Linearly Ordered Families of Baire 1 Functions

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Abstract

We consider the set of Baire 1 functions endowed with the pointwise partial ordering and investigate the structure of the linearly ordered subsets.

Introduction

Any set $\mathcal{F}$ of real valued functions defined on an arbitrary set X is partially ordered by the pointwise ordering, that is $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. In other words put $f < g$ iff $f(x) \leq g(x)$ for all $x \in X$ and $f(x) \neq g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible order types of the linearly ordered (or simply 'ordered' from now on) subsets of this partially ordered set, which is the same as to characterize the ordered sets that are similar to an ordered subset of $\mathcal{F}$. Here two ordered sets are said to be similar iff there exists an order preserving bijection between them, and such a bijection from an ordered set onto an ordered subset of $\mathcal{F}$ is often referred to as a 'representation' of the ordered set. We sometimes say that the set is represented 'on X'. An ordered set similar to a representable one is also representable, so we can talk about 'representable order types' as well.

Since the functions in an ordered set are somehow 'above each other', one could think that this ordered set must be similar to a subset of the real line. As we shall see this is far from being true.

The problem of finding long sequences in $\mathcal{F}$, that is representing big ordinals has been studied for a long time. It was Miklós Laczkovich who posed the question how one can characterize the representable ordered sets, particularly in the case when $X = \mathbb{R}$ and $\mathcal{F}$ is the set of Baire 1 functions. What makes this problem interesting is that the corresponding questions about continuous (that is Baire 0) and Baire α functions ($\alpha > 1$) are completely solved. In the continuous case an ordered set is representable iff it is similar to a subset of $\mathbb{R}$ (an easy exercise), and for $\alpha > 1$ the question has turned out to be independent of ZFC, that is the usual axioms of set theory [Ko].

The known facts about the case $\alpha = 1$ are the followings. The first is a classical theorem of Kuratowski asserting that there is no increasing or decreasing sequence of length ω_1 of real Baire 1 functions [Ku, §24. III.2'], that is ω_1 is not representable (in the sequel representable will always mean representable by real Baire 1 functions). The other is Péter Komjáth's Theorem stating that no Souslin line is representable [Ko]. (A Souslin line is a non-separable ordered set that does not contain an uncountable family of pairwise disjoint open intervals, that is ccc but not separable. The existence of Souslin lines is independent of ZFC [Je, Theorems 48,50].)

The main goal of this paper is to present a few constructions of representable ordered sets which show that Kuratowski's Theorem is 'not too far' from being a characterization. In Section 2 we prove that certain operations result representable order types, and then in Section 3 and 4 we show that everything is representable that can be built up by certain steps, like forming countable products or replacing points by ordered sets.

We would also like to point out that if we restrict ourselves to the case of characteristic functions, we arrive at the problem of families of sets linearly ordered by inclusion. Indeed, $\chi_A < \chi_B$ iff $A \subsetneqq B$. The case of real Baire 1 functions corresponds to the problem of representing ordered sets by ambiguous subsets of the real line. (A set is called ambiguous iff it is F_σ and G_δ at the same time.) It is not hard to check that almost everything proved in this paper is valid for this case as well, moreover, a kind of characterization of ordered sets that are representable by ambiguous sets is given in the last section.

For a topological space X the set of order types representable by real valued Baire 1 functions is denoted by $\mathcal{R}(X)$. The set of order types representable by ambiguous subsets is denoted by $\mathcal{R}_0(X)$.

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1 Preliminaries

We shall frequently use the following simple lemma.

Lemma 1.1

(i) Let X and Y be metric spaces, $f : X \to \mathbb{R}$ Baire 1 and $g : Y \to X$ continuous. Then $f \circ g : Y \to \mathbb{R}$ is Baire 1.

(ii) Let X be a metric space and $X_n \subset X$ $(n \in \mathbb{N})$ F_σ sets such that $X = \bigcup_{n=1}^{\infty} X_n$. If $f : X \to \mathbb{R}$ is relatively Baire 1 on each X_n $(n \in \mathbb{N})$ then f is Baire 1.

Let us first consider the following question, which shall be a useful tool in the sequel. Which Polish spaces are equivalent to the real line in the sense that the same ordered sets can be represented on them? We shall ignore the countable metric spaces as it is easy to see that if an order type is representable on such a space then it is similar to a subset of the real line. Denote by C the Cantor set.

Theorem 1.2 *$R(X) = R(C) = R(\mathbb{R})$ for any σ-compact uncountable metric space X.*

Proof It is obviously enough to prove the first equality. Let X be compact for the time being, then a classical theorem asserts that there exists a continuous surjection $F : C \to X$ [Ku, §41, VI.3a]. If $\{f_\alpha : \alpha \in \Gamma\}$ is an ordered set of Baire 1 functions defined on X, one can easily verify that $\{f_\alpha \circ F : \alpha \in \Gamma\}$ is also ordered, similar to the former ordered set as a consequence of the surjectivity of F and consists of Baire 1 functions defined on C by lemma 1.1.

