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Non-Trivial t-Intersection in the Function Lattice*

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Abstract. The function lattice, or generalized Boolean algebra, is the set of ℓ-tuples with the ith coordinate an integer between 0 and a bound n_i. Two ℓ-tuples t-intersect if they have at least t common nonzero coordinates. We prove a Hilton–Milner type theorem for systems of t-intersecting ℓ-tuples.

Keywords: generalized Boolean algebra, intersecting chains, Erdős-Ko-Rado theorem, Hilton-Milner theorem, kernel method

1. Introduction

Let t, ℓ, and $n_1 \leq n_2 \leq \cdots \leq n_\ell$ be positive integers. Denote by $\mathbb{F}_\ell(n_1, \ldots, n_\ell)$ the set of all ℓ-tuples

$$\{\mathbf{k} = (k_1, \ldots, k_\ell) \colon 0 \leq k_i \leq n_i,\ 1 \leq i \leq \ell\}.$$

The *support* of an ℓ-tuple $\mathbf{k}$ is the set of the non-zero coordinates: $\mathrm{supp}(\mathbf{k}) = \{i \colon k_i \neq 0\}$. We can define a partial ordering on $\mathbb{F}_\ell(n_1, \ldots, n_\ell)$ by $\mathbf{j} \leq \mathbf{k}$ if $\mathrm{supp}(\mathbf{j}) \subset \mathrm{supp}(\mathbf{k})$ and for all $i \in \mathrm{supp}(\mathbf{j})$ we have $j_i = k_i$. This partially ordered set is called the *function lattice* (see for example [5]). Another frequently used name is *generalized Boolean algebra*, because the case $n_1 = n_\ell = 1$, i.e., when all n_i are equal to 1, is just the case of (characteristic vectors of) set systems on an ℓ-element underlying set.

We say that two ℓ-tuples $\mathbf{j}$ and $\mathbf{k}$ are *t-intersecting* if there are at least t different integers $i \in \mathrm{supp}(\mathbf{j}) \cap \mathrm{supp}(\mathbf{k})$ such that $j_i = k_i$, or, with other words, if there is an ℓ-tuple $\mathbf{t}$ with support of size t such that $\mathbf{t} \leq \mathbf{k}$ and $\mathbf{t} \leq \mathbf{j}$. Denote by $m_t(n_1, \ldots, n_\ell)$ the

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maximum cardinality of t-intersecting ℓ-tuples in $\mathbb{F}_\ell(n_1,\ldots,n_\ell)$ and by $M_t(n_1,\ldots,n_\ell)$ the set of all t-intersecting families with this cardinality. The problems to determine the value $m_t(n_1,\ldots,n_\ell)$ and to describe the structures of the families in $M_t(n_1,\ldots,n_\ell)$, have a very long and notable history even in the case $n_\ell > 1$, and this is the case we are concentrating on in this note.

We start with the history of the case $t = 1$. C. Berge (1974, [4]) determined $m_t(n_1,\ldots,n_\ell)$ and $M_t(n_1,\ldots,n_\ell)$ when all ℓ-tuples have ℓ-element supports. Different proofs of Berge's result were given by Hsieh (1975, [19]) and by Livingston (1979, [21]) in the case when $n_1 = n_\ell$. The first result for set systems with uniform support size different from ℓ, but with $n_1 = n_\ell$, is due to Frankl (published in 1983, [9]). Moreover, Engel (1984, [10]) handled the case with $n_1 = n_\ell$, when the supports of the ℓ-tuples are arbitrary. In fact, Engel proved a Bollobás-type inequality (in the spirit of [8]) for the set of intersecting ℓ-tuples; a simpler proof of this last result is due to P.L. Erdős, U. Faigle and W. Kern (1992, [12]). In 2001 C. Bey gave a complete solution to the $t = 1$ case, for arbitrary n_i's and any uniform support size (2001, [6]), using his general weighted intersection theorem. This case shows interesting connections to the complete intersection theorem of R. Ahlswede and L. Khachatrian ([2]).

For arbitrary values of t, the first result is due to D. Kleitman (1966, [20]) in the case when $n_1 = n_\ell = 2$, and all supports are of size ℓ. Then P. Frankl and Z. Füredi handled the case $t \geq 15$, all supports are of size ℓ, and $n_1 = n_\ell$ (1980, [14]), using Frankl's version of the Erdős-Ko-Rado theorem (see [11]). Later A. Moon generalized this result for cross t-intersecting families (1982, [22]). The paper by Deza and Frankl (1983, [9]) also contains the solution for the case when all supports are of the same size k and $n_1 = n_\ell$, for ℓ large enough as a function of k and t. H-D. Gronau proved the first result for t-intersecting families with ℓ-element supports in the case of non-equal n_i's (1983, [16]). R. Ahlswede and L. Khachatrian (1998, [3]), and independently P. Frankl and N. Tokushige (1998, [15]), solved the t-intersecting problem for arbitrary t for ℓ-tuples with full support, applying Ahlswede and Khachatrian's seminal complete intersection theorem for set systems (1997, [2]). Finally C. Bey (1999, [5]) determined all parameters ℓ, k, t, n, for which "fixing t coordinates" yields the solution to the intersection problem.

All these results can be summarized in the following structural way: under some conditions for the parameter values, the (often unique) optimal t-intersecting family consists of all ℓ-tuples that are greater or equal than a fixed ℓ-tuple $\mathbf{t}$ with support size t. In the literature such set systems are called *trivially* t-intersecting families. As it is well known in the theory of t-intersecting set systems, there is a long-standing effort to solve the *nontrivial* t-intersection problem: what is the size and the structure of the maximum t-intersecting families where the total intersection of the sets has less than t elements. The first such result is due to A.J.H. Hilton and E.C. Milner (1967, [18]). The complete solution is again due to R. Ahlswede and L. Khachatrian (1996, [1]).

As far as these authors are aware, the only t-intersection result known for the function lattice $\mathbb{F}_\ell(n_1,\ldots,n_\ell)$ is due to C. Bey and K. Engel (2000, [7, Example 10, 11 and Lemma 18]): this is the complete solution to the non-trivial t-intersection problem in the case of equal n_i's.

