Chains of Baire class 1 functions and various notions of special trees

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Abstract

Following Laczkovich we consider the partially ordered set $\mathcal{B}_1(\mathbb{R})$ of Baire class 1 functions endowed with the pointwise order, and investigate the order types of the linearly ordered subsets. Answering a question of Komjáth and Kunen we show (in ZFC) that special Aronszajn lines are embeddable into $\mathcal{B}_1(\mathbb{R})$. We also show that under Martin's Axiom a linearly ordered set $\mathbb{L}$ with $|\mathbb{L}| < 2^\omega$ is embeddable into $\mathcal{B}_1(\mathbb{R})$ iff $\mathbb{L}$ does not contain a copy of ω_1 or ω_1^*. We present a ZFC-example of a linear order of size 2^ω showing that this characterisation is not valid for orders of size continuum.

These results are obtained using the notion of a compact-special tree; that is, a tree that is embeddable into the class of compact subsets of the reals partially ordered under reverse inclusion. We investigate how this notion is related to the well-known notion of an $\mathbb{R}$-special tree and also to some other notions of specialness.

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Introduction

Definition 0.1 Given two partial orders $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{P}', \leq_{\mathbb{P}'})$ the order $\mathbb{P}$ will be said to embed into $\mathbb{P}'$, denoted by $\mathbb{P} \hookrightarrow \mathbb{P}'$, if there is a mapping $\varphi \colon \mathbb{P} \to \mathbb{P}'$ such that $p_0 <_{\mathbb{P}} p_1$ implies $\varphi(p_0) <_{\mathbb{P}'} \varphi(p_1)$.

Note that this φ need not be one-to-one in general, but for a linear order $\mathbb{L}$ the relation $\mathbb{L} \hookrightarrow \mathbb{P}$ implies that there is an order-isomorphic copy of $\mathbb{L}$ in $\mathbb{P}$. As it is usual for trees, instead of $\mathbb{P} \hookrightarrow \mathbb{P}'$ we will sometimes say that $\mathbb{P}$ is $\mathbb{P}'$-special. From now on we will often write $\mathbb{P}$ instead of $(\mathbb{P}, \leq_{\mathbb{P}})$ when there is no danger of confusion.

$\mathcal{B}_1(\mathbb{R})$ is the class of Baire class 1 functions from $\mathbb{R}$ to $\mathbb{R}$; that is, pointwise limits of sequences of continuous real functions. This class is partially ordered under the usual pointwise ordering; that is, $f \leq g$ iff $f(r) \leq g(r)$ for every $r \in \mathbb{R}$. Note that $f < g$ iff $f \leq g$ and $f(r) \neq g(r)$ for some $r \in \mathbb{R}$. The following problem was posed by Laczkovich.

Problem 0.2 *Characterise those linear orders $\mathbb{L}$ for which $\mathbb{L} \hookrightarrow \mathcal{B}_1(\mathbb{R})$ holds.*

What makes the Baire class 1 case particularly interesting is that the corresponding questions for all other Baire classes are solved. In the Baire class 0; that is, continuous case it is easy to see that $\mathbb{L} \hookrightarrow \mathcal{B}_0(\mathbb{R})$ iff $\mathbb{L} \hookrightarrow \mathbb{R}$, while for $\alpha \geq 2$ Komjáth [6] showed that even the question whether $\omega_2 \hookrightarrow \mathcal{B}_\alpha(\mathbb{R})$ is independent of ZFC.

Another motivation may be that Problem 0.2 is apparently closely related to the theory of Rosenthal compacta, however, so far no direct connection has been found.

The earliest result concerning Problem 0.2 is a classical theorem of Kuratowski [8, 24.III.2'] stating that $\omega_1 \not\hookrightarrow \mathcal{B}_1(\mathbb{R})$. Note that $\alpha \hookrightarrow \mathbb{R} \hookrightarrow \mathcal{B}_1(\mathbb{R})$ for $\alpha < \omega_1$. For some related results see [3]. It is shown in [2] that, loosely speaking, starting from a class of simple linear orders, say the finite ones, and applying all sorts of countable operations one always obtains $\mathcal{B}_1(\mathbb{R})$-embeddable linear orders. Therefore it is quite natural to guess that Kuratowski's theorem is the only restriction; that is, $\mathbb{L} \hookrightarrow \mathcal{B}_1(\mathbb{R})$ iff $\omega_1, \omega_1^* \not\hookrightarrow \mathbb{L}$. (Here ω_1^* is the reversed ω_1.)