In the general case $X = \cup_{n=1}^{\infty} X_n$ where $X_n \subset X$ is compact and let again be $\{f_\alpha : \alpha \in \Gamma\}$ an ordered set of Baire 1 functions on X. We shall show that this set is representable on the interval $[0,1]$ and therefore on C as well, since $[0,1]$ is a compact metric space and we can apply what we have proven in the previous case.

Fix a set $H_n \subset (\frac{1}{n}, \frac{1}{n+1})$ for each $n \in \mathbb{N}$ homeomorphic to the Cantor set and also a homeomorphism $g_n : H_n \to C$. We can choose furthermore continuous surjections $F_n : C \to X_n$ $(n \in \mathbb{N})$ since X_n is a compact metric space. Now we represent the set in the following way. For each $\alpha \in \Gamma$ let

$$g_\alpha = \begin{cases} f_\alpha \circ F_n \circ g_n & \text{on } H_n \quad (n \in \mathbb{N}) \\ 0 & \text{on } [0,1] \setminus \cup_{n=1}^{\infty} H_n. \end{cases}$$

Indeed, the map $g_\alpha \mapsto f_\alpha$ $(\alpha \in \Gamma)$ turns out to be a similarity as $F_n \circ g_n$ is surjective and moreover in view of Lemma 1.1 it is straightforward to verify that g_α is a Baire 1 function on $[0,1]$ for each $\alpha \in \Gamma$.

In order to check the opposite direction let $\{f_\alpha : \alpha \in \Gamma\}$ be an ordered set of Baire 1 functions on the Cantor set. According to a classical theorem every uncountable compact metric space contains a subspace homeomorphic to C [Ku, §36, V.1], which easily generalizes to the case of uncountable σ-compact metric spaces since if $X = \cup_{n=1}^{\infty} X_n$, X_n compact, then at least one X_n is uncountable. We can therefore fix a homeomorphism $h : C \to Y \subset X$ and for $\alpha \in \Gamma$ let

$$g_\alpha = \begin{cases} f_\alpha \circ h^{-1} & \text{on } Y \\ 0 & \text{on } X \setminus Y. \end{cases}$$

One can easily prove in the above manner that this is an ordered set of Baire 1 functions similar to the above one. ■

The above theorem implies the surprising fact that all the complicated ordered sets represented in the following sections are also representable by functions of connected graphs.

Corollary 1.3 *A representable ordered set is also representable by Darboux Baire 1 functions and consequently by Baire 1 functions of connected graphs.*

Proof It is well-known that the graph of a Baire 1 function is connected iff it is Darboux [Br, II.1.1]. By the previous theorem we can assume that the set is represented on the Cantor set. It is not hard to extend the representing functions by a common continuous function to the complement of the Cantor set which makes the representing functions Darboux and Baire 1 by Lemma 1.1. ■

Next we show that there are at most two distinct possible sets $\mathcal{R}(X)$ for all uncountable Polish spaces X.

Theorem 1.4 $R(X) = R(\mathbb{R} \setminus \mathbb{Q})$ *for any non-σ-compact Polish space* X.

Proof We apply the argument of Theorem 1.2. In one direction we use that every Polish space is the continuous image of the irrationals [Ku, §36, II.1], while in the other direction we apply Hurewicz's Theorem [Ke, Theorem 7.10] asserting that every non-σ-compact Polish space contains a homeomorphic copy of the irrationals as a closed subspace. ■

This leaves the question open whether all uncountable Polish spaces are equivalent or not.

Question 1.5 *Does* $R(C) = R(\mathbb{R} \setminus \mathbb{Q})$ *hold?*

Remark In order to give an affirmative answer it would be enough to prove that every ordered set of Baire 1 functions on the irrationals can be represented by Baire 1 functions on the reals. Indeed, on one hand every uncountable Polish space contains a subset which is homeomorphic to the Cantor set [Ku, §36, V.1], and on the other hand every Polish space is the continuous image of $\mathbb{R} \setminus \mathbb{Q}$ hence the above argument works.

Moreover, it can be shown that a Baire 1 function defined on the irrationals can be extended to the reals as a Baire 1 function, but so far we were unable to do this in an order preserving way.

2 Operations on representable ordered sets

Now we investigate whether the class of representable sets are closed under certain operations. We shall make use of these operations when constructing complicated representable ordered sets.

Definition 2.1 For an arbitrary ordered set X we call $X \times \{0, 1\}$ with the lexicographical ordering the *duplication of* X.

