Covering locally compact groups by less than 2^ω many translates of a compact nullset

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Abstract

Gruenhage asked if it was possible to cover the real line by less than continuum many translates of a compact nullset. Under the Continuum Hypothesis the answer is obviously negative. Elekes and Steprāns [ES] gave an affirmative answer by showing that if C_{EK} is the well known compact nullset considered first by Erdős and Kakutani [EK] then $\mathbb{R}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of C_{EK}. As this set has no analogue in more general groups, it was posed as an open question in [ES] whether such a result holds for uncountable locally compact Polish groups. In this paper we give an affirmative answer in the abelian case.

More precisely, we show that if G is a nondiscrete locally compact abelian group in which every open subgroup is of index at most $\mathrm{cof}(\mathcal{N})$ then there exists a compact set C of Haar measure zero such that G can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of C. This result, which is optimal in a sense, covers the cases of uncountable compact abelian groups and of nondiscrete separable locally compact abelian groups.

We use Pontryagin's duality theory to reduce the problem to three special cases; the circle group, countable products of finite discrete abelian groups, and the groups of p-adic integers, and then we solve the problem on these three groups separately.

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In addition, using representation theory, we reduce the nonabelian case to the classes of Lie groups and profinite groups, and we also settle the problem for Lie groups.[1]

1 Introduction

Under the Continuum Hypothesis the real line obviously cannot be covered by less than 2^ω many translates of a set of Lebesgue measure zero. On the other hand, it is well known that in some models of set theory there exists such a covering [BJ]. Moreover, we can obviously assume that the set is G_δ. Gruenhage [Gr] asked whether such a covering can be constructed with an F_σ or closed or compact nullset (using of course some extra set-theoretic assumption).

Question 1.1 *(Gruenhage) Let $C \subset \mathbb{R}$ be a compact set of Lebesgue measure zero and $A \subset \mathbb{R}$ be of cardinality less than 2^ω. Does that imply $C + A \neq \mathbb{R}$?*

We remark that it is well known that in some models of set theory $\mathbb{R}$ can be covered by less than 2^ω many compact nullsets ([BJ] or [BS]), but in these coverings the sets are not translates of each other.

We also remark that already [Mi] considers cardinal invariants of closed measure zero sets, and [MS], [Pa] and [Sh] deals with 'translative cardinal invariants'; that is, when the small sets considered are translates of each other.

Gruenhage gave an affirmative answer to Question 1.1 when C is the classical Cantor set [Gr], and later Darji and Keleti [DK] generalized his results to the class of compact nullsets of so called packing dimension less than 1.

Then Elekes and Steprāns [ES] answered all versions of Gruenhage's question in the negative as follows.

[1] M. Abért recently gave an affirmative answer for profinite groups [Ab].

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Definition 1.2 Denote

$$C_{EK} = \left\{ \sum_{n=2}^{\infty} \frac{d_n}{n!} \;\middle|\; d_n \in \{0, 1, \ldots, n-2\} \; \forall n \right\}.$$

The letters E and K stand for Erdős and Kakutani.

Definition 1.3 Let $\mathcal{N}$ denote the set of Lebesgue nullsets of the real line, and let $\mathrm{cof}(\mathcal{N}) = \min\{|\mathcal{H}| : \mathcal{H} \subset \mathcal{N}, \forall N \in \mathcal{N}\ \exists H \in \mathcal{H}, N \subset H\}$.

It is not hard to see that $\omega < \mathrm{cof}(\mathcal{N}) \leq 2^{\omega}$ [BJ].

Theorem 1.4 *(Elekes-Steprāns) $\mathbb{R}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of the compact nullset C_{EK}.*

As $\mathrm{cof}(\mathcal{N}) < 2^{\omega}$ is consistent with the axioms of set theory [BJ], we obtain the following.

Corollary 1.5 *It is consistent with the axioms of set theory that less than continuum many translates of a compact set of measure zero cover the real line.*

As C_{EK} has no analogue in more general groups, it was posed as an open question in [ES] whether such a result holds for uncountable locally compact Polish groups. The main goal of this paper is to show that the answer is affirmative in the abelian case (Corollary 2.7). Note that countable locally compact groups are not interesting from this viewpoint, and also that the assumption that the group is Polish is also natural, since our problem actually considers a cardinal invariant (see [BJ]), and this topic is usually discussed in the framework of Polish spaces.

First we use Pontryagin's duality theory to reduce the problem to three special cases; the circle group, countable products of finite discrete groups, and the groups of p-adic integers, then we solve the problem separately for these groups.

In Section 3 we discuss the nonabelian version of our problem. We reduce the nonabelian case to the cases of Lie groups and profinite groups, and we show that every nondiscrete Lie group in which every open subgroup is of index at most $\mathrm{cof}(\mathcal{N})$ can be covered by $\mathrm{cof}(\mathcal{N})$ many left translates of a compact set of Haar measure zero.

Note that a set is of left Haar measure zero iff it is of right Haar measure zero.

All groups are tacitly assumed to be Hausdorff.

