measure and of cardinal $non^*\mathcal{N}$. Then the projection of the set $(B \times \mathbb{R}) \cap M$ to the second coordinate consists of $non^*\mathcal{N}$ zero sets, on the other hand, it is the whole line, a contradiction. $\square$

Remark The second direction of this proof (the last paragraph, in which we show a set without a measurable envelope) is due to D. Fremlin ([Fr]). In fact, it is not much harder to see that $non^*\mathcal{N} < cov\,\mathcal{N}$ implies the existence of subsets of $\mathbb{R}^n$ without $\mathcal{H}^d$-measurable envelopes for any $0 < d \leq [\frac{n}{2}]$ and $n \geq 2$. Indeed, we can replace $\mathbb{R} \times \mathbb{R}$ by the square of a d dimensional Cantor set in $\mathbb{R}^{[\frac{n}{2}]}$ of positive and finite $\mathcal{H}^d$-measure, and repeat the above argument.

However, we do not know the answer to the following question.

Question 2.6 *Is it consistent that there exists a subset of $\mathbb{R}^n$ without a $\mathcal{H}^d$-measurable envelope for $n = 1$, $0 < d < 1$ and for $n \geq 2$, $[\frac{n}{2}] < d < n$?*

3 Hausdorff measurable Sierpiński sets

As all subsets of the plane are $\mathcal{H}^d$-measurable for $d = 0$ or $d > 2$, the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of cardinality for these d-s is equivalent to the existence of Sierpiński sets in the sense of cardinality, which is known to be equivalent to CH (see [Tr]). The following theorem answers the question for the other d-s.

Theorem 3.1 *For $0 < d \leq 2$ there exists no $\mathcal{H}^d$-measurable Sierpiński set in the sense of cardinality.*

Proof For $d = 2$ the statement is obvious by the Fubini Theorem. Let $0 < d < 2$ and $C_{\frac{d}{2}}$ be a symmetric self-similar Cantor set in $[0,1]$ of dimension $\frac{d}{2}$.

Lemma 3.2 *There exists $0 < c < \infty$, such that*

$$\mathcal{H}^d|C_{\frac{d}{2}} \times C_{\frac{d}{2}} = c\ (\mathcal{H}^{\frac{d}{2}}|C_{\frac{d}{2}} \times \mathcal{H}^{\frac{d}{2}}|C_{\frac{d}{2}}).$$

Proof Since $K = C_{\frac{d}{2}} \times C_{\frac{d}{2}}$ is also self-similar, it is easy to see that $0 < \mathcal{H}^d(K) < \infty$, therefore if we let

$$c = \frac{\mathcal{H}^d(K)}{(\mathcal{H}^{\frac{d}{2}}(C_{\frac{d}{2}}))^2},$$

then $0 < c < \infty$ and the two above outer measures agree on K. By the self-similarity of K they also agree on the basic open sets, and as any open set is the disjoint union of countably many of these, the outer measures agree on all open sets. Hence (by finiteness) on all Borel sets as well, from which the lemma follows. $\square$

Now we can complete the proof of Theorem 3.1 as follows. Note that if S is an $\mathcal{H}^d$-measurable Sierpiński set in the sense of cardinality, then $S \cap K$ is a $\mathcal{H}^d$-measurable set, which is countable on each vertical section of K and co-countable on each horizontal section of K. But this gives a contradiction, once we apply the previous lemma and the Fubini Theorem. $\square$

The question concerning Sierpiński sets in the sense of measure is more complicated. Just as above, for $d = 0$ or $d > 2$ $\mathcal{H}^d$-measurability is not really a restriction, and we know that the existence of Sierpiński sets in the sense of measure is independent of ZFC (see [La, Theorem 2]). For $d = 2$ no such set can be $\mathcal{H}^d$-measurable by the Fubini Theorem, while in the remaining cases we have the following.

Theorem 3.3 *For $0 < d < 2$ the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of measure is independent of ZFC.*

Proof On one hand, for example $non^*\mathcal{N} < cov\,\mathcal{N}$ implies that there are no Sierpiński sets of any kind ([La, Theorem 2]).