The goal of this paper is to prove a more general non-trivial t-intersection result for the subset of the function lattice $\mathbb{F}_\ell(n_1,\ldots,n_\ell)$ consisting of ℓ-tuples with a fixed size k of the support, for some parameter values $t < k < \ell$ and $n_1 \le n_2 \le \cdots \le n_\ell$. The result is based on a Hilton–Milner type theorem for poset series, proved by the authors (2000, [13]). The proof of this latter uses the so-called *kernel method*, introduced by A. Hajnal and B. Rothschild (1973, [17]), therefore all of our results are valid only from a threshold for the parameters. We note that, perhaps surprisingly, the application of [13] is *not* for the natural partial order of $\mathbb{F}_\ell(n_1,\ldots,n_\ell)$. We shall investigate families of intersecting chains in the natural partial order of $\mathbb{F}_\ell(n_1,\ldots,n_\ell)$ in a forthcoming paper. Of course, a direct application of the kernel method may yield similar results, but citing [13] saves a lot of work. We admit that the methods of [7] are likely to allow generalization to the case of different n_i's.

In Section 2 we recall the necessary details from [13], while in Section 3 we reformulate the t-intersection problem of the function lattice and apply for it the method described in Section 2.

2. Non-Trivial t-Intersection Results for Posets

A *t-chain* $\mathcal{L}$ in a poset P is a strict chain of elements $\mathcal{L} = (x_1 < x_2 < \cdots < x_t)$. For a given t-chain $\mathcal{L} = (x_1 < x_2 < \cdots < x_t)$, let $\mathcal{T}_{P,k}(x_1, x_2, \ldots, x_t)$ denote the set of k-chains in P which contain $\mathcal{L}$ as a subset. Define $T_{P,k}(x_1, x_2, \ldots, x_t) = |\mathcal{T}_{P,k}(x_1, x_2, \ldots, x_t)|$. Sometimes we write T instead of $T_{P,k}$, when it does not cause ambiguity. Also define $r_t(P,k) = \max T_{P,k}(x_1, x_2, \ldots, x_t)$, where the maximum is taken for t-chains $x_1 < x_2 < \cdots < x_t$ in P. It follows from the definition that

$$r_i(P,k) \ge r_{i+1}(P,k). \tag{2.1}$$

For a t-chain $\mathcal{X} \subset P$ and $y \notin \mathcal{X}$, let $T(\mathcal{X}, y)$ denote the number of k-chains which contain $\mathcal{X}$ and y. For a t-chain $\mathcal{X}$ and a k-chain $\mathcal{L}$ in P, such that $|\mathcal{X} \cup \mathcal{L}| = k+1$, let $y^*_{\mathcal{L}} \in \mathcal{L} \setminus \mathcal{X}$ such that $T(\mathcal{X}, y^*_{\mathcal{L}})$ minimize $T(\mathcal{X}, y)$ for the elements $y \in \mathcal{L} \setminus \mathcal{X}$, and set

$$\tau(\mathcal{X}, \mathcal{L}) = \sum_{y \in \mathcal{L}\setminus\mathcal{X},\ y \ne y^*_{\mathcal{L}}} T(\mathcal{X}, y). \tag{2.2}$$

Also define

$$M_\tau(P,k) = \max_{\mathcal{X},\mathcal{L}} \tau(\mathcal{X}, \mathcal{L}), \tag{2.3}$$

and

$$M^*_\tau(P,k) = \max_{\substack{\mathcal{X},\mathcal{L} \\ \tau(\mathcal{X},\mathcal{L}) = M_\tau(P,k)}} T(\mathcal{X}, y^*_{\mathcal{L}}). \tag{2.4}$$

Now the following Hilton-Milner type theorem holds:

Theorem 2.1. *For fixed $1 \le t < k$, and a sequence of posets P_n, let us be given a maximum sized family $\mathcal{F}_n$ of non-trivially t-intersecting k-chains in P_n. Assume further that*

$$\lim_{n\to\infty} r_{t+2}(P_n,k)/M^*_\tau(P_n,k) = 0. \tag{2.5}$$

Then, for n sufficiently large, $\mathcal{F}_n$ has one of the following two descriptions:

(i) *there exists a t-chain $\mathcal{X}$ and a $(k+1-t)$-chain $\mathcal{Y}$, such that $\mathcal{X} \cap \mathcal{Y} = \emptyset$; and $\mathcal{F}_n$ is the following set of k-chains*:

$$\mathcal{F}(\mathcal{X}, \mathcal{Y}) = \{\mathcal{L} \colon\ \mathcal{X} \subseteq \mathcal{L} \text{ and } \ \mathcal{L} \cap \mathcal{Y} \neq \emptyset\} \cup \{\mathcal{L} \colon\ \mathcal{Y} \subseteq \mathcal{L} \text{ and } |\mathcal{L} \cap \mathcal{X}| = t-1\}, \tag{2.6}$$

where the second set of chains is non-empty;

(ii) *there exists a $(t+2)$-chain $\mathcal{Z}$, and $\mathcal{F}_n$ is the following set of k-chains*:

$$\mathcal{F}(\mathcal{Z}) = \{\mathcal{L} \colon\ |\mathcal{L} \cap \mathcal{Z}| \geq t+1\}, \tag{2.7}$$

and $|\bigcap_{\mathcal{L} \in \mathcal{F}_n} \mathcal{L} \cap \mathcal{Z}| \leq t-1$.

3. New Results

Let $t < k < \ell$ and $n_1 \leq \cdots \leq n_\ell$ be positive integers. We define two families $\mathcal{F}_1(t, k; n_1, \ldots, n_\ell)$ and $\mathcal{F}_2(t, k; n_1, \ldots, n_\ell)$ of non-trivially t-intersecting families in $\mathbb{F}_\ell(n_1, \ldots, n_\ell)$ with support size k as follows.