Remark 1.6 The referee pointed out the following interesting facts.
1. Our method of reducing the problem to some special groups is fairly general. Therefore it may well be applicable to show that all locally compact groups possess a certain property, supposing that whenever a factor group G/H has the property then G itself has it as well.
2. The use of $\mathrm{cof}(\mathcal{N})$ is not optimal, one can show that consistently it can be improved. In fact, it could be replaced with the the least cardinality κ for which for every pair $f, g : \omega \to \omega$ converging to infinity every f-slalom can be covered by κ many g-slaloms (see Definition 2.9). However, as this is not a very well known invariant (there is no abbreviation for it), and most probably this is also not optimal, we still prefer to use $\mathrm{cof}(\mathcal{N})$.
3. Question 1.1 is closely related to the following, which essentially asks whether the set of translations we use can be arbitrary. *Is it true that for every uncountable $X \subset \mathbb{R}$ there exists a countable set Y and a closed nullset F such that $(X + Y) + F = \mathbb{R}$?* On can easily show that this is in fact equivalent to the following. *Is it true that for every uncountable $X \subset \mathbb{R}$ there exists an F_σ nullset A so that $X + A = \mathbb{R}$?* On can very easily give a consistent negative answer to these questions (e.g. if $\mathrm{cov}(\mathcal{N}) = 2^\omega > \omega_1$), but a negative answer in ZFC would be interesting. On the other hand, a consistent affirmative answer would prove the consistency of the so called Borel Conjecture + Dual Borel Conjecture, which is a longstanding open problem.

2 The abelian case

Remark 2.1 It may be instructive to bear in mind that the proof (just as in Section 3 in the nonabelian case) will consist of two parts. First we prove a purely *analysis* result constructing a compact nullset and showing that every so called 'slalom' can be covered by a translate of the set, then we apply a purely *set-theoretic* result stating that consistently less than 2^ω many slaloms can cover the space.

A topological group is LCA if it is locally compact and abelian.

Definition 2.2 We say that a locally compact group G is *nice* if there exists a compact set $C \subset G$ of Haar measure zero such that G can be covered by $\mathrm{cof}(\mathcal{N})$ many left translates of C.

The aim of this section is to prove the following.

Theorem 2.3 *Suppose that G is a nondiscrete LCA group in which every open subgroup is of index at most* $\mathrm{cof}(\mathcal{N})$*. Then G is nice; that is, there exists a compact set $C \subset G$ of Haar measure zero such that G can be covered by* $\mathrm{cof}(\mathcal{N})$ *many translates of C.*

Remark 2.4 Both conditions of the theorem are necessary. First, if G is discrete then the only nullset is the emptyset, so no covering by nullsets exists. Secondly, if there is an open subgroup of index κ then at least κ many compact nullsets are needed to cover G, since a compact set can only intersect finitely many cosets.

In fact, as "$\mathrm{cof}(\mathcal{N}) = \omega_1$ and $2^\omega = \omega_2$" is consistent with the axioms of set theory [BJ], we actually obtain the following consistent characterisation.

Corollary 2.5 *It is consistent with the axioms of set theory that an LCA group G can be covered by less than 2^ω many translates of a compact nullset iff G is nondiscrete and has no open subgroup of index at least 2^ω.*

Before the proof of Theorem 2.3 we formulate two more corollaries.

Corollary 2.6 *Every uncountable compact abelian group and every nondiscrete separable LCA group is nice; that is, it can be covered by* $\mathrm{cof}(\mathcal{N})$ *many translates of a compact nullset.*

As $\mathrm{cof}(\mathcal{N}) < 2^\omega$ is consistent with the axioms of set theory [BJ], and as every Polish space is separable, we obtain the following, which answers Question 3.2 in [ES] in the abelian case.

Corollary 2.7 *It is consistent with the axioms of set theory that every uncountable locally compact abelian Polish group can be covered by less than 2^ω many translates of a compact nullset.*

In the rest of the section we prove Theorem 2.3. First we need two technical lemmas.

Lemma 2.8 *Let $n \geq 0$ be an integer, G be a finite group, A and S be subsets of G such that $\left(1 - \frac{1}{n+3}\right)|G| \leq |A|$ and $|S| \leq n+2$. Then there exists $g \in G$ such that $S \subset gA$.*

Proof. Clearly $S \not\subset gA$ iff $g \in S(G \setminus A)^{-1}$. So it is enough to check that $S(G \setminus A)^{-1} \neq G$, which is clear, since

$$|S(G \setminus A)^{-1}| \leq |S| \cdot |(G \setminus A)| \leq (n+2)\frac{|G|}{n+3} < |G|.$$

$\square$

For a sequence $(X_n)_{n\in\mathbb{N}}$ of set $\times_{n\in\mathbb{N}} X_n$ denotes the Cartesian product.

Definition 2.9 For every $n \in \mathbb{N}$ let X_n be an arbitrary set, and fix a function $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$. An *$f$-slalom* is a set of the form

$$S = \times_{n\in\mathbb{N}} S_n, \text{ where } S_n \subset X_n, \ |S_n| \leq f(n) \ \ (n \in \mathbb{N}).$$

Lemma 2.10 *Let $f_0 : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ be such that $\lim_\infty f_0 = \infty$, and let X_n $(n \in \mathbb{N})$ be countable sets. Then $\times_{n\in\mathbb{N}} X_n$ can be covered by $\mathrm{cof}(\mathcal{N})$ many f_0-slaloms.*