Question 2.2 *Is it true that the duplication of a representable set is also representable?*

In most cases this question can be replaced by the following statement.

Statement 2.3 *Let* X *be an ordered set such that the duplication of* X *is representable. Then so is the ordered set obtained by replacing every* $x \in X$ *by a representable set* Y_x*, that is* $\{(x, y) : x \in X, y \in Y_x\}$ *with the lexicographical ordering.*

Proof First we replace the points of the real line by uncountable closed sets in the following way. Let $P : [0,1] \to [0,1]^2$ be a Peano curve, that is a continuous surjection, and let P_1 be its first coordinate function. Then $P_1 : [0,1] \to [0,1]$ is also a continuous surjection, moreover the preimages $P_1^{-1}(\{c\})$ are uncountable closed sets for all $c \in [0,1]$. In virtue of Theorem 1.2 we may assume that the duplication of X is represented on $[0,1]$ by the pairs of functions $f_x < g_x$ $(x \in X)$. If we consider the functions $f_x \circ P_1$ and $g_x \circ P_1$ we obtain a similar ordered set of Baire 1 functions, but in the latter set any two distinct elements differ on an uncountable closed sets, for if f_x and g_x attained different values at c_x then $f_x \circ P_1$ and $g_x \circ P_1$ differ on $P_1^{-1}(\{c_x\})$. Since this is a compact metric space we may assume that Y_x is represented on it. By composing with a increasing homeomorphism between $\mathbb{R}$ and the interval $(f_x(c_x), g_x(c_x))$ we also can assume that the functions representing Y_x only attain values between $f_x(c_x)$ and $g_x(c_x)$.

Now we claim that the following representation will do. For $x \in X$ and $y \in Y_x$ let

$$h_{(x,y)} = \begin{cases} f_x \circ P_1 & \text{on } [0,1] \setminus P_1^{-1}(\{c_x\}) \\ \text{the function representing y} & \text{on } P_1^{-1}(\{c_x\}). \end{cases}$$

These functions are easily seen to be Baire 1 so what remains to show is that the representation is order preserving. In the first case $x_1 < x_2$ so $f_{x_1} < g_{x_2}$ hence

$$h_{(x_1,y_1)} < g_{x_1} \circ P_1 < f_{x_2} \circ P_1 < h_{(x_2,y_2)}.$$

Finally, in the second case $x_1 = x_2 = x$ and $y_1 < y_2$. Obviously $h_{(x,y_1)}$ and $h_{(x,y_2)}$ differ on $P_1^{-1}(\{c_x\})$ only, where they are defined according to the ordering of Y_x thus $h_{(x,y_1)} < h_{(x,y_2)}$. ■

Statement 2.4 *Let X be an ordered set such that the duplication of X is representable. Then X^ω endowed with the lexicographical ordering is also representable.*

Proof As in the previous proof we can represent the duplication of X such that for every $x \in X$ the representing functions $f_x,\ g_x : \mathbb{R} \to [0,1]$ are different constant functions on a suitable Cantor set C_x. Denote d_x the difference of these two values. In the next step, for every fixed $x_1 \in X$ let us represent the duplication of X on C_{x_1} in the same manner as above, that is for each $x_2 \in X$ let $f_{x_1,x_2},\ g_{x_1,x_2} : \mathbb{R} \to [0, \min(\frac{1}{2}, d_{x_1})]$ be zero outside C_{x_1} such that they are different constants on a suitable Cantor set $C_{x_1,x_2} \subset C_{x_1}$. Let d_{x_1,x_2} denote the difference of the two values. Then we proceed inductively and make sure that $0 \le f_{x_1,\dots,x_{n+1}},\ g_{x_1,\dots,x_{n+1}} \le \min(\frac{1}{2^n}, d_{x_1,\dots,x_n})$. It is not hard to see that

$$(x_1, x_2, \dots) \mapsto \sum_{n=1}^{\infty} f_{x_1,\dots,x_n}$$

is the required representation, as the uniform limit of Baire 1 functions is Baire 1 itself [Ku, §31, VIII.2]. ■

Remark Instead of using the same set X at each level, we can prove in exactly the same way that if the duplication of X_n is representable for every $n \in \mathbb{N}$ then so is $\prod_{n=1}^{\infty} X_n$, and more generally we can also use different sets at a level, that is we can correspond a set $X_{x_1,\ldots,x_n}$ to each $x_1, \ldots, x_n$.

However, we do not know the answer to the question concerning longer products. As a simple transfinite induction shows, the following two questions are equivalent.