(i) Let $j_1, j_2, \ldots, j_{k+1}$ be integers satisfying $1 \leq j_i \leq n_i$ for $i \in [1, k+1]$. We define $\mathcal{F}_1(t, k; n_1, \ldots, n_\ell)$ as the set of ℓ-tuples $\mathbf{k} = (k_1, \ldots, k_\ell)$ with support size k which belong to the set

$$\{\mathbf{k} \colon k_i = j_i \text{ for all } \ i \in [1, t] \text{ and for at least one } \ i \in [t+1, k+1]\} \cup \{\mathbf{k} \colon k_i = j_i \text{ for all } \ i \in [t+1, k+1] \text{ and for } t-1 \text{ values } i \in [1, t]\}. \tag{3.1}$$

(ii) Let $j_1, j_2, \ldots, j_{t+2}$ be integers satisfying $1 \leq j_i \leq n_i$ for $i \in [1, t+2]$. We define $\mathcal{F}_2(t, k; n_1, \ldots, n_\ell)$ as the set of ℓ-tuples $\mathbf{k} = (k_1, \ldots, k_\ell)$ with support size k which belong to the set

$$\{\mathbf{k} \colon k_i = j_i \text{ for at least } t+1 \text{ values } i \in [1, t+2]\}. \tag{3.2}$$

Note that $|\mathcal{F}_1(t, k; n_1, \ldots, n_\ell)|$ and $|\mathcal{F}_2(t, k; n_1, \ldots, n_\ell)|$ do not depend on the particular choices of the j_i. Our goal is to give sufficient conditions for the parameter values $t, k, \ell, n_1, \ldots, n_\ell$ which ensure that either $\mathcal{F}_1$ or $\mathcal{F}_2$ is of maximum size among the non-trivially t-intersecting families of ℓ-tuples with support size k.

Given $n_1 \leq \cdots \leq n_\ell$, we define a partially ordered set $(\mathcal{P}(n_1, \ldots, n_\ell), \prec)$ as follows. The underlying set is $\mathcal{P}(n_1, \ldots, n_\ell) := \{(i, j) \colon 1 \leq i \leq \ell, \ \ 1 \leq j \leq n_i\}$, and $(i_1, j_1) \prec (i_2, j_2)$ if and only if $i_1 < i_2$. The map $\mathbf{k} = (k_1, \ldots, k_\ell) \mapsto \{(i, k_i) \in \mathcal{P}(n_1, \ldots, n_\ell) \colon k_i \neq 0\}$ is obviously a bijection between $\mathbb{F}_\ell(n_1, \ldots, n_\ell)$ and the chains in the poset $(\mathcal{P}(n_1, \ldots, n_\ell), \prec)$, and ℓ-tuples with support size k are mapped to k-chains. Therefore, t-intersecting families of ℓ-tuples in $\mathbb{F}_\ell(n_1, \ldots, n_\ell)$ with support size k correspond to t-intersecting k-chains in $(\mathcal{P}(n_1, \ldots, n_\ell), \prec)$. For a subset $\mathcal{Y} \subseteq \mathcal{P}(n_1, \ldots, n_\ell)$, we define the *support* of $\mathcal{Y}$ as the set of first coordinates of the elements of $\mathcal{Y}$; namely, $\operatorname{supp}(\mathcal{Y}) = \{i \leq \ell \colon \exists j \leq n_i \ (i, j) \in \mathcal{Y}\}$. We start with the determination of the quantities r_{t+2}, M_τ, and

M^*_τ defined in Section 2. Note that for any m-chain $\mathcal{L}$ in $\mathcal{P} = (\mathcal{P}(n_1,\dots,n_\ell),\prec)$, we have

$$T_{\mathcal{P},k}(\mathcal{L}) = \sum_{\substack{A\subset[1,\ell]\setminus\mathrm{supp}(\mathcal{L})\\ |A|=k-m}} \prod_{i\in A} n_i. \tag{3.3}$$

Proposition 3.1. *Let $t<k<\ell$, let $\mathcal{P} = (\mathcal{P}(n_1,\dots,n_\ell),\prec)$ and let $\mathcal{L}$ be an m-chain in $\mathcal{P}$. Suppose that $(i,k_i)\in\mathcal{L}$ and $j\notin\mathrm{supp}(\mathcal{L})$ with $j<i$, and let $\mathcal{L}^* = (\mathcal{L}\setminus\{(i,k_i)\})\cup\{(j,k_j)\}$ for some $k_j\le n_j$. Then $T_{\mathcal{P},k}(\mathcal{L}^*)\ge T_{\mathcal{P},k}(\mathcal{L})$, with equality if and only if $n_j = n_{j+1} = \cdots = n_i$.*

Proof. We obtain $T_{\mathcal{P},k}(\mathcal{L}^*)$ from $T_{\mathcal{P},k}(\mathcal{L})$ by replacing each occurrence of n_j by n_i in the sum in (3.3). Hence the inequalities $n_j\le n_{j+1}\le\cdots\le n_i$ imply both assertions of the proposition. ∎

Let $\sigma_i(x_1,x_2,\dots,x_m)$ denote the ith elementary symmetric polynomial in variables $x_1,x_2,\dots,x_m$. We define $\sigma_0(x_1,x_2,\dots,x_m)=1$.

Lemma 3.2. *Let $t<k<\ell$ and let $\mathcal{P} = (\mathcal{P}(n_1,\dots,n_\ell),\prec)$. Then*

$$r_{t+2}(\mathcal{P},k) = \sum_{\substack{A\subset[t+3,\ell]\\ |A|=k-t-2}} \prod_{i\in A} n_i = \sigma_{k-t-2}(n_{t+3},\dots,n_\ell). \tag{3.4}$$

Proof. Proposition 3.1 implies that for $(t+2)$-chains $\mathcal{L}$ in $\mathcal{P}$, the quantity $T_{\mathcal{P},k}(\mathcal{L})$ is maximized when $\mathrm{supp}(\mathcal{L}) = [1,t+2]$. ∎