Proof. [BJ, 2.3.9] states that there exist a system of functions $f_\alpha : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ $(\alpha < \mathrm{cof}(\mathcal{N}))$ with $\sum_{n\in\mathbb{N}^+} f_\alpha(n)/n^2 < \infty$, and for every $\alpha < \mathrm{cof}(\mathcal{N})$ there exists an f_α-slalom $S_\alpha = \times_{n\in\mathbb{N}}(S_\alpha)_n \subset \mathbb{N}^\mathbb{N}$ such that these slaloms cover $\mathbb{N}^\mathbb{N}$ mod finite; that is, for every $g \in \mathbb{N}^\mathbb{N}$ there exists $\alpha < \mathrm{cof}(\mathcal{N})$ such that $\{n \in \mathbb{N} : g(n) \notin (S_\alpha)_n\}$ is finite. For an f-slalom $S \subset \mathbb{N}^\mathbb{N}$ let $\mathcal{S}_S = \{S' \subset \mathbb{N}^\mathbb{N} : S' \text{ is an } f\text{-slalom, and } \{n \in \mathbb{N} : S_n \neq S'_n\} \text{ is finite}\}$. Clearly, every $\mathcal{S}_S$ is countable, and hence $\cup_{\alpha<\mathrm{cof}(\mathcal{N})} \mathcal{S}_{S_\alpha}$ is easily seen to be a set of $\mathrm{cof}(\mathcal{N})$ many slaloms actually covering $\mathbb{N}^\mathbb{N}$. So we can assume that $\cup_{\alpha<\mathrm{cof}(\mathcal{N})} S_\alpha = \mathbb{N}^\mathbb{N}$. Put $f(n) = n^2 + 1$. Clearly, $\{n \in \mathbb{N} : f_\alpha(n) > f(n)\}$ is finite for every α, and therefore an argument similar to the previous one yields that every f_α-slalom can be covered by countably many f-slaloms. So $\mathbb{N}^\mathbb{N}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many f-slaloms.

[GL, 2.10.] states that if $f, g : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ are such that $\lim_\infty f = \lim_\infty g = \infty$, then the minimal number of f-slaloms needed to cover $\mathbb{N}^\mathbb{N}$ equals the minimal number of g-slaloms needed to cover $\mathbb{N}^\mathbb{N}$. Therefore $\mathbb{N}^\mathbb{N}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many f_0-slaloms, hence $\times_{n\in\mathbb{N}} X_n$ can also be covered by $\mathrm{cof}(\mathcal{N})$ many f_0-slaloms. $\square$

In order to prove Theorem 2.3 first we need to prove it in two special cases; for countable products of finite discrete (abelian) groups and for the groups of p-adic integers.

For a sequence $(G_n)_{n\in\mathbb{N}}$ of compact groups $\otimes_{n\in\mathbb{N}} G_n$ is the (Cartesian) product group with the product topology.

Theorem 2.11 *For every $n \in \mathbb{N}$ let G_n be a discrete finite group of at least 2 elements. Then $\otimes_{n\in\mathbb{N}} G_n$ is nice.*

Proof. Write $\mathbb{N}$ as the disjoint union of finite sets N_n such that $2^{|N_n|} > 2(n+3)$, and define $G'_n = \otimes_{k\in N_n} G_k$. Then $\otimes_{n\in\mathbb{N}} G_n = \otimes_{n\in\mathbb{N}} G'_n$ and $|G'_n| > 2(n+3)$. Hence for every $n \in \mathbb{N}$ we can find an $A_n \subset G'_n$ such that

$$\left(1 - \frac{1}{n+3}\right) |G'_n| \leq |A_n| \leq \left(1 - \frac{1}{2(n+3)}\right) |G'_n|.$$

Define $C = \times_{n\in\mathbb{N}} A_n$. Then C is clearly compact, and $\Pi_{n\in\mathbb{N}} \left(1 - \frac{1}{2(n+3)}\right) = 0$ implies that C is of Haar measure zero.

Put $f_0(n) = n+2$ ($n \in \mathbb{N}$). By Lemma 2.10 $\otimes_{n\in\mathbb{N}} G'_n$ can be covered by $\mathrm{cof}(\mathcal{N})$ many f_0-slaloms. We will complete the proof by showing that every f_0-slalom $S = \times_{n\in\mathbb{N}} S_n \subset \otimes_{n\in\mathbb{N}} G'_n$ can be covered by a left translate of C.

For every $n \in \mathbb{N}$ we can apply Lemma 2.8 to n, G'_n, A_n and S_n, and so we obtain a $g_n \in G'_n$ such that $S_n \subset g_n A_n$. But then for $g = (g_n)_{n\in\mathbb{N}} \in \otimes_{n\in\mathbb{N}} G'_n$ we have $S \subset \times_{n\in\mathbb{N}}\, gA_n = gC$. $\square$

We need certain properties of the p-adic integers $\mathbb{Z}_p$ that we collect here for the convenience of the reader. For a precise treatment see e.g. [Ro]. The underlying topological space is $\{0, 1, \ldots, p-1\}^{\mathbb{N}}$ equipped with the product topology (each factor is considered discrete). Addition is coordinatewise with carried digits from the n^{th} coordinate to the $n+1^{st}$; that is, if $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p$ then $(x+y)_0 = x_0 + y_0$ if $x_0 + y_0 \leq p-1$ while $(x+y)_0 = x_0 + y_0 - p$ if $x_0 + y_0 \geq p$. In the second case when calculating $(x+y)_1$ we add 1 to $x_1 + y_1$ and then check whether the sum is greater than $p-1$, etc, recursively.