However, Komjáth [6] gave a consistent counterexample by showing that if $\mathbb{L}$ is a Souslin line then $\mathbb{L} \not\hookrightarrow \mathcal{B}_1(\mathbb{R})$. But this still leaves open the possibility that the above answer to Laczkovich's problem is consistent with ZFC.

Question 0.3 *Is it consistent that a linear order* $\mathbb{L} \hookrightarrow \mathcal{B}_1(\mathbb{R})$ *iff* $\omega_1, \omega_1^* \not\hookrightarrow \mathbb{L}$*?*

Komjáth and Kunen independently asked the following natural question.

Question 0.4 *Is there an Aronszajn line* $\mathbb{A}$ *such that* $\mathbb{A} \hookrightarrow \mathcal{B}_1(\mathbb{R})$*?*

In this paper we answer Question 0.3 and Question 0.4.

First we establish our basic tool in Section 1, then make some preparations in Section 2 by proving that nine notions of specialness coincide for countably branching trees. Then we answer Question 0.4 in the positive in Section 3. More precisely, we show that special Aronszajn lines are $\mathcal{B}_1(\mathbb{R})$-embeddable, hence there exists (in ZFC) a $\mathcal{B}_1(\mathbb{R})$-embeddable Aronszajn line, and consistently all Aronszajn lines are $\mathcal{B}_1(\mathbb{R})$-embeddable. We also show in this section that under Martin's Axiom the characterisation in Question 0.3 is valid for linear orders of cardinality strictly less than the continuum. In Section 4 we answer Question 0.3 in the negative (in ZFC). Finally, in Section 5 we formulate some open problems.

The set-theoretic terminology followed in this paper can be found e.g. in [4] and [7]. For an element t of a tree $\mathbb{T}$ denote $\operatorname{succ}(t)$ the set of immediate successors of t. We say that a tree $\mathbb{T}$ is countably branching, if $|\operatorname{succ}(t)| \leq \omega$ for every $t \in \mathbb{T}$. All trees in this paper are considered to be normal; that is, for $t_0, t_1 \in \mathbb{T}$ the equation $\{t \in \mathbb{T} \colon t <_{\mathbb{T}} t_0\} = \{t \in \mathbb{T} \colon t <_{\mathbb{T}} t_1\}$ implies $t_0 = t_1$. The basic facts about Baire class 1 functions can be found e.g. in [5] or [8]. An F_σ set is a set that is the union of countably many closed sets, a G_δ set is a set that is the intersection of countably many open sets. The symbols $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of natural, rational and real numbers, respectively.

1 The main lemma

For a linear order $\mathbb{L}$, denote $\mathbb{T}_{\mathbb{L}}$ its (binary) partition tree (see [9]), which is constructed as follows. Denote by $\mathbb{T}_\alpha$ the α^{th} level of a tree $\mathbb{T}$. Elements of the partition tree will be nonempty intervals; that is, convex subsets of $\mathbb{L}$, and the ordering will be reverse inclusion. Set $(\mathbb{T}_{\mathbb{L}})_0 = \{\mathbb{L}\}$. Once $(\mathbb{T}_{\mathbb{L}})_\alpha$ is given, split every $I \in (\mathbb{T}_{\mathbb{L}})_\alpha$ of at least two elements into two disjoint nonempty intervals I_0^+ and I_1^+, and put $(\mathbb{T}_{\mathbb{L}})_{\alpha+1} = \{I_i^+ \colon I \in (\mathbb{T}_{\mathbb{L}})_\alpha,\ |I| \geq 2,\ i \in 2\}$. We tacitly assume that I_0^+ is the 'left' interval; that is, for every $l_0 \in I_0^+$ and $l_1 \in I_1^+$ we have $l_0 \leq_{\mathbb{L}} l_1$. For α limit put $(\mathbb{T}_{\mathbb{L}})_\alpha = \{\cap_{\beta<\alpha} I_\beta \colon I_\beta \in (\mathbb{T}_{\mathbb{L}})_\beta,\ \cap_{\beta<\alpha} I_\beta \neq \emptyset\}$.

