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On the average rank of LYM-sets

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Abstract

Let S be a finite set with some rank function r such that the Whitney numbers $w_i = |\{x \in S \mid r(x) = i\}|$ are log-concave. Given $k, m \in \mathbb{N}$ so that $w_{k-1} < w_k \leqslant w_{k+m}$, set $W = w_k + w_{k+1} + \cdots + w_{k+m}$. Generalizing a theorem of Kleitman and Milner, we prove that every $F \subseteq S$ with cardinality $|F| \geqslant W$ has average rank at least $(kw_k + \cdots + (k+m)w_{k+m})/W$, provided the normalized profile vector $(x_1, \ldots, x_n)$ of F satisfies the following LYM-type inequality: $x_0 + x_1 + \cdots + x_n \leqslant m + 1$.

1. Introduction

Extremal set theory studies the combinatorial structure of sets that maximize certain parameters under various constraints. A fundamental result in this direction is due to Sperner [11], who proved that in the ordered set of all subsets of a finite set no antichain is larger than the level corresponding to the largest binomial coefficient. It was subsequently discovered that Sperner's result is a consequence of the fact that profile vectors of antichains in Boolean algebras satisfy a basic linear inequality [12, 10, 9] known as *LYM-inequality*. We will come back to a more detailed discussion of these concepts in Section 3.

Many important ordered sets (e.g., lattices of subspaces of finite vector spaces, lattices of subspaces of finite affine spaces, divisor lattices of integers) share with Boolean algebras the property that their antichains satisfy the LYM-condition. Hence analogous results follow for these ordered sets.

Kleitman and Milner [8] determined the best-possible lower bound on the *average rank* of an antichain in a Boolean algebra if the antichain has at least size $\binom{n}{k}$. Odlyzko pointed out that the theorem of Kleitman and Milner actually may be deduced by solving an associated linear program (cf. [6]). The solution of the latter only makes

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use of the LYM-inequality and the fact that the binomial coefficients are *logarithmically concave.*

It is the purpose of this note to derive an m-analogue of the theorem of Kleitman and Milner by looking at the average rank of sets with cardinality at least the sum of $m + 1$ successive level numbers. We start out with a very general model: a finite set S with some 'rank' function r. No special property of S or r is assumed at the outset. r induces a partition of the ground set S into blocks

$$P_i = \{x \in S \mid r(x) = i\}.$$

We study the profile vectors of subsets $F \subset S$ and normalize these relative to the rank function r. By definition, an $(m + 1)$-set F is a subset of S whose normalized profile lies in the simplex $S_m \subseteq \mathbb{R}^{n+1}$ with $(0, 1)$-vertices, which we call the *Sperner polytope* (see Section 2). Re-interpretation of the vertices of S_m immediately yields well-known results on Sperner $(m + 1)$-families in a wider context (see Section 3).

In order to obtain results on the average rank of an $(m + 1)$-set, we generalize Odlyzko's linear program accordingly. The difficulty consists in the determination of the optimal solution of the associated dual program. Therefore, we study the feasibility region of the dual program and find the optimal solution from our geometric analysis in Section 2. Our argument is valid if the *Whitney numbers* $w_i = |P_i|$, associated with the rank function r, are logarithmically concave.

It is curious to note that, from an algebraic point of view, our argument in Section 2 essentially establishes a new non-linear inequality for log-concave sequences of numbers (Proposition 2). While this inequality follows directly from the geometry, it appears to be more involved to give an ad hoc purely algebraic proof.

2. A linear programming model

For the integer parameters $0 \leqslant m \leqslant n$, we define the *m-Sperner polytope* $S_m \subseteq \mathbb{R}^{n+1}$ to be the collection of all vectors $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ that satisfy the linear inequalities

$$x_0 + x_1 + \cdots + x_n \leqslant m + 1, \tag{2.1}$$

$$0 \leqslant x_i \leqslant 1 \quad (i = 0, 1, \ldots, n). \tag{2.2}$$

The following observation is then immediate.