Theorem 2.12 *For every prime p the group of p-adic integers $\mathbb{Z}_p$ is nice.*

Proof. If we forget about the group operation then we can write $\mathbb{Z}_p = \times_{n\in\mathbb{N}} X_n$, where $X_n = \{0, 1, \ldots, p-1\}$ for every $n \in \mathbb{N}$.

Write $\mathbb{N}$ as the disjoint union of the finite *intervals* $[k_n, k_{n+1})$, where $\{k_n\}_{n\in\mathbb{N}}$ is a strictly monotone increasing sequence of nonnegative integers such that $p^{k_{n+1}-k_n} > 2(n+3)$. Define $X'_n = \times_{k\in[k_n,k_{n+1})} X_k$. Then $\times_{n\in\mathbb{N}} X_n = \times_{n\in\mathbb{N}} X'_n$ and $|X'_n| > 2(n+3)$. As above, for every $n \in \mathbb{N}$ we can find an $A_n \subset X'_n$ such that

$$\left(1 - \frac{1}{n+3}\right) |X'_n| \leq |A_n| \leq \left(1 - \frac{1}{2(n+3)}\right) |X'_n|.$$

Let $C = \times_{n\in\mathbb{N}} A_n$. Again, C is compact and is of Haar measure zero.

Put $f_0(n) = \lfloor \frac{n+2}{2} \rfloor$ $(n \in \mathbb{N})$. ($\lfloor x \rfloor$ is the integer part of x.) By Lemma 2.10 $\times_{n\in\mathbb{N}} X'_n$ can be covered by $\mathrm{cof}(\mathcal{N})$ many f_0-slaloms. We will complete the proof by showing that every f_0-slalom $S = \times_{n\in\mathbb{N}} S_n \subset \times_{n\in\mathbb{N}} X'_n$ can be covered by a translate of C.

For every $n \in \mathbb{N}$ we define a new group G_n (*not* a subgroup of $\mathbb{Z}_p$) as follows. Let $G_n = X'_n = \times_{k\in[k_n,k_{n+1})} X_k$, and for $x = (x_k)_{k\in[k_n,k_{n+1})} \in G_n$ and $y = (y_k)_{k\in[k_n,k_{n+1})} \in G_n$ put

$$(x +_{G_n} y)_k = (x +_{\mathbb{Z}_p} y)_k \text{ for every } k \in [k_n, k_{n+1});$$

that is, we always forget about the last carried digit. One can check that G_n with this addition is indeed a group. For example, to avoid all calculations, it is easy to see that this group is (canonically isomorphic to) $p^{k_n}\mathbb{Z}_p/p^{k_{n+1}}\mathbb{Z}_p$ and also to $p^{k_n}\mathbb{Z}/p^{k_{n+1}}\mathbb{Z}$, but we will not use this fact.

Put $1_n = \chi_{\{k_n\}}$. (χ_H is the characteristic function of the set H.) Fix $n \in \mathbb{N}$, and set $\tilde{S}_n = S_n \cup (S_n +_{G_n} 1_n)$. As $|S_n| \le \lfloor \frac{n+2}{2} \rfloor$, we clearly have $|\tilde{S}_n| \le n + 2$, hence we can apply Lemma 2.8 to n, G_n, A_n and $\tilde{S}_n$, and so we obtain a $g_n \in G_n = \times_{k\in[k_n,k_{n+1})} X_k$ such that $\tilde{S}_n \subset A_n +_{G_n} g_n$. Let x_n be the inverse of g_n in G_n, then $\tilde{S}_n +_{G_n} x_n \subset A_n$. Put $x = (x_n)_{n\in\mathbb{N}} \in \times_{n\in\mathbb{N}} X'_n$. We claim that $S +_{\mathbb{Z}_p} x \subset C$, which will complete the proof. Fix $s = (s_n)_{n\in\mathbb{N}} \in S$. When we recursively calculate the digits of $s +_{\mathbb{Z}_p} x$, we need to show that for every $n \in \mathbb{N}$ we have $\left((s +_{\mathbb{Z}_p} x)_k\right)_{k\in[k_n,k_{n+1})} \in A_n$, but this is clear, as $\left((s +_{\mathbb{Z}_p} x)_k\right)_{k\in[k_n,k_{n+1})}$ equals either $\left((s_n +_{G_n} x_n)_k\right)_{k\in[k_n,k_{n+1})}$ or $(s_n +_{G_N} x_n +_{G_n} 1_n)_k)_{k\in[k_n,k_{n+1})}$, depending on whether there is a carried digit at k_n or not. $\square$

Before proving Theorem 2.3 we need an algebraic fact about abelian groups. It is formulated in Theorem 2.16, which is well known, e.g. a more general version appears in [KR], but for the sake of completeness we include a proof below.

Definition 2.13 Let G be a group. For every $n \in \mathbb{N}$ let $G_{p^n} = \{g \in G : p^n g = 0\}$, and also let $G_{p^\infty} = \cup_{n\in\mathbb{N}} G_{p^n}$. We say that G is a *p-group* if $G = G_{p^\infty}$.