Lemma 3.3. *Let $t<k<\ell$ and let $\mathcal{P} = (\mathcal{P}(n_1,\dots,n_\ell),\prec)$. Then for any t-chain $\mathcal{X}$ and k-chain $\mathcal{L}$ in $\mathcal{P}$ with $|\mathcal{X}\cup\mathcal{L}| = k+1$, we have $M_\tau(\mathcal{P},k) = \tau(\mathcal{X},\mathcal{L})$ if and only if the multiset relations $\{n_i\colon i\in\mathrm{supp}(\mathcal{X})\} = \{n_i\colon 1\le i\le t\}$ and $\{n_i\colon i\in\mathrm{supp}(\mathcal{L})\}\supseteq\{n_i\colon t+1\le i\le k\}$ hold.*

Proof. We first note that the condition $|\mathcal{X}\cup\mathcal{L}| = k+1$ implies that $\mathcal{X}$ and $\mathcal{L}$ have $t-1$ common elements and $|\mathcal{L}\setminus\mathcal{X}| = k-t+1$. Moreover, since $\tau(\mathcal{X},\mathcal{L})$ is the sum of only $k-t$ values $T(\mathcal{X},y)$ with $y\in\mathcal{L}\setminus\mathcal{X}$, it is possible that for a fixed t-chain $\mathcal{X}$, $\tau(\mathcal{X},\mathcal{L})$ is maximized for some $\mathcal{L}$ even though $T(\mathcal{X},y)=0$ for some $y\in\mathcal{L}\setminus\mathcal{X}$.

For a fixed t-chain $\mathcal{X}$, Proposition 3.1 implies that $\tau(\mathcal{X},\mathcal{L})$ is maximized for a k-chain $\mathcal{L}$ whose support contains the $k-t$ smallest elements of $[1,\ell]\setminus\mathrm{supp}(\mathcal{X})$. Moreover, another application of Proposition 3.1 shows that if $\mathcal{X}'$ is obtained by replacing an element $(i_1,j_1)\in\mathcal{X}$ with some (i_2,j_2) satisfying $i_2<i_1$ and i_2 the smallest number not in $\mathrm{supp}(\mathcal{X})$ then $\tau(\mathcal{X}',\mathcal{L}')\ge\tau(\mathcal{X},\mathcal{L})$ for an optimal $\mathcal{L}'$ constructed in the way described in the previous sentence. Hence $M_\tau(\mathcal{P},k) = \tau(\mathcal{X},\mathcal{L})$ for $\mathcal{X},\mathcal{L}$ with $\mathrm{supp}(\mathcal{X}) = [1,t]$ and $\mathrm{supp}(\mathcal{L})\supseteq[t+1,k]$. Finally, Proposition 3.1 also implies that if $\mathrm{supp}(\mathcal{X}')\ne[1,t]$ or $\mathrm{supp}(\mathcal{L}')\not\supseteq[t+1,k]$ then $\tau(\mathcal{X}',\mathcal{L}')<M_\tau(\mathcal{P},k)$, unless the condition about the multiset of n_i values described in the statement of the lemma holds. ∎

Lemma 3.4. *Let $t<k<\ell$ and let $\mathcal{P} = (\mathcal{P}(n_1,\dots,n_\ell),\prec)$. Then*

$$M^*_\tau(\mathcal{P},k) = \sum_{\substack{A\subset[t+1,\ell]\setminus\{k+1\}\\ |A|=k-t-1}} \prod_{i\in A} n_i = \sigma_{k-t-1}(n_{t+1},\dots,\widehat{n_{k+1}},\dots,n_\ell). \tag{3.5}$$

Proof. Let $\mathcal{X}$ be a t-chain and $\mathcal{L}$ be a k-chain with $|\mathcal{X} \cup \mathcal{L}| = k+1$ and $\tau(\mathcal{X}, \mathcal{L}) = M_\tau(\mathcal{P}, k)$. Then, by Lemma 3.3, we have the multiset relations $\{n_i\colon i \in \operatorname{supp}(\mathcal{X})\} = \{n_i\colon 1 \le i \le t\}$ and $\{n_i\colon i \in \operatorname{supp}(\mathcal{L})\} \supseteq \{n_i\colon t+1 \le i \le k\}$. Also, we have $k \le |\operatorname{supp}(\mathcal{X} \cup \mathcal{L})| \le k+1$. If $|\operatorname{supp}(\mathcal{X} \cup \mathcal{L})| = k$ then there exists $y^*_{\mathcal{L}} = (i, k_i) \in \mathcal{L} \setminus \mathcal{X}$ with $i \in \operatorname{supp}(\mathcal{X})$ and so $T(\mathcal{X}, y^*_{\mathcal{L}}) = 0$. If $|\operatorname{supp}(\mathcal{X} \cup \mathcal{L})| = k+1$ then Proposition 3.1 implies that $T(\mathcal{X}, y)$ is minimized in $\mathcal{L} \setminus \mathcal{X}$ for the $y^*_{\mathcal{L}} = (i, k_i) \in \mathcal{L} \setminus \mathcal{X}$ with $i = \max \operatorname{supp}(\mathcal{L} \setminus \mathcal{X})$ and, in order to maximize $T(\mathcal{X}, y^*_{\mathcal{L}})$, we have to choose $\max \operatorname{supp}(\mathcal{L} \setminus \mathcal{X})$ as small as possible. Combining these observations, we obtain that $\max T(\mathcal{X}, y^*_{\mathcal{L}})$ is achieved in the case $\operatorname{supp}(\mathcal{X}) = [1, t]$, $\operatorname{supp}(\mathcal{L} \setminus \mathcal{X}) = [t+1, k+1]$, and $\operatorname{supp}(y^*_{\mathcal{L}}) = \{k+1\}$, leading to (3.5). ∎

The following two lemmas will be useful at the comparison of r_{t+2} and M^*_τ.