Proposition 1. *The vertices of S_m are exactly the $(0, 1)$-vectors $v \in \mathbb{R}^{n+1}$ with at most $m + 1$ non-zero components.*

Let now $w_0, w_1, \ldots, w_n$ be $n + 1$ (strictly) positive real weights. For technical reasons, it is convenient to define $w_{-1} = 0$. We call the number

$$w(x) = w_0 x_0 + \cdots + w_0 x_n,$$

the *weight* of the vector $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$, while

$$\bar{w}(x) = w_1 x_1 + 2w_2 x_2 + \cdots + iw_i x_i + \cdots + nw_n x_n$$

is the *weighted rank* of x. Given $W \in \mathbb{R}$, we want to determine a lower bound on the average rank $\bar{w}(x)/w(x)$ of an arbitrary $x \in S_m$ with $w(x) \geqslant W$.

Note that, in view of Proposition 1, our problem is only meaningful if W does not exceed the sum of the $m + 1$ largest weights. Assuming feasibility, we hence want to solve the linear program

$$(\mathrm{P}_m) \qquad \begin{aligned} &\min \sum_{i=0}^{n} iw_i x_i \\ &\text{s.t.} \sum_{i=0}^{n} x_i \leqslant m + 1, \\ &\sum_{i=0}^{n} w_i x_i \geqslant W, \\ &0 \leqslant x_i \leqslant 1. \end{aligned}$$

Equivalently, we may study the linear programming dual

$$(\mathrm{D}_m) \qquad \begin{aligned} &\max -(m+1)u + Wv - \sum_{i=0}^{n} z_i \\ &\text{s.t.} -u + w_i v - z_i \leqslant iw_i \quad (i = 0, 1, \ldots, n), \\ &u, v, z_i \geqslant 0. \end{aligned}$$

From an algorithmic point of view, our problem just amounts of solving (P_m). If we wish to derive qualitative statements, we may study (D_m) as any feasible solution of (D_m) yields a lower bound for the objective function in (P_m). We will also impose more structure on our set of weights.

2.1. The case $m = 0$ and log-concavity

In this subsection, we will assume throughout $m = 0$. Thus the restrictions $x_i \leqslant 1$ in (P_0) are redundant and the dual (D_0) becomes

$$(\mathrm{D}_0) \qquad \begin{aligned} &\max -u + Wv \\ &\text{s.t.} -u + w_i v \leqslant iw_i \quad (i = 0, 1, \ldots, n), \\ &u, v \geqslant 0. \end{aligned}$$

The feasibility region R_0 of (D_0) is the area in the non-negative orthant of $\mathbb{R}^2$ bounded by the lines

$$L_i = \{(u, v) \in \mathbb{R}^2 \mid -u + w_i v = iw_i\}.$$

Assume for the moment $0 < w_0 < w_1 < \cdots < w_n$. Then the slopes of the lines L_i are strictly positive and monotonically decreasing. In particular, $L_{i-1} \cap L_i \neq \emptyset$ for $i = 1, \ldots, n$. Setting $V_0 = (0, 0)$, we denote furthermore by $V_i = (u_i, v_i)$ the point L_{i-1} and L_i have in common. Are the V_i's the vertices of the feasibility region R_0? Clearly, this will be the case if the V_i's are increasing in their second component, say (because then the corresponding segments of the lines L_i must be part of the boundary of the feasibility region).

To formulate a sufficient condition for the latter, recall that the w_i's are said to be *log-concave* if for $i = 1, \ldots, n-1$,

$$w_i^2 \geqslant w_{i-1} w_{i+1}.$$

Theorem 1. *Let the weights w_i be log-concave and strictly increasing. Then the points V_i defined above are the vertices of the feasibility region R_0 of the linear program (D_0). Moreover, for any $0 \leqslant W \leqslant w_n$, the primal program (P_0) has a unique optimal solution.*

Proof. The second component of $V_i = (u_i, v_i)$ is computed as

$$v_i = i + \frac{w_{i-1}}{w_i - w_{i-1}} \qquad (i = 1, \ldots, n).$$

Hence

$$v_i - v_{i-1} = 1 + \frac{w_{i-1}^2 - w_{i-2} w_i}{(w_i - w_{i-1})(w_{i-1} - w_{i-2})} > 0,$$

proving that $V_0, \ldots, V_n$ are the vertices of R_0.