Definition 2.14 Let p be a prime. An abelian group G is called *quasicyclic* if it is generated by a sequence $(g_n)_{n\in\mathbb{N}}$ with the property that $g_0 \neq 0$ and $pg_{n+1} = g_n$ for every $n \in \mathbb{N}$. For a fixed prime p the unique (up to isomorphism) quasicyclic group is denoted by C_{p^∞}.

Note that $C_{p^\infty} = (\mathbb{Q}/\mathbb{Z})_{p^\infty} = (\mathbb{R}/\mathbb{Z})_{p^\infty}$.

Lemma 2.15 *Let p be a prime and G be an infinite abelian p-group such that G_{p^n} is finite for every $n \in \mathbb{N}$. Then G contains C_{p^∞} as a subgroup.*

Proof. We define a graph on G as follows. For every nonzero $g \in G$ we connect pg with g. The obtained graph is clearly a tree (with root 0) in which each node has finitely many immediate successors by the finiteness of the G_{p^n}'s. So by König's lemma [Ku, 5.7] the tree has an infinite branch, which clearly generates a quasicyclic subgroup. $\square$

For a sequence $(G_n)_{n\in\mathbb{N}}$ of abelian groups $\oplus_{n\in\mathbb{N}} G_n$ is the direct sum group (that is, those elements of the product that only have finitely many nonzero coordinates) with the discrete topology.

Theorem 2.16 *Every infinite abelian group G contains a subgroup isomorphic to one of the following.*

1. *$\mathbb{Z}$,*
2. *$\oplus_{n\in\mathbb{N}} G_n$, where each G_n is a finite abelian group of at least two elements,*
3. *C_{p^∞} for some prime p.*

Proof. If G contains an element of infinite order then G contains $\mathbb{Z}$ as a subgroup. Therefore we may assume that G is a torsion group.

Every torsion group is the direct sum of p-groups; $G = \oplus_{p \text{ prime}} G_{p^\infty}$ [Fu, 2.1].

Suppose that $|G_{p^\infty}| \geq 2$ for infinitely many primes p. For every such p we can find a finite nontrivial subgroup of G_{p^∞}, and hence we have a sequence $(G_n)_{n\in\mathbb{N}}$ of finite nontrivial groups such that $\oplus_{n\in\mathbb{N}} G_n \subset G$. So we may assume that $|G_{p^\infty}| = 1$ for all but finitely many primes. As G is infinite, there is a prime p for which G_{p^∞} is infinite.

Assume that G_p is infinite. Then G_p is clearly an infinite dimensional vector field over $\mathbb{F}_p$, therefore it contains $\oplus_{n\in\mathbb{N}} C_p$ as a subgroup. (C_p is the cyclic group of p elements.)

So we may assume that G_p is finite. Then we claim that G_{p^n} is also finite for every $n \in \mathbb{N}$. We prove this by induction on n. The map $g \mapsto pg$ is a homomorphism of $G_{p^{n+1}}$ into G_{p^n} with kernel G_p, so $|G_{p^{n+1}}| \leq |G_{p^n}| \cdot |G_p|$, which finishes the induction. Hence we can apply Lemma 2.15 to G_{p^∞}, and we obtain that $C_{p^\infty} \subset G_{p^\infty} \subset G$. This finishes the proof. $\square$

The following lemma is crucial.

Lemma 2.17 *Let G be a locally compact group and $H \subset G$ a compact normal subgroup. If G/H is nice then G is also nice.*

Proof. Let μ_G be a left Haar measure on G, and let $\pi : G \to G/H$ be the canonical homomorphism. Then by [Ha, §63. Thm. C.] $\mu_G \circ \pi^{-1}$ is a left Haar measure on G/H. This shows that the inverse image of a nullset in G/H under π is a nullset in G. Moreover, [Ha, §63. Thm. B.] states that the inverse image of a compact set under π is also compact.

Hence if $C \subset G/H$ is a compact nullset witnessing that G/H is nice then $\pi^{-1}(C) \subset G$ is a compact nullset witnessing that G is also nice. $\square$

Remark 2.18 The following example shows that the lemma does not hold in general, that is, when H is a closed normal subgroup. Let H be a discrete group of cardinality greater than $\mathrm{cof}(\mathcal{N})$ and let $G = H \times \mathbb{R}$. Then G/H is nice by Theorem 1.4, but G is not nice as every compact set intersects only finitely many cosets.

Now we are ready to prove our main theorem.

Proof. (Theorem 2.3) By the Principal Structure Theorem of LCA Groups [Ru, 2.4.1], G has an open subgroup H which is of the form $H = K \otimes \mathbb{R}^n$, where K is a compact subgroup and $n \in \mathbb{N}$. By assumption the index of H is at most $\mathrm{cof}(\mathcal{N})$, so it suffices to prove that H is nice, so we can assume $G = H$.

Suppose $n \geq 1$. By [ES, 2.1] $\mathbb{R}$ is nice, let C be the compact nullset witnessing this fact. Then it is easy to see that $K \times C \times [0,1]^{n-1}$ witnesses that $G = K \otimes \mathbb{R}^n$ is nice. Hence we can assume $n = 0$, so G is compact.