On the other hand, we assume $add\ \mathcal{N} = 2^\omega$ and prove the existence of $\mathcal{H}^d$-measurable Sierpiński sets in the sense of measure, separately for all $0 < d < 1$, $d = 1$ and all $1 < d < 2$.

If $d = 1$, then our statement is a consequence of [DP, 3.10], but we present another proof here. By [La, Theorem 2] and $add\ \mathcal{N} = 2^\omega$ we can find a Sierpiński set in the sense of measure, and by 2.1 this set has a $\mathcal{H}^d$-measurable envelope. It is not hard to check, that this envelope possesses the required properties.

Now let $0 < d < 1$. Enumerate the Borel subsets of $\mathbb{R}^2$ of positive finite $\mathcal{H}^d$-measure as $\{B_\alpha : \alpha < 2^\omega\}$ and also $\mathbb{R}$ as $\{x_\alpha : \alpha < 2^\omega\}$. We can assume, that S is a Sierpiński set in the sense of measure, such that the cardinality of every vertical section is less than 2^ω (the proof in [La] provides such a set).

Then put

$$S_1 = S \cup \bigcup_{\alpha < 2^\omega} [B_\alpha \setminus (\bigcup_{\beta<\alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}))].$$

S_1 is a Sierpiński set in the sense of measure as its horizontal sections contain the horizontal sections of S, the vertical section over x is still of Lebesgue measure zero, since $add\,\mathcal{N} = 2^\omega$, and $x = x_\alpha$ for some $\alpha < 2^\omega$ so this section is increased only in the first α steps, and always by a set of finite $\mathcal{H}^d$-measure, therefore of zero Lebesgue measure. What remains to check is that S_1 is $\mathcal{H}^d$-measurable. Like above, by the Borel regularity of $\mathcal{H}^d$ it is sufficient to show that

$$\mathcal{H}^d(B) = \mathcal{H}^d(S_1 \cap B) + \mathcal{H}^d(S_1^C \cap B)$$

holds for every Borel set B of finite and positive $\mathcal{H}^d$-measure, therefore we only have to prove that $S_1 \cap B_\alpha$ is $\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$. We show this by induction on α as follows. Put

$$A_\alpha = \bigcup_{\beta<\alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}).$$

Then

$$S_1 \cap B_\alpha = S_1 \cap [(B_\alpha \setminus A_\alpha) \cup (B_\alpha \cap A_\alpha)] =$$

$$[S_1 \cap (B_\alpha \setminus A_\alpha)] \cup [S_1 \cap (B_\alpha \cap A_\alpha)] = [B_\alpha \setminus A_\alpha] \cup [S_1 \cap (B_\alpha \cap A_\alpha)] =$$

$$[B_\alpha \setminus A_\alpha] \cup [\bigcup_{\beta<\alpha} (B_\alpha \cap B_\beta \cap S_1)] \cup [\bigcup_{\beta<\alpha} (B_\alpha \cap (\{x_\beta\} \times \mathbb{R}) \cap S_1)].$$

Now we apply Lemma 2.2 to B_α in view of $add\,\mathcal{N} = 2^\omega$. Then the first expression in the last line is clearly $\mathcal{H}^d$-measurable, and the same holds for the second expression by our inductional hypotheses. In order to see this for the last one we note that $B_\alpha \cap (\{x_\beta\} \times \mathbb{R}) \cap S$ is of cardinal less than 2^ω for every $\beta < \alpha$, thus $\mathcal{H}^d$-negligible, but when we construct $B_\alpha \cap (\{x_\beta\} \times \mathbb{R}) \cap S_1$ out of this set, we increase it only in the first β steps, and always by a $\mathcal{H}^d$-measurable set.