Assume now, w.l.o.g., $w_{k-1} < W \leqslant w_k$. Then the vertex $V_k = (u_k, v_k)$ optimizes (D_0) as one can easily see by comparing the slopes of the lines L_{k-1} and L_k with the slope of the (dual) objective function. V_k satisfies the dual restrictions

$$-u_k + w_i v_k \begin{cases} = i w_i & \text{if } i = k-1, k, \\ < i w_i & \text{if } i \neq k-1, k, \end{cases}$$

$$u_k > 0, \quad v_k > 0.$$

From complementary slackness, we therefore conclude that every optimal solution $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_n)$ of (P_0) in turn must satisfy

$$\bar{x}_0 + \bar{x}_1 + \cdots + \bar{x}_n = 1,$$

$$w_0 \bar{x}_0 + w_1 \bar{x}_1 + \cdots + w_n \bar{x}_n = W,$$

$$\bar{x}_i = 0 \quad \text{for all } i \neq k-1, k.$$

These conditions determine $\bar{x}$: $\bar{x}_{k-1}$ and $\bar{x}_k$ are unique coefficients for representing W as a convex combination of w_{k-1} and w_k. □

As a consequence, we obtain the generalization of Greene and Kleitman [6] of a theorem due to Kleitman and Milner [8]:

Corollary 1.1. *Let the weights w_i be log-concave and strictly increasing. Then every $x \in S_0$ with weights $w(x) \geqslant w_k$ has average rank*

$$\frac{\bar{w}(x)}{w(x)} \geqslant k.$$

Moreover, $\bar{w}(x)/w(x) = k$ if and only if x is the (0, 1)*-vector with component values* '1' *in position k and* '0' *otherwise.*

It is useful to observe that also the concept of *normalized matching* (see, e.g., [6]) becomes simple in the present context. We will discuss it in the present abstract setting and refer the reader to Section 3 for an interepretation in more familiar terms.

If $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in S_0$ is an arbitrary Sperner vector, we consider the components x_k and x_{k+l}. Writing $y_{k+l} = 1 - x_{k+l}$ and using the inequality $x_k + x_{k+l} \leqslant 1$, we derive the *normalized matching property*

$$x_k \leqslant y_{k+l}. \tag{2.3}$$

If the weights w_i are strictly positive (not necessarily monotone), we may write $x_k = A/w_k$ and $y_{k+l} = A^*/w_{k+l}$ and obtain the equivalent expression

$$\frac{A}{w_k} \leqslant \frac{A^*}{w_{k+l}}.$$

We want to analyze the optimal solutions of (P_0) from the point of view of (2.3). So assume that $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \in S_0$ satisfies $w(x) \geqslant W$ and that $w_k \geqslant w_{k+l}$ for some $l \geqslant 1$. Suppose $x_{k+l} > 0$ and define the vector $\boldsymbol{x}' = (x'_0, x'_1, \ldots, x'_n) \in S_0$ as follows:

$$x'_i = \begin{cases} x_i & \text{if } i \neq k,\ k+l, \\ x_k + \dfrac{w_{k+l}}{w_k} x_{k+l} & \text{if } i = k, \\ 0 & \text{if } i = k+l. \end{cases}$$

Then we have $w(\boldsymbol{x}') = w(\boldsymbol{x}) \geqslant W$. Moreover, the weighted ranks satisfy

$$\bar{w}(\boldsymbol{x}') = \bar{w}(\boldsymbol{x}) - l w_{k+l} x_{k+l}, \tag{2.4}$$

contradicting the optimality of the solution. Let $h \in \{0, \ldots, n\}$ be such that

$$w_h = \max_i w_i.$$

Then (2.4) implies that necessarily $x_{h+l} = 0$ must hold whenever $\boldsymbol{x} = (x_0, \ldots, x_{h+l}, \ldots, x_n)$ is optimal for (P_0). Consequently, for h as above, (P_0) is equivalent to the linear program

$$\min \sum_{i=0}^{h} i w_i x_i$$

$$\text{s.t. } \sum_{i=0}^{h} x_i \leqslant 1$$

$$\sum_{i=0}^{h} w_i x_i \geqslant W$$

$$x_i \geqslant 0 \quad (i = 0, \ldots, h).$$

In the case of positive log-concave weights w_i, there is an index h such that $w_i < w_{i+1}$ for $i < h$ and $w_i \geqslant w_{i+1}$ for $i \geqslant h$. Hence the assumption of monotone weights in Theorem 1 is not necessary.