By Lemma 2.17 it is sufficient to find a closed subgroup $H \subset G$ such that G/H is nice. By [Ru, 2.1.2] (and by the Pontryagin Duality Theorem [Ru, 1.7.2]) factors of G are the same as (isomorphically homeomorphic to) dual groups of closed subgroups of $\hat{G}$. As G is compact, $\hat{G}$ is discrete [Ru, 1.2.5]. Hence it suffices to find a subgroup $M \subset \hat{G}$ such that $\hat{M}$ is nice.

By Theorem 2.16 $\hat{G}$ has a subgroup isomorphic either to $\mathbb{Z}$, or to $\oplus_{n\in\mathbb{N}} G_n$, where each G_n is a finite abelian group of at least two elements, or to C_{p^∞} for some prime p. We need to show that the duals of these groups are nice.

By [ES, 2.1] $\mathbb{R}$ is nice, which easily implies that the circle group $\mathbb{T}$ is also nice, so we are done in the first case, since $\hat{\mathbb{Z}} = \mathbb{T}$.

In the second case note that $\hat{G}$ is finite iff G is finite, hence each $\hat{G}_n$ is finite. By [Ru, 2.2.3] the dual of a direct sum (equipped with the discrete topology) is the direct product of the dual groups (equipped with the product topology), so $(\oplus_{n\in\mathbb{N}} G_n)\hat{} = \otimes_{n\in\mathbb{N}} \hat{G}_n$, which is nice by Theorem 2.11.

Finally, the third case is settled by Theorem 2.12, since by [HR, 25.2] $\hat{C}_{p^\infty} = \mathbb{Z}_p$. □

3 The nonabelian case

The aim of this section is to reduce the general case to the case of profinite groups; that is, inverse limits of finite groups.[2]

Theorem 3.1 *Suppose that every infinite profinite group is nice.*[2] *Then every nondiscrete locally compact group in which every open subgroup is of index at most* $\mathrm{cof}(\mathcal{N})$ *is also nice; that is, there exists a compact set* C *of Haar measure zero such that the group can be covered by* $\mathrm{cof}(\mathcal{N})$ *many left translates of* C.

Similarly to Corollary 2.5 we also have the following.

Corollary 3.2 *Suppose that every infinite profinite group is nice. Then it is consistent with the axioms of set theory that a locally compact group* G *can be covered by less than* 2^ω *many translates of a compact nullset iff* G *is nondiscrete and has no open subgroup of index at least* 2^ω.

The main goal of this section is to prove Theorem 3.1. We start with the Lie case. We use [MZ] as the main reference, so note that Lie groups are *not* assumed to be second countable.

Theorem 3.3 *Every nondiscrete Lie group in which the identity component has index at most* $\mathrm{cof}(\mathcal{N})$ *is nice; that is, it can be covered by* $\mathrm{cof}(\mathcal{N})$ *many left translates of a compact set of Haar measure zero.*

[2]We have been informed by M. Abért that he recently proved that every infinite profinite group is nice [Ab].

Proof. Let G be a Lie group as in the theorem. We can clearly assume that G is connected. Every compact neighbourhood of e (the identity of G) generates an open σ-compact subgroup, moreover, every open subgroup is actually clopen. As G is connected, we obtain that G is σ-compact, hence it possesses the Lindelöf property. Therefore it suffices to show that there is a neighbourhood of the identity that can be covered by $\mathrm{cof}(\mathcal{N})$ many left translates of a compact set of Haar measure zero.

Every nondiscrete Lie group contains one-parameter subgroups; that is; a continuous homomorphic (not necessarily closed) image of $\mathbb{R}$, see e.g. [MZ, 2.22.]. Let $H \subset G$ be the closure of such a subgroup, then H is a closed connected commutative subgroup of G. By [MZ, 4.11.] each closed subgroup of a Lie group is itself a Lie group, and so H is actually a submanifold. If $G = H$, we can apply Theorem 2.3, so we can assume that H is a proper subgroup. Let M be a submanifold transversal to H so that $dim(H) + dim(M) = dim(G)$ and

$$H \cap M = \{e\}. \tag{1}$$

Lemma 3.4 *There is a compact set $K \subset M$ which is a neighbourhood of e (in M), so that if $m : H \times K \to G$ is the restriction of the multiplication map then*

(i) $m(H \times K) = HK$ is a neighbourhood of e

(ii) $m : H \times K \to HK$ is a homeomorphism.

Proof. It is well-known that if we use the exponential map as a chart then the derivative of the $G \times G \to G$ multiplication map attains the form $(x, y) \mapsto x + y$ in the tangent spaces [Wa]. This implies that the derivative of m is nonsingular at (e, e). Hence by the inverse function theorem m is a diffeomorphism in a neighbourhood of (e, e). More precisely, there exist open neighbourhoods U, V and W of e in H, M and G, respectively, so that the restriction of m is a smooth bijection of $U \times V$ onto $UV = W$.

This shows that (i) holds for any choice of K that is a neighbourhood of e.

Now we claim that

$$H \cap W \subset U. \tag{2}$$

Indeed, if $h = uv \in H \cap UV$ then $v = u^{-1}h$. As $U \subset H$ we obtain $v \in H$, and as $V \subset M$ by (1) we get $v = e$, so $h = u$.

Choose a compact neighbourhood $K \subset V$ of e in M so that

$$KK^{-1} \subset W. \tag{3}$$

We claim that K satisfies (ii), which will finish the proof of the lemma.