Finally, let $1 < d < 2$. As above, let $\{B_\alpha : \alpha < 2^\omega\} = \{B \subset \mathbb{R} : B \text{ Borel}, \mathcal{H}^d(B) < \infty\}$ and also $\{x_\alpha : \alpha < 2^\omega\} = \mathbb{R}$. Put

$$D_\alpha = B_\alpha \setminus [\bigcup_{\beta<\alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}) \cup (\mathbb{R} \times \{x_\beta : \beta < \alpha\})]$$

for every $\alpha < 2^\omega$. Since $D_\alpha \subset B_\alpha$, D_α is (two dimensional) Lebesgue negligible, therefore $\{x \in \mathbb{R} : \lambda((\{x\} \times \mathbb{R}) \cap D_\alpha) > 0\}$ is Lebesgue negligible, thus contained in a Borel set N_α of Lebesgue measure zero. Define

$$S_1 = S \cup [\bigcup_{\alpha<2^\omega} (D_\alpha \setminus (N_\alpha \times \mathbb{R}))] \setminus [\bigcup_{\alpha<2^\omega} (D_\alpha \cap (N_\alpha \times \mathbb{R}))].$$

First we check that S_1 is a Sierpiński set in the sense of measure. If $\mathbb{R} \times \{x\}$ is a horizontal line, then $x = x_\alpha$ for some $\alpha < 2^\omega$. D_ξ and $\mathbb{R} \times \{x_\alpha\}$ are disjoint for every $\xi > \alpha$, therefore our set is not modified after the first α steps. Hence

$$[S \setminus S_1] \cap (\mathbb{R} \times \{x_\alpha\}) \subset [\bigcup_{\xi\leq\alpha} (D_\xi \cap (N_\xi \times \mathbb{R}))] \cap (\mathbb{R} \times \{x_\alpha\}),$$

which is Lebesgue negligible on this horizontal line by $add\,\mathcal{N} = 2^\omega$. Similarly, on a vertical line $\{x_\alpha\} \times \mathbb{R}$

$$S_1 \cap (\{x_\alpha\} \times \mathbb{R}) \subset [S \cup \bigcup_{\xi\leq\alpha} (D_\xi \setminus (N_\xi \times \mathbb{R}))] \cap (\{x_\alpha\} \times \mathbb{R}),$$

which is again a zero set. What remains to show is the $\mathcal{H}^d$-measurability of S_1. It is again sufficient to prove by induction on α that $S_1 \cap B_\alpha$ is

$\mathcal{H}^d$-measurable for every $\alpha < 2^\omega$. Just like above, we apply Lemma 2.2 to B_α.

$$S_1 \cap B_\alpha =$$

$$[S_1 \cap D_\alpha] \cup [S_1 \cap B_\alpha \cap (\bigcup_{\beta<\alpha} B_\beta \cup (\{x_\beta : \beta < \alpha\} \times \mathbb{R}) \cup (\mathbb{R} \times \{x_\beta : \beta < \alpha\}))],$$

where the expression in the first brackets equals $D_\alpha \setminus (N_\alpha \times \mathbb{R})$, which is clearly $\mathcal{H}^d$-measurable, while the second expression equals

$$B_\alpha \cap ([\bigcup_{\beta<\alpha} (B_\beta \cap S_1)] \cup [(\{x_\beta : \beta < \alpha\} \times \mathbb{R}) \cap S_1] \cup [(\mathbb{R} \times \{x_\beta : \beta < \alpha\}) \cap S_1]),$$

from which the $\mathcal{H}^d$-measurability follows, since $B_\beta \cap S_1$ is $\mathcal{H}^d$-measurable for every $\beta < \alpha$ by the inductional hypothesis, moreover $\{x_\beta : \beta < \alpha\} \times \mathbb{R}$ as well as $\mathbb{R} \times \{x_\beta : \beta < \alpha\}$ is of $\mathcal{H}^d$-measure zero for every $\beta < \alpha$. $\square$

However, we do not know the answer to the following.

Question 3.4 *Let $0 < d < 2$. Is it consistent, that there exists a Sierpiński sets in the sense of measure, but it cannot be $\mathcal{H}^d$-measurable?*

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