Question 2.5 *Is it true, that if the duplication of X is representable, then the duplication of X^ω is also representable? Or equivalently, is it true, that if the duplication of X is representable, then so is X^α for every $\alpha < \omega_1$?*

Corollary 2.6 *Suppose that the duplications of representable orderings are also representable. Then X^α is representable for every representable X and $\alpha < \omega_1$.*

Proof We prove this by induction on α. If $\alpha = \beta + 1$ then X^α is similar to $X^\beta \times X$. But X^β is representable by the inductional hypothesis, so is its duplication by our assumption, therefore we can apply Statement 2.3 and we are done.

If α is a limit ordinal, then $[0, \alpha)$ can be written as the disjoint union of $[\alpha_n, \alpha_{n+1})$ for a suitable sequence α_n $(n \in \mathbb{N})$. The interval $[\alpha_n, \alpha_{n+1})$ is similar to an ordinal $\beta_n < \alpha$, so X^α is similar to $\prod_{n=1}^{\infty} X^{\beta_n}$, and we are again done by the previous remark. ■

Remark As above, we can generalize this result as well to $\prod_{\beta<\alpha} X^\beta$ and also to the case when at each level we correspond an arbitrary representable set to each point.

Next we pose another question.

Question 2.7 *Is it true that the completion (as an ordered set) of a representable ordered set is also representable?*

Definition 2.8 Let X and X_n $(n \in \mathbb{N})$ be ordered sets. We say that X is *a blend of the sets X_n* if there exist pairwise disjoint subsets $H_n \subset X$ $(n \in \mathbb{N})$ such that $X = \cup_{n=1}^{\infty} H_n$ and H_n is similar to X_n.

Statement 2.9 *Suppose that duplications and completions of representable sets are also representable. Then so is a blend X of the representable sets X_n.*

Proof Let H_n be as in the definition. By the hypothesis the completion of $H_n \times \{0, 1\}$ is representable for each $n \in \mathbb{N}$ and we may assume that it is represented on the interval $(n, n + 1)$. Let $x \in X$, that is $x \in H_n$ for exactly one n, and let

$$f_x = \begin{cases} \text{the function representing } (x, 0) & \text{on } (n, n+1) \\ \text{the function representing} & \\ \sup\{(y, i) \in H_m \times \{0, 1\} : y \leq x\} & \text{on } (m, m+1) \text{ } if \text{ } m \neq n \\ 0 & \text{elsewhere,} \end{cases}$$

where 'sup' means supremum according to the ordering of the completion of $H_m \times \{0,1\}$. f_x is Baire 1 as the usual argument shows so we only have to check that this latter set of functions is similar to the original one. Let $x, y \in X$, $x < y$ and $x \in H_k$, $y \in H_l$ for some k and l. If $k = l$ then $f_x < f_y$ is obvious while if $k \neq l$ then one can easily check that $f_x \leq f_y$ on $(k, k+1)$, $(l, l+1)$ and on the complement of their union, moreover $f_x \neq f_y$ on $(k, k+1)$ since f_y is not less here then the function representing $(x, 1)$. ■

3 The first construction

In the sequel we present a few constructions of representable sets which have such a rich structure in some sense that we may hope to be able to produce all the representable order types this way.

Definition 3.1 Let α be an ordinal number and $I = [0,1]$. We denote by I^α the set of transfinite sequences in I of length α with the lexicographical ordering (i.e. $I^\alpha = \{f : f : \alpha \to I\}$ and $f < g$ iff $f(\gamma) = g(\gamma)$ and $f(\beta) < g(\beta)$ for some β and every $\gamma < \beta$).

When $\alpha \geq \omega_1$, then due to Kuratowski's Theorem [Ku, §24, III.2'], I^α is not representable as it contains a subset of type ω_1. However the following holds.

Theorem 3.2 *I^α is representable for all $\alpha < \omega_1$.*

Proof For $\alpha < \omega$ the assertion follows from Statement 2.3 by induction. Denote by $H = \prod_{n=0}^{\infty}[0,1]$ the Hilbert cube, that is the topological product of countably many copies of the closed unit interval. It is well-known that H is a compact metric space so it is sufficient to represent I^α on H. We show that this is possible even by characteristic functions, in other words there exists a system of ambiguous subsets of H which is of order type I^α when ordered by inclusion. First we define an ordering of type I^α on H. As $\alpha < \omega_1$ there exists a bijection $\varphi : \mathbb{N} \to \alpha$ so we can assign to each element $a = (a_1, a_2, \dots) \in H$ a transfinite sequence $x = (a_{\varphi(n)} : n \in \mathbb{N})$. Since this is a bijection between H and I^α it induces an ordering of type I^α on H which we shall denote by $<_H$. We claim that the sets of the form $H_x = \{y \in H : y <_H x\}$ constitute a system of sets possessing all the properties we need. First of all $H_x \subsetneqq H_y$ iff $x <_H y$ thus $\{H_x : x \in H\}$ is of order type I^α. We still have to check that $H_x \subset H$ is ambiguous for all $x \in H$. First we show that it is F_σ. Indeed,

$$H_x = \bigcup_{\beta<\alpha} \left(\bigcap_{\gamma<\beta} \left\{(y_1, y_2, \dots) \in H : y_{\varphi^{-1}(\gamma)} = x_{\varphi^{-1}(\gamma)}\right\} \cap \left\{y_{\varphi^{-1}(\beta)} < x_{\varphi^{-1}(\beta)}\right\} \right)$$

so it is sufficient to check that the members of the union are F_σ sets, but this is obvious as they are intersections of certain closed sets and an open set.