Lemma 3.5. *Let t, k, ℓ satisfy $k \ge t+2$ and $\ell \ge 2k-t-1$, and let $\mathcal{P} = (\mathcal{P}(n_1, \ldots, n_\ell), \prec)$. Then*

$$r_{t+2}(\mathcal{P}, k) \le \left(1 + \frac{k-t-2}{\ell-2k+t+2}\right) \sigma_{k-t-2}(n_{t+1}, \ldots, \widehat{n_{k+1}}, \ldots, n_\ell).$$

Proof. On one hand, if $A \subseteq [t+1, \ell]$ satisfies $|A| = k-t-2$ and $k+1 \in A$ then

$$\prod_{i \in A} n_i \le \frac{\sum_{s \in [k+2, \ell] \setminus A} n_s}{(\ell-k-1) - (k-t-3)} \prod_{i \in A \setminus \{k+1\}} n_i.$$

On the other hand, any $(k-t-2)$-element subset B of $[t+1, \ell] \setminus \{k+1\}$ can be obtained at most $k-t-2$ ways by replacing $k+1$ by an element $j \ge k+2$ of B. Hence Lemma 3.2 implies

$$\begin{aligned} r_{t+2}(\mathcal{P}, k) &= \sigma_{k-t-2}(n_{t+3}, \ldots, n_\ell) \\ &\le \sigma_{k-t-2}(n_{t+1}, \ldots, n_\ell) \\ &\le \left(1 + \frac{k-t-2}{\ell-2k+t+2}\right) \sigma_{k-t-2}(n_{t+1}, \ldots, \widehat{n_{k+1}}, \ldots, n_\ell). \end{aligned}$$ ∎

Lemma 3.6. *Let t, k, ℓ satisfy $k \ge t+2$ and let $\mathcal{P} = (\mathcal{P}(n_1, \ldots, n_\ell), \prec)$. Then*

$$M^*_\tau(\mathcal{P}, k) \ge n_{t+1} \, \frac{\ell-k+1}{k-t-1} \, \sigma_{k-t-2}(n_{t+1}, \ldots, \widehat{n_{k+1}}, \ldots, n_\ell). \tag{3.6}$$

Proof. Using the fact that any $(k-t-2)$-element subset B of $[t+1, \ell] \setminus \{k+1\}$ can be obtained $(\ell-t-1) - (k-t-2) = \ell-k+1$ ways by deleting an element different from $k+1$ from a $(k-t-1)$-element subset of $[t+1, \ell] \setminus \{k+1\}$, we have

$$\begin{aligned} &(k-t-1)\sigma_{k-t-1}(n_{t+1}, \ldots, \widehat{n_{k+1}}, \ldots, n_\ell) \\ &= \sum_{\substack{s=t+1 \\ s \ne k+1}}^{\ell} n_s \sigma_{k-t-2}(n_{t+1}, \ldots, \widehat{n_s}, \ldots, \widehat{n_{k+1}}, \ldots, n_\ell) \\ &\ge n_{t+1}(\ell-k+1)\sigma_{k-t-2}(n_{t+1}, \ldots, \widehat{n_{k+1}}, \ldots, n_\ell). \end{aligned}$$

Hence Lemma 3.4 implies (3.6). ∎

Lemma 3.7. *Let $t < k < \ell$ and $\mathcal{P} = (\mathcal{P}(n_1, \dots, n_\ell), \prec)$. If $\mathcal{X}$ is a t-chain and $\mathcal{Y}$ is a $(k+1-t)$-chain with $\mathcal{X} \cap \mathcal{Y} = \emptyset$ then $|\mathcal{F}(\mathcal{X}, \mathcal{Y})| \le |\mathcal{F}_1(t, k; n_1, \dots, n_\ell)|$ for the families of chains defined in* (2.6) *and* (3.1)*, respectively.*

Proof. First note that $|\mathrm{supp}(\mathcal{X}) \cap \mathrm{supp}(\mathcal{Y})| \le 1$, because otherwise there is no k-chain containing $\mathcal{Y}$ and $t-1$ elements of $\mathcal{X}$ as required in (2.6). If $|\mathrm{supp}(\mathcal{X}) \cap \mathrm{supp}(\mathcal{Y})| = 1$, say $(i, f_i) \in \mathcal{X}$ and $(i, g_i) \in \mathcal{Y}$ for some $f_i \ne g_i$, then there exists exactly one k-chain in $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ which contains (i, g_i), namely, $(\mathcal{Y} \cup \mathcal{X}) \setminus \{(i, f_i)\}$. Hence, if we define $\mathcal{Y}_1 = (\mathcal{Y} \setminus \{(i, g_i)\}) \cup \{(j, 1)\}$ for some $j \notin \mathrm{supp}(\mathcal{X} \cup \mathcal{Y})$ then $|\mathcal{F}(\mathcal{X}, \mathcal{Y})| \le |\mathcal{F}(\mathcal{X}, \mathcal{Y}_1)|$, because $\mathcal{F}(\mathcal{X}, \mathcal{Y}_1)$ contains all but one chain from $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ and it contains t chains not in $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ (the chains obtained by deleting an element of $\mathcal{X}$ from $\mathcal{X} \cup \mathcal{Y}$). Therefore, it is enough to prove that $|\mathcal{F}(\mathcal{X}, \mathcal{Y})| \le |\mathcal{F}_1(t, k; n_1, \dots, n_\ell)|$ for chains $\mathcal{X}, \mathcal{Y}$ with $\mathrm{supp}(\mathcal{X}) \cap \mathrm{supp}(\mathcal{Y}) = \emptyset$.