First we show that the map $m : H \times K \to HK$ is injective. If $h_1k_1 = h_2k_2$ then $h_2^{-1}h_1 = k_2k_1^{-1}$. Set $h = h_2^{-1}h_1 = k_2k_1^{-1}$, then clearly $h \in H$, and by (3) we also have $h \in W$, hence by (2) we obtain $h \in U$.

Now we apply the fact that m is a bijection between $U \times K$ and UK (as $K \subset V$) to the elements $hk_1 = ek_2$. Indeed, $h, e \in U$ and $k_1, k_2 \in K$, so $k_1 = k_2$ and $h_1 = h_2$, proving that M is injective.

Finally, we show that the inverse of m is also continuous. We use again the fact that m is a smooth bijection between $U \times K$ and UK. So let U_1, K_1 be neighbourhoods of some $h \in H$ and $k \in K$, respectively, then $h^{-1}U_1 \times K_1k^{-1}$ is a neighbourhood of (e, e). Hence its image $h^{-1}U_1K_1k^{-1}$ contains a neighbourhood W_1 of e. Thus hW_1k is a neighbourhood of hk contained in the image of $U_1 \times K_1$ under m, proving that the inverse of m is also continuous. $\square$

Now we complete to proof of Theorem 3.3.

Fix a Haar measure μ_H on H, and consider a compact nullset C in H as in Theorem 2.3. The set CK is compact, and $\mathrm{cof}(\mathcal{N})$ many left translates of CK cover HK, which is a neighbourhood of e in G. Therefore the proof of the theorem will be complete once we show the following.

Lemma 3.5 *CK is of μ_G-measure zero, where μ_G is a left Haar measure on G.*

Proof. By the above lemma the multiplication map $H \times K \to HK$ is a homeomorphism, hence BK is Borel for every Borel set $B \subset H$. So we can define the set-function

$$\mu : B \mapsto \mu_G(BK), \quad (B \subset H \text{ Borel}).$$

It is easy to see that this is a left-invariant measure which is finite for compact sets. We check that if $A \subset H$ is a nonempty open (in H) set then $\mu(A) > 0$. Let $a \in A$, then $a^{-1}A$ is a neighbourhood of e in H. Clearly

$\mu(A) = \mu_G(AK) = \mu_G(a^{-1}AK) > 0$, where the last expression is positive, since $a^{-1}AK$ is a neighbourhood of e in G.

By the uniqueness of Haar-measure [Ke, 17.B], there exists $c > 0$, so that $\mu = c\mu_H$, and so $\mu_G(CK) = \mu(C) = c\mu_H(C) = 0$. This concludes the proof of the lemma, and hence of the theorem. $\square$

$\square$

Remark 3.6 Lemma 3.5 also follows from the construction of Haar measure via an invariant smooth volume form, but we decided to use this alternative approach, which establishes the lemma in a more direct fashion.

The proof of the above theorem with minor modifications shows that if H is a closed subgroup of a separable Lie group G, and H can be covered by κ many left translates of a compact nullset then G can also be covered by κ many left translates of a compact nullset. It would be interesting to see if this remains true in general, and if so, if it could be used to establish our main theorem for profinite groups.

Next we consider the compact case. The following fact is most probably well known. It was communicated to us by Ken Kunen.

Statement 3.7 *Every infinite compact group has a factor which is either an infinite Lie group or an infinite profinite group.*

Proof. More precisely we show that if G is an infinite compact group then either it has an infinite Lie group factor or G itself is profinite.

Denote by $U(n)$ the unitary group on $\mathbb{C}^n$. By the Peter-Weyl theorem [HR, 27.40] the set of all representations of G in the $U(n)$'s separate points of G, hence G is (isomorphic to) the inverse limit of the images of these representations. If all these images are finite then G is profinite, and we are done. Otherwise G has a factor that is an infinite compact subgroup of some $U(n)$. But by [MZ, 4.11] each closed subgroup of a Lie group is itself a Lie group, so we are done. $\square$

Now we are ready to prove Theorem 3.1.

Definition 3.8 We say that a topological group *does not contain arbitrarily small subgroups* if there is a neighbourhood of the identity that contains no subgroup apart from the one formed by the identity element itself.

The identity component of G is denoted by G_0.

Proof. (Theorem 3.1) First note that if H is a closed normal subgroup in a topological group G and every open subgroup of G is of index at most $\mathrm{cof}(\mathcal{N})$ then the same is true for G/H and also for every open subgroup of G.

Suppose that G is a nondiscrete locally compact group in which every open subgroup is of index at most $\mathrm{cof}(\mathcal{N})$. We have to cover G by at most $\mathrm{cof}(\mathcal{N})$ many left translates of a compact left nullset. By [MZ, 4.5. Cor.] (actually, in every locally compact group G) there exists an open subgroup $G' \subset G$ and a compact normal subgroup H of G' such that G'/H does not contain arbitrarily small subgroups.

G' is clearly nondiscrete, since G is nondiscrete and G' is open. As the index of G' is at most $\mathrm{cof}(\mathcal{N})$, it is sufficient to cover G' by at most $\mathrm{cof}(\mathcal{N})$ many left translates of a compact left nullset, hence we can assume $G = G'$.

So H is a compact subgroup of G such that G/H does not contain arbitrarily small subgroups.