Similarly $\{y \in H : x <_H y\}$ is also F_σ, and as $\{x\}$ is F_σ, H_x is the complement of an F_σ set hence G_δ. ■

In view of Kuratowski's Theorem it is natural to ask whether every representable set can be embedded into I^α for a suitable $\alpha < \omega_1$. We show in two steps that this is not true.

Lemma 3.3 *$I^{\alpha+1}$ cannot be embedded into I^α for any $\alpha < \omega_1$.*

Proof Suppose indirectly that $f : I^{\alpha+1} \to I^\alpha$ is an order-preserving injection and let $f = (f_0, f_1, \ldots, f_\beta, \ldots)$ where $f_\beta : I^{\alpha+1} \to I$ $(\beta < \alpha)$ are the coordinate functions. As $f_0 : I^{\alpha+1} \to I$ is monotone, and for distinct values of $c \in I$ the convex hulls of the sets $f_0(\{x_0, \ldots, x_\beta, \ldots, x_\alpha : x_0 = c\})$ are non-overlapping intervals in I, all but countably many of them are singletons. Therefore we can fix a_0 such that $f_0((a_0, x_1, \ldots, x_\beta, \ldots, x_\alpha))$ is constant. Once we have already chosen a_γ for each $\gamma < \beta$ such that $f_\gamma((a_0, \ldots, a_\gamma, x_{\gamma+1}, \ldots, x_\alpha))$ is constant then as before for distinct values of x_β we obtain essentially pairwise disjoint image sets and thus we can fix $a_\beta \in I$ such that $f_\beta((a_0, \ldots, a_\beta, x_{\beta+1}, \ldots, x_\alpha))$ is constant. But then eventually we get

$$f((a_0, \ldots, a_\beta, \ldots, 0)) = f((a_0, \ldots, a_\beta, \ldots, 1)),$$

contradicting the injectivity of f. ■

Statement 3.4 *There exists a representable set that is not embeddable into I^α for any $\alpha < \omega_1$.*

Proof The duplication of the real line is representable as it is similar to a subset of I^2, hence if we replace $\aleph_1$ arbitrary points of $\mathbb{R}$ by the sets I^α $(\alpha < \omega_1)$ we obtain a representable set. In virtue of the previous lemma and Statement 2.3 this set possesses the required property. ■

This negative result shows how to go on to find new representable sets by iteration.

Definition 3.5 Let $\mathcal{H}$ be an arbitrary set of ordered sets. We define an increasing transfinite sequence S_α $(\alpha \in On)$ of sets as follows.

Let $S_0 = \mathcal{H} \cup \{\emptyset\}$ and S_α be the set of ordered sets that can be obtained by replacing the points of a set $X \in \bigcup_{\beta<\alpha} S_\beta$ by sets $Y_x \in \bigcup_{\beta<\alpha} S_\beta$ $(x \in X)$.

Finally, let $\mathcal{S}(\mathcal{H})$ denote the set of order types of $\bigcup_{\alpha \in On} S_\alpha$.

Lemma 3.6 *$\mathcal{S}(\mathcal{H})$ is a set indeed as there exists an ordinal α such that $S_\beta = S_\alpha$ for every $\beta \geq \alpha$.*

Proof Let κ be a infinite cardinal such that $|H| \leq \kappa$ for every $H \in \mathcal{H}$. A simple transfinite induction shows that $|X| \leq \kappa$ for all $X \in S_\alpha$ and $\alpha \in On$. We choose a cardinal μ of cofinality greater than κ (e.g. 2^κ), and claim that $\alpha = \mu$ will do.

First we show that $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. Choose $X \in S_\alpha$, that is $Y, Z_y \in \bigcup_{\beta<\alpha} S_\beta$ and fix $\beta, \beta_y < \alpha$ $(y \in Y)$ such that $Y \in S_\beta$ and $Z_y \in S_{\beta_y}$ $(y \in Y)$. The set $\{\beta\} \cup \{\beta_y : y \in Y\}$ is at most of power κ which is less then the cofinality of α thus we can find a $\beta^* < \alpha$ such that $\beta, \beta_y < \beta^*$ $(y \in Y)$. But then $X \in S_{\beta^*} \subset \bigcup_{\beta<\alpha} S_\beta$.