Suppose now that $\mathrm{supp}(\mathcal{X}) \cap \mathrm{supp}(\mathcal{Y}) = \emptyset$. There are exactly t chains in $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ containing $\mathcal{Y}$ and there are t chains in $\mathcal{F}_1(t, k; n_1, \dots, n_\ell)$ with support containing $[t+1, k+1]$; hence it is enough to show that for the set of chains

$$\mathcal{F}^*(\mathcal{X}, \mathcal{Y}) = \{\mathcal{L} \colon \mathcal{X} \subseteq \mathcal{L} \text{ and } \mathcal{L} \cap \mathcal{Y} \ne \emptyset\},$$

and

$$\mathcal{F}_1^*(t, k; n_1, \dots, n_\ell) = \{\mathcal{L} \in \mathcal{F}_1(t, k; n_1, \dots, n_\ell) \colon \mathrm{supp}(\mathcal{L}) \supseteq [1, t]\},$$

we have $|\mathcal{F}^*(\mathcal{X}, \mathcal{Y})| \le |\mathcal{F}_1^*(t, k; n_1, \dots, n_\ell)|$. If $\mathrm{supp}(\mathcal{X}) \ne [1, t]$ then we define a new set of chains by the following shifting operation. Let $i_1 \in [1, t]$ be the smallest number not in $\mathrm{supp}(\mathcal{X})$ and let $i_2 \in \mathrm{supp}(\mathcal{X})$ with $i_2 > i_1$, say $(i_2, k_{i_2}) \in \mathcal{X}$. For a k-chain $\mathcal{L} \in \mathcal{F}^*(\mathcal{X}, \mathcal{Y})$, let

$$f(\mathcal{L}) = \begin{cases} (\mathcal{L} \setminus \{(i_2, k_{i_2})\}) \cup \{(i_1, 1)\}, & \text{if } i_1 \notin \mathrm{supp}(\mathcal{L}), \\ (\mathcal{L} \setminus \{(i_1, k_{i_1}), (i_2, k_{i_2})\}) \cup \{(i_1, 1), (i_2, k_{i_1})\}, & \text{if } (i_1, k_{i_1}) \in \mathcal{L} \text{ for some} \\ & \quad k_{i_1} \le n_{i_1}. \end{cases} \tag{3.7}$$

Moreover, define $\mathcal{X}' = (\mathcal{X} \setminus \{(i_2, k_{i_2})\}) \cup \{(i_1, 1)\}$ and

$$\mathcal{Y}' = \begin{cases} \mathcal{Y}, & \text{if } i_1 \notin \mathrm{supp}(\mathcal{Y}), \\ (\mathcal{Y} \setminus \{(i_1, k_{i_1})\}) \cup \{(i_2, k_{i_1})\}, & \text{if } (i_1, k_{i_1}) \in \mathcal{Y} \text{ for some } k_{i_1} \le n_{i_1}. \end{cases} \tag{3.8}$$

Then it is clear that f is an injection from $\mathcal{F}^*(\mathcal{X}, \mathcal{Y})$ into $\mathcal{F}^*(\mathcal{X}', \mathcal{Y}')$, and so $|\mathcal{F}^*(\mathcal{X}, \mathcal{Y})| \le |\mathcal{F}^*(\mathcal{X}', \mathcal{Y}')|$ and $|\mathcal{F}(\mathcal{X}, \mathcal{Y})| \le |\mathcal{F}(\mathcal{X}', \mathcal{Y}')|$. Repeating this procedure, we arrive to some t-chain $\mathcal{X}''$ and $(k+1-t)$-chain $\mathcal{Y}''$ such that $|\mathcal{F}^*(\mathcal{X}, \mathcal{Y})| \le |\mathcal{F}^*(\mathcal{X}'', \mathcal{Y}'')|$ and $\mathrm{supp}(\mathcal{X}'') = [1, t]$ and $\mathrm{supp}(\mathcal{X}'') \cap \mathrm{supp}(\mathcal{Y}'') = \emptyset$. It is enough to show that $|\mathcal{F}^*(\mathcal{X}'', \mathcal{Y}'')| \le |\mathcal{F}_1^*(t, k; n_1, \dots, n_\ell)|$.

If $\mathrm{supp}(\mathcal{Y}'') \ne [t+1, k+1]$ then let $i_1 \in [t+1, k+1]$ be the smallest number not in $\mathrm{supp}(\mathcal{Y}'')$ and let $i_2 \in \mathrm{supp}(\mathcal{Y}'')$ with $i_2 > i_1$, say $(i_2, k_{i_2}) \in \mathcal{Y}''$. By renumbering the i_2th coordinate, we may assume that $k_{i_2} \le n_{i_1}$. We apply the following modification of the shifting operation described in the previous paragraph. For a k-chain

$\mathcal{L} \in \mathcal{F}^*(\mathcal{X}'', \mathcal{Y}'')$, let

$$g(\mathcal{L}) = \begin{cases} (\mathcal{L} \setminus \{(i_2, j_2)\}) \cup \{(i_1, j_2)\}, & \text{if } i_1 \notin \operatorname{supp}(\mathcal{L}) \text{ and} \\ & \quad (i_2, j_2) \in \mathcal{L} \text{ with } j_2 \leq n_1, \\ (\mathcal{L} \setminus \{(i_1, j_1)\}) \cup \{(i_2, j_1)\}, & \text{if } i_2 \notin \operatorname{supp}(\mathcal{L}) \text{ and} \\ & \quad (i_1, j_1) \in \mathcal{L}, \\ (\mathcal{L} \setminus \{(i_1, j_1), (i_2, j_2)\}) \cup \{(i_1, j_2), (i_2, j_1)\}, & \text{if } (i_1, j_1), (i_2, j_2) \in \mathcal{L} \text{ and} \\ & \quad j_2 \leq n_1, \\ \mathcal{L}, & \text{otherwise.} \end{cases} \tag{3.9}$$

Moreover, define $\mathcal{Y}''' = (\mathcal{Y}'' \setminus \{(i_2, k_{i_2})\}) \cup \{(i_1, k_{i_2})\}$. Then g is an injection from $\mathcal{F}^*(\mathcal{X}'', \mathcal{Y}'')$ into $\mathcal{F}^*(\mathcal{X}''', \mathcal{Y}''')$, and so $|\mathcal{F}^*(\mathcal{X}'', \mathcal{Y}'')| \leq |\mathcal{F}^*(\mathcal{X}''', \mathcal{Y}''')|$ and $|\mathcal{F}(\mathcal{X}'', \mathcal{Y}'')| \leq |\mathcal{F}(\mathcal{X}''', \mathcal{Y}''')|$. Repeating this procedure, we arrive to a member of the family $\mathcal{F}_1^*(t, k; n_1, \ldots, n_\ell)$. ∎