Now we separate two cases. First assume that H is open. It suffices to show that H is nice. As above, H cannot be discrete, so it is infinite. By Statement 3.7 either H has an infinite profinite factor, in which case we are done by assumption (and by Lemma 2.17), or H has a factor which is an infinite Lie group. But an infinite compact Lie group is clearly nondiscrete, and every open subgroup has finite index, so we are done with this case by Theorem 3.3 (and again by Lemma 2.17).

So we can assume that H is not open, hence G/H is not discrete. By Lemma 2.17 it is sufficient to show that G/H is nice. By [MZ, 4.2. Cor. 2] if a locally compact group does not contain arbitrarily small subgroups then the identity component is open, hence $(G/H)_0$ is open in G/H. By the remark at the beginning of the proof the index condition holds for G/H too, so it is sufficient to show that $(G/H)_0$ is nice.

As G/H does not contain arbitrarily small subgroups, the same holds for the subgroup $(G/H)_0$. By [MZ, 4.4. Thm.] a connected locally compact group that does not contain arbitrarily small subgroups is a Lie group, and clearly all these requirements hold for $(G/H)_0$. Moreover, as G/H is nondiscrete, the same holds for the open subgroup $(G/H)_0$. Hence Theorem 3.3 shows that $(G/H)_0$ is nice, finishing the proof. $\square$

We conclude with some natural questions. Theorem 3.1 shows that the first two are equivalent.

Question 3.9 *Can we drop the assumption in Theorem 2.3 that the group* *is abelian?*[3]

Or equivalently,

Question 3.10 *Suppose G is an infinite profinite group. Is G nice? That is, can G be covered by* $\mathrm{cof}(\mathcal{N})$ *many left translates of a compact set of Haar measure zero?*[3]

Of course in both questions it is also natural to replace $\mathrm{cof}(\mathcal{N})$ by $< 2^\omega$. In that case one can show that these questions are also equivalent to the original Question 3.2 in [ES].

Our last question is a reformulation of [ES, Question 3.4].

Question 3.11 *Suppose that κ is a cardinal and G_1, G_2 are uncountable locally compact (abelian) separable (Polish) groups such that G_1 can be covered by κ many translates of a suitably chosen compact nullset. Is the same true for G_2?*

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References

[Ab] M. Abért, Less than continuum many translates of a compact nullset may cover any infinite profinite group, preprint.

[BJ] T. Bartoszyński and H. Judah: *Set Theory: On the Structure of the Real Line.* A. K. Peters, Wellesley, Massachusetts, 1995.

[BS] T. Bartoszyński and S. Shelah, Closed measure zero sets, *Ann. Pure Appl. Logic* **158** (1992), no. 2, 93–110.

[DK] U. B. Darji and T. Keleti, Covering $\mathbb{R}$ with translates of a compact set, *Proc. Amer. Math. Soc.* **131** (2003), no. 8, 2593–2596.

[3] We have been informed by M. Abért that he recently proved that every infinite profinite group is nice, which answers these questions affirmatively [Ab].

[ES] M. Elekes and J. Steprāns, Less than 2^ω many translates of a compact nullset may cover the real line, *Fund. Math.* **181** (2004), no. 1, 89–96.

[EK] P. Erdős and S. Kakutani, On a perfect set, *Colloquium Math.* **4** (1957), 195–196.

[Fu] L. Fuchs: *Abelian Groups.* Pergamon Press, 1960.

[Gr] G. Gruenhage, unpublished notes and personal communication.

[GL] G. Gruenhage and R. Levy, Covering ${}^\omega\omega$ by special Cantor sets, *Comment. Math. Univ. Carolin.* **43** (2002), no. 3, 497–509.

[Ha] P. R. Halmos: *Measure theory.* Springer, 1974.

[HR] E. Hewitt and K. A. Ross: *Abstract harmonic analysis.* Vol. I., Springer, 1963.

[Ke] A. S. Kechris: *Classical Descriptive Set Theory.* Springer, 1995.

[Ku] K. Kunen: *Set theory. An introduction to independence proofs.* North-Holland, 1983.

[KR] K. Kunen and W. Rudin, Lacunarity and the Bohr topology, *Math. Proc. Cambridge Philos. Soc.* **126** (1999), no. 1, 117–137.

[Mi] A. W. Miller, Some properties of measure and category, *Trans. Amer. Math. Soc.* **266** (1981), no. 1, 93–114.

[MS] A. W. Miller and J. Steprāns, The number of translates of a closed nowhere dense set required to cover a Polish group, preprint, `www.math.wisc.edu/~miller/res/index.html` or `www.math.yorku.ca/~steprans/Research/menu.html`.

[MZ] D. Montgomery and L. Zippin: *Topological transformation groups.* Interscience Publishers, 1955.

[Pa] J. Pawlikowski, Powers of transitive bases of measure and category, *Proc. Amer. Math. Soc.* **93** (1985), no. 4, 719–729.

[Ro] A. M. Robert: *A course in p-adic analysis.* Springer, 2000.

[Ru] W. Rudin: *Fourier analysis on groups.* John Wiley & Sons, 1990.

[Sh] S. Shelah, Every null-additive set is meager-additive, *Israel J. Math.* **89** (1995), no. 1-3, 357–376.

[Wa] F. W. Warner: *Foundations of differentiable manifolds and Lie groups. Corrected reprint of the 1971 edition.* Springer, 1983.

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