Secondly, we check by transfinite induction that $S_\beta = S_\alpha$ for all $\beta \geq \alpha$. Suppose $S_\gamma = S_\alpha$ for $\alpha \leq \gamma < \beta$ and let $X \in S_\beta$, that is $Y, Z_y \in \bigcup_{\gamma<\beta} S_\gamma$. However,

$$\bigcup_{\gamma<\beta} S_\gamma = \bigcup_{\gamma<\beta} S_\alpha = S_\alpha = \bigcup_{\delta<\alpha} S_\delta$$

which implies $X \in S_\alpha$ by repeating the above argument. ■

Theorem 3.7 *If $\mathcal{H}$ is a set of ordered sets such that the duplications of the elements of $\mathcal{H}$ are representable, then the elements of $\mathcal{S}(\mathcal{H})$ are also representable.*

Proof We prove by transfinite induction on α the seemingly stronger statement that even the duplications of elements of $\mathcal{S}(\mathcal{H})$ are representable. For $\alpha = 0$ this is just a reformulation of our assumption. Suppose now that the statement holds for all $\beta < \alpha$ and let $X \in S_\alpha$, that is $Y, Z_y \in \bigcup_{\beta<\alpha} S_\beta$. As $Z_y \in \bigcup_{\beta<\alpha} S_\beta$ $Z_y \times \{0,1\}$ is representable by the inductional hypothesis. Moreover if we replace the points of Y by the sets $Z_y \times \{0,1\}$ what we obtain is exactly the duplication of X, which therefore turns out to be representable as by the inductional hypothesis $Y \times \{0,1\}$ is representable and so we can apply Statement 2.3. ■

Definition 3.8 If $\mathcal{H}$ is a set of ordered sets, then let

$$\mathcal{H}^\omega = \{Y : Y \subset X^\omega, X \in \mathcal{H}\},$$

and let $\mathcal{H}^*$ be the closure of $\mathcal{H}$ under the operations $X \mapsto X^\alpha$ $(\alpha < \omega_1)$. (This closure can be formed by a similar transfinite construction as $\mathcal{S}(\mathcal{H})$.)

Corollary 3.9 *If $\mathcal{H}$ is a set of ordered sets such that the duplications of the elements of $\mathcal{H}$ are representable, then the elements of $\mathcal{S}(\mathcal{H})^\omega$ are also representable. This holds even for $\mathcal{S}(\mathcal{H})^*$, assuming that the duplications of representable sets are representable.*

Remark (a) We could define similar notions with products instead of powers, or even with the more complex constructions mentioned in the remark following Statement 2.4, but in fact we would not get more, as in the case we are interested in, there are always at most continuum many sets involved, thus we can put them together (e.g. replace the points of $\mathbb{R}$ by them) to form a huge set X that contains each of them, and so the power of this set X contains subsets similar to all these above constructions.

(b) If we begin our procedure of building large representable orderings, we can start with some set of simple ordered sets, for example the ones representable by constants or even continuous functions. In both cases we have $\mathcal{H} = \{\mathbb{R}\}$. It is not hard to prove that we will not get too far this way as I^ω will not be in $\mathcal{S}(\mathcal{H})$. (The proof goes by transfinite induction. Note that any non-trivial subinterval of I^ω contains a copy of I^ω and that building up a set X by replacing each element y of a set Y by X_y is the same as partitioning X into subintervals that are ordered similarly to Y such that each subinterval is similar to the corresponding X_y.) Therefore we prefer starting with the set of 'unboundedly wide trees', $\{I^\alpha : \alpha < \omega_1\}$.

(c) According to the previous theorems $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})$ contains order types of representable duplication only, as the duplication of I^α is a subset of $I^{\alpha+1}$. However, $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\}) \neq \mathcal{R}(\mathbb{R})$ as every element of the former set contains a non-trivial subinterval that is similar to a subset of I^α for some α, while if X is as in the proof of Statement 3.4, then X^ω does not. Therefore $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^\omega$ is a strictly larger class of representable orderings. This holds for $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^*$ as well, under the assumption about duplications.

It seems quite plausible that if we are allowed to replace points by arbitrarily large sets of the form I^α (of course $\alpha < \omega_1$), and allowed to form countable products, then we can build up every set not containing a sequence of length ω_1. Moreover it can be shown that $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^*$ is closed under duplication, completion and blends. (The definition of these notions for order types instead of ordered sets is obvious.) Together with Kuratowski's Theorem this motivates the following question.