Lemma 3.8. *Let $t < k < \ell$ and let $\mathcal{P} = (\mathcal{P}(n_1, \ldots, n_\ell), \prec)$. If $\mathcal{Z}$ is a $(t+2)$-chain then $|\mathcal{F}(\mathcal{Z})| \leq |\mathcal{F}_2(t, k; n_1, \ldots, n_\ell)|$ for the families of chains defined in* (2.7) *and* (3.2)*, respectively.*

Proof. Given $\mathcal{F}(\mathcal{Z})$, if $\operatorname{supp}(\mathcal{Z}) \neq [1, t+2]$ then we can apply the shifting procedure described in (3.9), not decreasing the size of $\mathcal{F}(\mathcal{Z})$, and eventually arriving to a set of chains in the family $\mathcal{F}_2(t, k; n_1, \ldots, n_\ell)$. ∎

Lemma 3.9. *For $\mathcal{F}_1$ and $\mathcal{F}_2$ from* (3.1) *and* (3.2)*,*

$$|\mathcal{F}_1| = \sigma_{k-t}(n_{t+1}, \ldots, n_\ell) - \sigma_{k-t}(n_{t+1} - 1, \ldots, n_{k+1} - 1, n_{k+2}, \ldots, n_\ell) + t,$$

$$|\mathcal{F}_2| = \sum_{i=1}^{t+2} \sigma_{k-t-1}(n_i, n_{t+3}, \ldots, n_\ell) - (t+1)\sigma_{k-t-2}(n_{t+3}, \ldots, n_\ell).$$

Proof. Explanation for $|\mathcal{F}_1|$. The second line of (3.1) yields the term t, and the cardinality arising from the first line of (3.1) is obtained as a difference, counting all functions $\mathbf{k}$ with $k_i = j_i$ for all $i \in [1, t]$, and subtracting the number of functions $\mathbf{k}$ with $k_i = j_i$ for all $i \in [1, t]$ that have no $i \in [t+1, k+1]$ with $k_i = j_i$.

Explanation for $|\mathcal{F}_2|$. Fix a $(t+2)$-chain $\mathcal{Z}$ with support $[1, t+2]$. For $i \in [1, t+2]$, the number of k-chains intersecting $\mathcal{Z}$ in coordinates $1, 2, \ldots, i-1, i+1, \ldots, t+2$ is $\sigma_{k-t-1}(n_i, n_{t+3}, \ldots, n_\ell)$. Adding these expressions for all $i \in [1, t+2]$, the k-chains intersecting $\mathcal{Z}$ in exactly $t+1$ coordinates are counted once, and the k-chains intersecting $\mathcal{Z}$ in $t+2$ coordinates are counted $t+2$ times. The negative term reduces the multiplicity of the latter ones to one. ∎

In order to apply Theorem 2.1, we have to find values of the parameters $t, k, \ell, n_1, \ldots, n_\ell$ such that the hypothesis of the theorem is satisfied.

Theorem 3.10. *Let $t < k < \ell$ be fixed. Then there exists a bound $n(t, k, \ell)$ such that if $n > n(t, k, \ell)$ then for any non-trivially t-intersecting family $\mathcal{F}$ of ℓ-tuples with support*

k in $\mathbb{F}_\ell(n,\dots,n)$ we have

$$|\mathcal{F}| \le \max\{|\mathcal{F}_1(t,k;n,\dots,n)|, |\mathcal{F}_2(t,k;n,\dots,n)|\}.$$

Moreover, if $k > 2t+1$ then for large enough n we have

$$|\mathcal{F}_1(t,k;n,\dots,n)| > |\mathcal{F}_2(t,k;n,\dots,n)|$$

and if $t+1 < k \le 2t+1$ then for large enough n we have

$$|\mathcal{F}_1(t,k;n,\dots,n)| < |\mathcal{F}_2(t,k;n,\dots,n)|.$$

Proof. Let $\mathcal{P}_n = (\mathcal{P}(n,\dots,n), \prec)$. By Lemmas 3.2 and 3.4, we have $r_{t+2}(\mathcal{P}_n,k) = \binom{l-t-2}{k-t-2}n^{k-t-2}$ and $M^*_\tau(\mathcal{P}_n,k) = \binom{l-t-1}{k-t-1}n^{k-t-1}$. Hence

$$\lim_{n\to\infty}\frac{r_{t+2}(\mathcal{P}_n,k)}{M^*_\tau(\mathcal{P}_n,k)} = \lim_{n\to\infty}\frac{k-t-1}{l-t-1}\cdot\frac{1}{n} = 0 \tag{3.10}$$

and so Theorem 2.1, together with Lemmas 3.7 and 3.8, implies that for large enough n one of the maximum sized families of t-intersecting ℓ-tuples with support k in $\mathbb{F}_\ell(n,\dots,n)$ is $\mathcal{F}_1 = \mathcal{F}_1(t,k;n,\dots,n)$ or $\mathcal{F}_2 = \mathcal{F}_2(t,k;n,\dots,n)$.

Our final task is to compare $|\mathcal{F}_1(t,k;n,\dots,n)|$ and $|\mathcal{F}_2(t,k;n,\dots,n)|$. From Lemma 3.9 we have

$$|\mathcal{F}_1| = t + \binom{l-t}{k-t}n^{k-t} - \sum_{i=0}^{k-t}\binom{k+1-t}{i}\binom{l-k-1}{k-t-i}(n-1)^i n^{k-t-i} \tag{3.11}$$

and

$$|\mathcal{F}_2| = (t+2)\binom{l-t-1}{k-t-1}n^{k-t-1} - (t+1)\binom{l-t-2}{k-t-2}n^{k-t-2}. \tag{3.12}$$