Denote by $\mathcal{K}(\mathbb{R})$ the set of compact subsets of $\mathbb{R}$ ordered under reverse inclusion.

Definition 1.1 We say that $\mathbb{T} \hookrightarrow \mathcal{K}(\mathbb{R})$ *strongly*, if there exists an embedding which maps incomparable elements to disjoint sets; that is, there exists an embedding $\varphi \colon \mathbb{T} \to \mathcal{K}(\mathbb{R})$ such that $\varphi(t_0) \cap \varphi(t_1) = \emptyset$ for every $t \in \mathbb{T}$ and distinct $t_0, t_1 \in \mathrm{succ}(t)$.

Main Lemma 1.2 *Let $\mathbb{L}$ be a linear order such that $\mathbb{T}_{\mathbb{L}} \hookrightarrow \mathcal{K}(\mathbb{R})$ strongly. Then $\mathbb{L} \hookrightarrow \mathcal{B}_1(\mathbb{R})$.*

Proof. Let $\varphi \colon \mathbb{T}_{\mathbb{L}} \to \mathcal{K}(\mathbb{R})$ be a strong embedding. For every $l \in \mathbb{L}$ define

$$A^l = \cup\{\varphi(I_0^+) \colon I \in \mathbb{T}_{\mathbb{L}},\ |I| \geq 2,\ l \in I_1^+\}.$$

We claim that $\psi \colon \mathbb{L} \to \mathcal{B}_1(\mathbb{R})$

$$\psi(l) = \chi_{A^l}$$

is the required embedding, where χ_H is the characteristic function of the set H. As $\chi_{H_0} < \chi_{H_1}$ iff $H_0 \subsetneq H_1$, we first have to show that for $l_0 <_{\mathbb{L}} l_1$ the strict inclusion $A^{l_0} \subsetneq A^{l_1}$ holds.

Fix $l_0 <_{\mathbb{L}} l_1$. First we show $A^{l_0} \subseteq A^{l_1}$. Suppose $I \in (\mathbb{T}_{\mathbb{L}})_\alpha,\ |I| \geq 2$ and $l_0 \in I_1^+$. We have to show that $\varphi(I_0^+) \subseteq A^{l_1}$. There is a first level where l_0 and l_1 are not in the same element of $\mathbb{T}_{\mathbb{L}}$, moreover, this is necessarily a successor level, say $l_0, l_1 \in I^* \in (\mathbb{T}_{\mathbb{L}})_{\alpha *}$, $l_0 \in (I^*)_0^+$ and $l_1 \in (I^*)_1^+$. Clearly, $\varphi((I^*)_0^+) \subseteq A^{l_1}$. If $\alpha < \alpha^*$ then $l_0 \in I_1^+$

implies $l_1 \in I_1^+$, hence $\varphi(I_0^+) \subseteq A^{l_1}$. If $\alpha \geq \alpha^*$ then $I \subseteq I^*$, hence $\varphi(I_0^+) \subseteq \varphi((I^*)_0^+) \subseteq A^{l_1}$.

Now we show $A^{l_0} \neq A^{l_1}$. By compactness, $C = \cap\{\varphi(I) \colon l_0 \in I \in \mathbb{T}_{\mathbb{L}}\} \neq \emptyset$. Using $\varphi((I^*)_0^+) \subseteq A^{l_1}$ again, we obtain $C \subseteq A^{l_1}$. We claim that $C \cap A^{l_0} = \emptyset$. In order to show this we have to check that $l_0 \in I_1^+$ implies $\varphi(I_0^+) \cap C = \emptyset$. But this is clear, as $C \subseteq \varphi(I_1^+)$ and φ is a strong embedding.

What remains to be shown is that $\chi_{A^l} \in \mathcal{B}_1(\mathbb{R})$ for every $l \in \mathbb{L}$. A characteristic function χ_H is of Baire class 1 iff H is simultaneously F_σ and G_δ, hence we have to check this for A^l. It is well known (see [5, 22.27] or [8, 24.III.1]) that if for some $\xi < \omega_1$ the nonincreasing transfinite sequences $\{F_\alpha\}_{\alpha<\xi}$ and $\{H_\alpha\}_{\alpha<\xi}$ of closed subsets of $\mathbb{R}$ satisfy $F_\alpha \supseteq H_\alpha$ for every $\alpha < \xi$ and $H_\alpha \supseteq F_\beta$ for every $\alpha < \beta < \xi$, then the set

$$\bigcup_{\alpha<\xi}(F_\alpha \setminus H_\alpha)$$

is simultaneously F_σ and G_δ.