Question 3.10 *Does either $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^\omega = \mathcal{R}(\mathbb{R})$ or $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^* = \mathcal{R}(\mathbb{R})$ hold?*

4 The second construction

Now we turn to an other approach of the problem which results in a notion very similar to $\mathcal{S}(\mathcal{H})$.

Statement 4.1 *Let $\{f_\alpha : \alpha \in \Gamma\}$ be an ordered set of functions defined on a second countable topological space and possessing the Baire property. If any two functions differ on a set of second category then the ordered set is similar to a subset of the real line.*

Proof Recall that an ordered set is similar to a subset of $\mathbb{R}$ iff it is separable and does not contain more than countably many pairs of consecutive elements.

First we prove separability. Let X be the second countable space and suppose for the time being that X is a Baire space, that is every non-empty open subset is of second category. Denote by B a countable base of the space not containing the empty set. We construct a countable dense subset M of $\{f_\alpha : \alpha \in \Gamma\}$ in the following way. If for $U, V \in B$ and $p, q \in \mathbb{Q}$ there exists $h \in \{f_\alpha : \alpha \in \Gamma\}$ such that $p < h$ on a residual subset of U and $h < q$ on a residual subset of V then we choose such an h. M is obviously countable and to verify that it is dense let (f, g) be an open interval of the ordered set. If this interval is empty then we are done so we may assume that there exists an element h_0 of the ordered set in the interval. Obviously

$$X(f < h_0) = \bigcup_{p \in \mathbb{Q}} X(f < p < h_0)$$

and

$$X(h_0 < g) = \bigcup_{q \in \mathbb{Q}} X(h_0 < q < g),$$

where the sets on the left hand side are by assumption of second category hence for some p and q $X(f < p < h_0)$ and $X(h_0 < q < g)$ are of second category as well. It is easy to see that a set of second category which also possesses the Baire property is residual in some non-empty open subset, moreover this open set can be chosen to be an element of B. As f, g and h_0 have the Baire property $X(f < p < h_0)$ and $X(h_0 < q < g)$ have it as well so we can find $U, V \in B$ in which these sets are residual respectively. But this means that for $U, V \in B$ and $p, q \in \mathbb{Q}$ there exists an element of the ordered set, namely h_0, satisfying all the conditions of the definition of M so there must be such an element $h \in M$ as well. We show that $h \in (f, g)$. X is a Baire space hence U is not of first category therefore there exists $x \in U$ for which $f(x) < p < h(x)$ and similarly $y \in V$ for which $h(y) < q < g(y)$. But this implies $f < h < g$ proving the separability.

Let now $f_i < g_i$ $(i \in I)$ be distinct consecutive elements in the ordered set. Like above, for every $i \in I$

$$X(f_i < g_i) = \bigcup_{p \in \mathbb{Q}} X(f_i < p < g_i)$$

hence for a suitable p_i $X(f_i < p_i < g_i)$ is of second category and we can thus fix $U_i \in B$ in which this set is residual. We show that the map $i \mapsto (p_i, U_i)$ is injective which implies that I is countable. Indeed, if $i \neq i'$ and $(p_i, U_i) = (p_{i'}, U_{i'}) = (p, U)$ than, as U is of second category, we obtain that for some $x \in U$ $f_i(x) < p < g_i(x)$ and $f_{i'}(x) < p < g_{i'}(x)$ contradicting the consecutiveness of the pairs.

Finally, if X is not a Baire space than as a consequence of Banach's Union Theorem [Ku, §10, III] we can write it as $X = G \cup A$ where G is an open subset which is a Baire space as a subspace and A is of first category. If we consider the restrictions of the functions to G we obtain a similar ordered set as any two functions differ on a set of second category in X hence they can not coincide on G. In fact, by the same argument they differ in G on a set of second category and thus we can apply what we have proven in the previous case. ■

This statement enables us to simplify the structure of a represented set X in the following way. Zorn's lemma implies that we can find a maximal subset of X in which every two elements differ on a set of second category. As this subset must be separable we can choose a countable dense subset M of it. The maximal intervals of $X \setminus M$ are of a simpler structure than X since any two elements of such an interval coincide on a residual set, moreover it follows from Kuratowski's Theorem that all elements of the interval coincide on a common residual set. We can thus go on and repeat this procedure inside this residual set. This motivates the following.

Definition 4.2 Let $\mathcal{H}$ be an arbitrary set of ordered sets. We call elements of $\mathcal{H}$ and the empty set sets of rank 0. For an ordinal α we say that an ordered set X is of rank at most α if there exists a countable subset $M \subset X$ such that all maximal intervals I of $X \setminus M$ are of rank at most β for some $\beta < \alpha$ where β may depend on I. The class of ordered sets of rank at most α is denoted by T_α.

Finally, let $\mathcal{T}(\mathcal{H})$ be the set of order types of $\bigcup_{\alpha \in On} T_\alpha$.