Suppose now that $t+2 \le k$. For fixed t,k,ℓ, as $n\to\infty$, we expand (3.11) and (3.12) as polynomials of n. There is nothing to do with (3.12), as it is already written in polynomial form. In (3.11), the coefficient of n^{k-t} in $|\mathcal{F}_1|$ is

$$\binom{l-t}{k-t} - \sum_{i=0}^{k-t}\binom{k+1-t}{i}\binom{l-k-1}{k-t-i} = 0,$$

the coefficient of n^{k-t-1} in $|\mathcal{F}_1|$ is

$$\begin{aligned}\sum_{i=1}^{k-t} i\binom{k+1-t}{i}\binom{l-k-1}{k-t-i} &= \sum_{i=1}^{k-t}(k+1-t)\binom{k-t}{i-1}\binom{l-k-1}{k-t-i}\\ &= (k+1-t)\binom{l-t-1}{k-t-1},\end{aligned}$$

and similarly the coefficient of n^{k-t-2} in $|\mathcal{F}_1|$ is

$$-\sum_{i=1}^{k-t}\binom{i}{2}\binom{k+1-t}{i}\binom{l-k-1}{k-t-i} = -\frac{(k+1-t)(k-t)}{2}\binom{l-t-2}{k-t-2}.$$

We compare $|\mathcal{F}_1|$ and $|\mathcal{F}_2|$ for large n. The leading term in both is n^{k-t-1}, with coefficients $(k+1-t)\binom{l-t-1}{k-t-1}$ and $(t+2)\binom{l-t-1}{k-t-1}$. Therefore, if $k+1-t > t+2$, i.e. $k > 2t+1$, then for large enough n we have $|\mathcal{F}_1| > |\mathcal{F}_2|$ and if $k < 2t+1$ then for large enough n we have $|\mathcal{F}_1| < |\mathcal{F}_2|$. If $k-t-1 = t+2$, i.e. $k = 2t+1$, then the main terms have equal coefficients. We compare the coefficients of the next term, $n^{k-t-2} = n^{t-1}$ in $|\mathcal{F}_1|$ and $|\mathcal{F}_2|$, which are $-\frac{(t+2)(t+1)}{2}\binom{l-t-2}{t-1}$ and $-(t+1)\binom{l-t-2}{t-1}$, respectively. We have $|\mathcal{F}_1| < |\mathcal{F}_2|$. ∎

Theorem 3.11. *Let $t < k$ be fixed. Then there exists a bound $\ell(t,k)$ such that if $\ell > \ell(t,k)$ then for any non-trivially t-intersecting family $\mathcal{F}$ of ℓ-tuples with support k in $\mathbb{F}_\ell(n_1,\dots,n_\ell)$ we have*

$$|\mathcal{F}| \leq \max\{|\mathcal{F}_1(t,k;n_1,\dots,n_\ell)|, |\mathcal{F}_2(t,k;n_1,\dots,n_\ell)|\}.$$

Proof. Let $\mathcal{P}_\ell = (\mathcal{P}(n_1,\dots,n_\ell),\prec)$. If $k = t+1$ then $r_{t+2}(\mathcal{P}_\ell,k) = 0$ and $M^*_\tau(\mathcal{P}_\ell,k) > 0$. If $k \geq t+2$ then by Lemmas 3.5 and 3.6, for $\ell \geq 2k-t-1$ we have

$$\frac{r_{t+2}(\mathcal{P}_\ell,k)}{M^*_\tau(\mathcal{P}_\ell,k)} \leq \left(1+\frac{k-t-2}{\ell-2k+t+2}\right)\cdot\frac{1}{n_{t+1}}\cdot\frac{k-t-1}{\ell-k+1} \tag{3.13}$$

and therefore

$$\lim_{\ell\to\infty}\frac{r_{t+2}(\mathcal{P}_\ell,k)}{M^*_\tau(\mathcal{P}_\ell,k)} = 0.$$

So Theorem 2.1, together with Lemmas 3.7 and 3.8, implies that for large enough ℓ one of the maximum sized families of t-intersecting ℓ-tuples with support k in $\mathbb{F}_\ell(n_1,\dots,n_\ell)$ is $\mathcal{F}_1 = \mathcal{F}_1(t,k;n_1,\dots,n_\ell)$ or $\mathcal{F}_2 = \mathcal{F}_2(t,k;n_1,\dots,n_\ell)$. ∎

Theorem 3.12. *Let $t < k < \ell$ be fixed, satisfying $\ell \geq 2k-t-1$. Then there exists a bound $n(t,k,\ell)$ such that if $n_{t+1} > n(t,k,\ell)$ then for any non-trivially t-intersecting family $\mathcal{F}$ of ℓ-tuples with support k in $\mathbb{F}_\ell(n_1,\dots,n_\ell)$ we have $|\mathcal{F}| \leq \max\{|\mathcal{F}_1(t,k;n_1,\dots,n_\ell)|, |\mathcal{F}_2(t,k;n_1,\dots,n_\ell)|\}$.*

Proof. Let $\mathcal{P}_{n_{t+1}} = (\mathcal{P}(n_1,\dots,n_{t+1},\dots,n_\ell),\prec)$. If $k = t+1$ then $r_{t+2}(\mathcal{P}_{n_{t+1}},k) = 0$ and $M^*_\tau(\mathcal{P}_{n_{t+1}},k) > 0$. If $k \geq t+2$ then, analogously to (3.13) in the proof of Theorem 3.11,

$$\frac{r_{t+2}(\mathcal{P}_{n_{t+1}},k)}{M^*_\tau(\mathcal{P}_{n_{t+1}},k)} \leq \left(1+\frac{k-t-2}{\ell-2k+t+2}\right)\cdot\frac{1}{n_{t+1}}\cdot\frac{k-t-1}{\ell-k+1} \tag{3.14}$$

and therefore

$$\lim_{n_{t+1}\to\infty}\frac{r_{t+2}(\mathcal{P}_{n_{t+1}},k)}{M^*_\tau(\mathcal{P}_{n_{t+1}},k)} = 0.$$

So Theorem 2.1, together with Lemmas 3.7 and 3.8, implies that for large enough n_{t+1} one of the maximum sized families of t-intersecting ℓ-tuples with support k in $\mathbb{F}_\ell(n_1,\dots,n_\ell)$ is $\mathcal{F}_1 = \mathcal{F}_1(t,k;n_1,\dots,n_\ell)$ or $\mathcal{F}_2 = \mathcal{F}_2(t,k;n_1,\dots,n_\ell)$. ∎

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