Fix $l \in \mathbb{L}$. Let ξ^l be the ordinal for which $\{l\} \in (\mathbb{T}_{\mathbb{L}})_{\xi^l}$ holds. As every strictly decreasing transfinite sequence of compact subsets of $\mathbb{R}$ is countable, $\xi^l < \omega_1$. For $\alpha < \xi^l$ the unique interval $I \in (\mathbb{T}_{\mathbb{L}})_\alpha$ with $l \in I$ has at least two elements, so define

$$F^l_{\alpha+1} = H^l_{\alpha+1} = \varphi(I_0^+) \cup \varphi(I_1^+),$$

if $l \in I_0^+$, and

$$F^l_{\alpha+1} = \varphi(I_0^+) \cup \varphi(I_1^+),$$
$$H^l_{\alpha+1} = \varphi(I_1^+)$$

if $l \in I_1^+$. For $\alpha < \xi^l$ limit, which includes the case $\alpha = 0$, define

$$F^l_\alpha = H^l_\alpha = \varphi(I).$$

Clearly, $F^l_\alpha \supseteq H^l_\alpha$ for every $\alpha < \xi^l$ and it is easy to see that $F^l_\alpha \supseteq F^l_\beta$ and $H^l_\alpha \supseteq H^l_\beta$ for every $\alpha < \beta < \xi^l$. Using that F^l_α is monotone nonincreasing, in order to obtain that $H^l_\alpha \supseteq F^l_\beta$ for every $\alpha < \beta < \xi^l$ it is sufficient to check that $H^l_\alpha \supseteq F^l_{\alpha+1}$ for every $\alpha < \xi^l$, which is straightforward. Therefore $\bigcup_{\alpha<\xi^l}(F^l_\alpha \setminus H^l_\alpha)$ is F_σ and G_δ. Using that our embedding φ is strong we obtain

$$A^l = \bigcup_{\alpha<\xi^l}(F^l_\alpha \setminus H^l_\alpha),$$

so the proof is complete. $\square$

2 Various notions of special trees

In this section we prove that the relation $\mathbb{T}_\mathbb{L} \hookrightarrow \mathcal{K}(\mathbb{R})$ strongly can be translated to $\mathbb{T}_\mathbb{L} \hookrightarrow \mathbb{R}$. As specialness of trees is interesting in its own right, we prove that, at least for countably branching trees, this is also equivalent to specialness in certain other senses. Let $\mathbb{C}$ denote the Cantor set with its inherited ordering as a subset of $\mathbb{R}$. The Prikry-Silver partial order will be denoted by $\mathbb{S}$ – it consists of all partial functions $f : \mathbb{N} \to 2 = \{0, 1\}$ with co-infinite domain ordered under inclusion.

Definition 2.1 We say that $\mathbb{T} \hookrightarrow \mathbb{S}$ *strongly*, if there exists an embedding which maps incomparable elements to incompatible functions; that is, there exists an embedding $\varphi \colon \mathbb{T} \to \mathbb{S}$ such that for every $t \in \mathbb{T}$ and distinct $t_0, t_1 \in \mathrm{succ}(t)$ there exists $n \in \mathrm{dom}(\varphi(t_0)) \cap \mathrm{dom}(\varphi(t_1))$ such that $\varphi(t_0)(n) \neq \varphi(t_1)(n)$.

Theorem 2.2 *Let $\mathbb{T}$ be a countably branching tree, e.g. a partition tree. Then the following are equivalent.*

(1) $\mathbb{T}$ is $\mathbb{C}$-special

(2) $\mathbb{T}$ is $\mathbb{R}$-special

(3) $\mathbb{T}$ is strongly $\mathbb{S}$-embeddable

(4) $\mathbb{T}$ is strongly $\mathcal{K}(\mathbb{C})$-embeddable

(5) $\mathbb{T}$ is $\mathcal{K}(\mathbb{C})$-special

(6) $\mathbb{T}$ *is strongly* $\mathcal{K}(\mathbb{R})$*-embeddable*

(7) $\mathbb{T}$ *is* $\mathcal{K}(\mathbb{R})$*-special*

(8) $\mathbb{T}$ *is* $(\mathcal{P}(\mathbb{N}), \subseteq)$*-special*

(9) $\mathbb{T}$ *is* $\mathbb{S}$*-special*

Proof. (1) $\Rightarrow$ (2): This is immediate.