Lemma 4.3 *If X is a set of rank at most α then it is similar to a set obtained by replacing the points of $\mathbb{R}$ by elements of $\bigcup_{\beta<\alpha} T_\beta$.*

Proof Let $M \subset X$ be the countable subset as in the definition. Recall that every countable ordered set can be embedded into $\mathbb{Q}$ and fix a $\varphi : M \to \mathbb{Q}$ order preserving injective map.

A maximal interval I of $X \setminus M$ splits M into two parts M_1 and M_2 in a natural way. Define

$$F(I) = \sup\{\varphi(x) : x \in M_1\},$$

where we may assume the supremum to be finite as we may attach a first and a last element to X which may also be elements of M. Now if I_1, I_2 and I_3 are distinct maximal intervals following each other in this order then we can find an element $x \in M$ between I_1 and I_2 and $y \in M$ between I_2 and I_3 therefore $F(I_1) < F(I_3)$ as $\varphi(x) < \varphi(y)$. Similarly, $F(I_1) = F(I_2)$ implies that there is exactly one $x \in M$ between I_1 and I_2. Consequently we can map X to the real line via φ and F in an order preserving way such that the preimage of a real number is one of the followings: the empty set, a single point, a maximal interval, a maximal interval plus an extra point to the left or right or two intervals and a point in between. But these sets are obviously elements of $\bigcup_{\beta<\alpha} T_\beta$ hence the lemma follows. ■

Corollary 4.4 *If $\mathbb{R} \in \mathcal{H}$ then $\mathcal{T}(\mathcal{H}) \subset \mathcal{S}(\mathcal{H})$ thus $\mathcal{T}(\mathcal{H})$ is a set indeed.*

Corollary 4.5 *If the duplication of every element of rank 0 is representable then so is every element of $\mathcal{T}(\mathcal{H})$.*

Remark $\mathcal{T}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$ fails in general as the examples $\mathcal{H} = \{\mathbb{R}\}$ or $\mathcal{H} = \{X : X \subset I^\omega\}$ show, since in both cases $\mathcal{T}(\mathcal{H})$ is a subset of the order types of $\{X : X \subset I^\omega\}$.

However, the following question is open.

Question 4.6 *Does $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\}) = \mathcal{T}(\{I^\alpha : \alpha < \omega_1\})$ or $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^\omega = \mathcal{T}(\{I^\alpha : \alpha < \omega_1\})^\omega$ or $\mathcal{S}(\{I^\alpha : \alpha < \omega_1\})^* = \mathcal{T}(\{I^\alpha : \alpha < \omega_1\})^*$ hold?*

5 Final remarks

First we give a characterization of $\mathcal{R}_0(\mathbb{R})$, which in fact does not show too much about the structure of these orderings. This is motivated by the way our constructions worked.

Theorem 5.1 *An ordered set X is representable by ambiguous sets iff there exists an ordering on a compact metric space such that certain initial segments are ambiguous and ordered similarly to X by inclusion.*

Proof If we have such an ordering then of course the initial segments will do. Conversely, let $\{H_x : x \in X\}$ be a representation by ambiguous sets. Let

$$a \prec b \text{ iff } \exists x \in X \text{ such that } a \in H_x \text{ and } b \notin H_x.$$

One can easily see that this is a partial ordering on the compact metric space. By Zorn's lemma every partial ordering can be extended to an ordering, thus denote $\prec^*$ such an extension. We only have to show that H_x is an initial segment indeed of $\prec^*$ for each $x \in X$. So let $a \in H_x$, $b \prec^* a$ and show that $b \in H_x$. If this was not true then $b \notin H_x$, $a \in H_x$ and $b \prec^* a$ would hold, which contradicts the definition of $\prec^*$. ∎

Question 5.2 *Does $\mathcal{R}(\mathbb{R}) = \mathcal{R}_0(\mathbb{R})$ hold?*

To summarize our results we may say that the class of representable ordered sets seems to be quite close to the ones not containing sequences of length ω_1. Our last theorem asserts that one actually can not prove in ZFC that these two classes coincide.

Theorem 5.3 *The statement that a set is representable iff it does not contain a sequence of length ω_1 is not provable in ZFC.*

Proof A Souslin line does not contain such a long increasing sequence otherwise $\{(x_\alpha, x_{\alpha+2}) : \alpha < \omega_1 \text{ is a limit ordinal}\}$ would be an uncountable system of pairwise disjoint non-empty open intervals. The case of decreasing sequences is similar. Therefore in view of Komjáth's Theorem and the independence of the existence of Souslin lines the theorem follows. ∎

Finally we pose a fundamental question.

Question 5.4 *Is it consistent with ZFC that an ordered set is representable iff it does not contain a sequence of length ω_1?*

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