(2) $\Rightarrow$ (3): Let $\varphi\colon \mathbb{T} \to \mathbb{R}$ be an embedding, and let $\{q_n \colon n \in \mathbb{N}\}$ enumerate $\mathbb{Q}$. Set $\mathrm{dom}(\psi(t)) = \{n \in \mathbb{N}\colon q_n < \varphi(t)\}$ and define $\psi(t)\colon \{n \in \mathbb{N}\colon q_n < \varphi(t)\} \to 2$ by induction along $\mathbb{T}$ as follows. At limit nodes simply let $\psi(t)$ be the union of all $\psi(s)$ such that $s \subset t$. Given that $\psi(t)$ is defined, enumerate $\mathrm{succ}(t)$ as $\{t_k \colon k \in \mathbb{N}\}$, and by induction on k pick distinct $n_k \in \mathbb{N}$ such that $\varphi(t) \leq q_{n_k} < \varphi(t_k)$. For $n \in \mathbb{N}$ such that $q_n < \varphi(t_k)$ set $\psi(t_k)(n) = \psi(t)(n)$ if $q_n < \varphi(t)$, $\psi(t_k)(n_k) = 1$, and $\psi(t_k)(n) = 0$ otherwise. It is easy to check that $\psi\colon \mathbb{T} \to \mathbb{S}$ is a strong embedding.

(3) $\Rightarrow$ (4): Let $\varphi\colon \mathbb{T} \to \mathbb{S}$ be a strong embedding. Identify $\mathbb{C}$ with $2^{\mathbb{N}}$; that is, the set of functions from $\mathbb{N}$ to 2. For $t \in \mathbb{T}$ define $\psi(t) = \{f \in 2^{\mathbb{N}}\colon \varphi(t) \subseteq f\}$. Then $\psi\colon \mathbb{T} \to \mathcal{K}(\mathbb{C})$ is a strong embedding.

(4) $\Rightarrow$ (5): Obvious.

(5) $\Rightarrow$ (1): Again, identify $\mathbb{C}$ with $2^{\mathbb{N}}$. Let $\{g_n\}_{n=1}^{\infty}$ enumerate all $g\colon k \to 2$ where $k \in \mathbb{N}$ and send $K \in \mathcal{K}(\mathbb{C})$ to

$$\sum_{\substack{n \in \mathbb{N} \\ \nexists f \in K \ g_n \subseteq f}} \frac{2}{3^{n+1}}.$$

(4) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (7): Obvious.

(7) $\Rightarrow$ (2): Enumerate $\{(p,q)\colon p, q \in \mathbb{Q},\ p < q\}$ as $\{(p_n, q_n)\colon n \in \mathbb{N}\}$, and send $K \subseteq \mathbb{R}$ to $\sum_{(p_n,q_n)\cap K = \emptyset} \frac{1}{2^n}$.

(2) $\Rightarrow$ (8): Enumerate $\mathbb{Q}$ as $\{q_n \colon n \in \mathbb{N}\}$, and send $r \in \mathbb{R}$ to $\{n \in \mathbb{N} : q_n < r\}$.

(8) $\Rightarrow$ (9): Send $H \subseteq \mathbb{N}$ to the function that is constant 0 on $\{2n \colon n \in H\}$ and undefined elsewhere.

(9) $\Rightarrow$ (2): Send $f \in \mathbb{S}$ to $-\sum_{n \notin \mathrm{dom}(f)} \frac{1}{2^n}$. $\square$

Remark 2.3 The assumption that the tree $\mathbb{T}$ is countably branching cannot be dropped, as if $\operatorname{succ}(t)$ has cardinality larger than the continuum for some $t \in \mathbb{T}$ then $\mathbb{T}$ is clearly not strongly $\mathcal{K}(\mathbb{C})$-special but it can be $\mathbb{R}$-special.

3 Consequences for $\mathcal{B}_1$-embeddability

In this section we answer Question 0.4 and give an affirmative answer to Question 0.3 in the case $|\mathbb{L}| < 2^\omega$.

Theorem 3.1 *Let $\mathbb{A}$ be a special Aronszajn line; that is, $\mathbb{T}_\mathbb{A} \hookrightarrow \mathbb{Q}$. Then $\mathbb{A} \hookrightarrow \mathcal{B}_1(\mathbb{R})$.*

Proof. Clearly, $\mathbb{T}_\mathbb{A} \hookrightarrow \mathbb{R}$, hence Theorem 2.2 yields $\mathbb{T}_\mathbb{A} \hookrightarrow \mathcal{K}(\mathbb{R})$ strongly, therefore by the Main Lemma 1.2 we obtain $\mathbb{A} \hookrightarrow \mathcal{B}_1(\mathbb{R})$. $\square$

Theorem 3.2 *Assume Martin's Axiom. Then for a linear order $\mathbb{L}$ with $|\mathbb{L}| < 2^\omega$ the relation $\mathbb{L} \hookrightarrow \mathcal{B}_1(\mathbb{R})$ holds iff $\omega_1, \omega_1^* \not\hookrightarrow \mathbb{L}$.*

Proof. First suppose $\omega_1 \hookrightarrow \mathbb{L}$ or $\omega_1^* \hookrightarrow \mathbb{L}$. By the theorem of Kuratowski [8, 24.III.2'] every strictly monotone transfinite sequence in $\mathcal{B}_1(\mathbb{R})$ is countable, hence $\mathbb{L} \not\hookrightarrow \mathcal{B}_1(\mathbb{R})$. Now suppose $\omega_1, \omega_1^* \not\hookrightarrow \mathbb{L}$. It follows that there is no strictly decreasing sequence of subintervals of $\mathbb{L}$ of length ω_1, hence $\mathbb{T}_\mathbb{L}$ has at most ω_1 levels. Each level is a disjoint family of nonempty intervals of $\mathbb{L}$, so $|\mathbb{L}| < 2^\omega$ implies $|(\mathbb{T}_\mathbb{L})_\alpha| < 2^\omega$ for every α. By Martin's Axiom $\omega_1 < 2^\omega$ and 2^ω is regular, therefore $|\mathbb{T}_\mathbb{L}| < 2^\omega$. Under Martin's Axiom every tree of cardinality less than 2^ω with no branch of length ω_1 is $\mathbb{Q}$-special [1], hence $\mathbb{T}_\mathbb{L} \hookrightarrow \mathbb{Q}$, and we can repeat the previous proof. $\square$

4 Answer to Question 0.3

Now we answer Question 0.3 in the negative, using some ideas from [9].

Theorem 4.1 *There exists a linear order* $\mathbb{L}$ *such that* $\omega_1, \omega_1^* \not\hookrightarrow \mathbb{L}$ *but still* $\mathbb{L} \not\hookrightarrow \mathcal{B}_1(\mathbb{R})$.

Proof. Define

$$\sigma\mathcal{B}_1(\mathbb{R}) = \{l\colon \xi < \omega_1,\ l\colon \xi \to \mathcal{B}_1(\mathbb{R}) \text{ strictly increasing}\}.$$

This set becomes a tree if we partially order it by extension; that is, $l_0 \leq_{\mathbb{T}} l_1$ iff $l_0 \subseteq l_1$.

Lemma 4.2 $(\sigma\mathcal{B}_1(\mathbb{R}), \leq_{\mathbb{T}}) \not\hookrightarrow \mathcal{B}_1(\mathbb{R})$.

Proof. Suppose $\varphi\colon \sigma\mathcal{B}_1(\mathbb{R}) \to \mathcal{B}_1(\mathbb{R})$ is an embedding. Then the transfinite recursion

$$l^*(\alpha) = \varphi(l^* \restriction \alpha)$$

produces a strictly increasing sequence of length ω_1 in $\mathcal{B}_1(\mathbb{R})$, which is impossible by Kuratowski's theorem [8, 24.III.2']. $\square$

This lemma shows that in order to finish the proof of Theorem 4.1 it is sufficient to construct a linear order $\leq_{\mathbb{L}}$ on $\sigma\mathcal{B}_1(\mathbb{R})$ extending $\leq_{\mathbb{T}}$ such that $\omega_1, \omega_1^* \not\hookrightarrow (\sigma\mathcal{B}_1(\mathbb{R}), \leq_{\mathbb{L}})$. So fix an arbitrary bijection $\Phi\colon \mathcal{B}_1(\mathbb{R}) \to \mathbb{R}$ and define $\leq_{\mathbb{L}}$ to be the usual lexicographical ordering as follows. The functions $l_0\colon \xi^{l_0} \to \mathcal{B}_1(\mathbb{R})$ and $l_1\colon \xi^{l_1} \to \mathcal{B}_1(\mathbb{R})$ are incomparable with respect to $\leq_{\mathbb{T}}$ iff there exists $\alpha < \xi^{l_0}, \xi^{l_1}$ such that $l_0(\alpha) \neq l_1(\alpha)$. In such a case choose the minimal such α and define $l_0 <_{\mathbb{L}} l_1$ iff $\Phi(l_0(\alpha)) < \Phi(l_1(\alpha))$.

Now we prove that $\omega_1, \omega_1^* \not\hookrightarrow (\sigma\mathcal{B}_1(\mathbb{R}), \leq_{\mathbb{L}})$. Suppose $\{l_\eta\}_{\eta<\omega_1}$ is strictly monotonic. We prove by induction on $\beta < \omega_1$ that there exists $l^*\colon \omega_1 \to \mathcal{B}_1(\mathbb{R})$ such that for every $\beta < \omega_1$ there exists η_β such that for $\eta \geq \eta_\beta$

$$l_\eta(\beta) = l^*(\beta).$$

Suppose this holds for every $\gamma < \beta$. If $\eta \geq \sup\{\eta_\gamma\colon \gamma < \beta\}$ then $l_\eta \restriction \beta = l^* \restriction \beta$, and hence $\Phi(l_\eta(\beta))$ is monotonic in $\mathbb{R}$, and therefore is constant above some η_β. As Φ is a bijection, $l_\eta(\beta)$ is also constant for $\eta \geq \eta_\beta$. Defining $l^*(\beta) = l_{\eta_\beta}(\beta)$ finishes the induction. But once again, the existence of the strictly monotone sequence $\{l^*(\alpha)\}_{\alpha<\omega_1}$ contradicts Kuratowski's theorem. $\square$

5 Open questions

The fundamental open problem is still of course Problem 0.2. However, we formulate here a couple of related questions.

We mentioned in the Introduction that, starting from some simple linear orders, countable operations always result in $\mathcal{B}_1(\mathbb{R})$-embeddable orders. However, we do not know whether the class of $\mathcal{B}_1(\mathbb{R})$-embeddable orders itself is closed under these operations. It is shown in [2] that the answer is affirmative for all these operations provided that it is affirmative for the simplest such operation, namely, for the operation that doubles the points of the order. That is why we are particularly interested in the following.

Question 5.1 *Suppose* $\mathbb{L} \hookrightarrow \mathcal{B}_1(\mathbb{R})$*, where* $\mathbb{L}$ *is a linear order. Does* $\mathbb{L} \times \{0,1\} \hookrightarrow \mathcal{B}_1(\mathbb{R})$*, where the ordering of* $\mathbb{L} \times \{0,1\}$ *is the usual lexicographical order?*

Denote $\Delta^0_2(\mathbb{R})$ the class of subsets of $\mathbb{R}$ that are simultaneously F_σ and G_δ. The ordering is reverse inclusion. Clearly,

$$\mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \Delta^0_2(\mathbb{R}) \hookrightarrow \mathcal{B}_1(\mathbb{R}),$$

and it can be shown that the first two arrows cannot be reversed. How about the third one?

Question 5.2 $\mathcal{B}_1(\mathbb{R}) \hookrightarrow \Delta^0_2(\mathbb{R})$*?*

Question 5.3 *Suppose* $\mathbb{L} \hookrightarrow \mathcal{B}_1(\mathbb{R})$*, where* $\mathbb{L}$ *is a linear order. Does* $\mathbb{L} \hookrightarrow \Delta^0_2(\mathbb{R})$*? How about trees instead of linear orderings?*

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