2.2. *The case $m \geqslant 1$*

We will now assume that the weights w_i are (strictly positive and) log-concave. In particular, there is an index $0 \leqslant h \leqslant n$ such that

$$w_0 < w_1 < \cdots < w_h \geqslant w_{h+1} \geqslant \cdots \geqslant w_n.$$

Given an arbitrary index $0 \leqslant k \leqslant h$, we set

$$K := \max\{i \mid w_i > w_{k-1}\},$$

where $w_{-1} = 0$. As before, we are interested in the lines

$$L_i = \{(u, v) \in \mathbb{R}^2 \mid -u + w_i v = i w_i\}.$$

Fixing the line L_{k-1}, we denote by $V_i^{k-1} = (u_i^{k-1}, v_i^{k-1})$ the intersection of L_{k-1} with L_i for $i = k, \ldots, K$.

Lemma 1. *$v_{i+1}^{k-1} > v_i^{k-1}$ for $i = k, \ldots, K-1$.*

Proof. The statement is clear if $h = 0$ and thus $k = 0$. Assume now $k \leqslant i < h$ and recall from the proof of Theorem 1 in this case

$$v_{i+1} > v_i.$$

Since $v_k = v_k^{k-1}$ and the slopes of the lines L_i are positive and decreasing for $i \leqslant h$, we must have

$$v_{i+1}^{k-1} > v_i^{k-1} \text{ as long as } w_{i+1} \geqslant w_i.$$

In the case $w_{i+1} < w_i$, we compute from the definition

$$v_i^{k-1} = i - k + 1 + (k-1)\frac{w_i}{w_i - w_{k-1}}.$$

Hence

$$v_{i+1}^{k-1} > v_i^{k-1} = 1 + (k-1)\frac{w_k(w_i - w_{i+1})}{(w_{i+1} - w_{k-1})(w_i - w_{k-1})} > 0. \qquad \square$$

Lemma 2. *Let $m \geqslant 0$ be such that $k+m \leqslant K$ and consider the point $V_{k+m}^{k-1} = (u_{k+m}^{k-1}, v_{k+m}^{k-1})$. Then*

$$-u_{k+m}^{k-1} + w_i v_{k+m}^{k-1} \begin{cases} \geqslant iw_i & \text{for } i = k, \ldots, k+m-1, \\ \leqslant iw_i & \text{otherwise.} \end{cases}$$

Proof. We consider several cases.

(1) $0 \leqslant i \leqslant k-2$: By Theorem 1, the point V_k is a vertex of the feasibility region R_0. In particular V_k satisfies the restriction $-u + w_i v \leqslant iw_i$. Because the slope of L_{k-1} is smaller than the slope of L_i and $v_k < v_{k+m}^{k-1}$ holds, also V_{k+m}^{k-1} must satisfy the restriction.

(2) $k \leqslant i \leqslant k+m-1$: We know from Lemma 1 that $v_i^{k-1} < v_{k+m}^{k-1}$ holds. Because the slope of L_i is not bigger than the slope of L_{k-1} and both slopes are positive, we obtain

$$-u_{k+m}^{k-1} + w_i v_{k+m}^{k-1} \geqslant iw_i.$$

(3) $k+m+1 \leqslant i \leqslant K$: The line L_i goes through the points $(0, i)$ and V_i^{k-1} while L_{k-1} connects $(0, k-1)$ and V_i^{k-1}. Because $i > k-1$ and $v_i^{k-1} > v_{k+m}^{k-1}$, the point V_{k+m}^{k-1} must lie in the half-plane determined by $-u + w_i v < iw_i$.

(4) $K+1 \leqslant i \leqslant n$: L_i intersects the v-axis of the (u, v)-plane in $(0, i)$ and has slope $(w_i)^{-1} \geqslant (w_K)^{-1}$. L_K passes through $(0, K)$ with slope $(w_K)^{-1}$. By case (3) above, V_{k+m}^{k-1} satisfies the restriction

$$-u + w_K v \leqslant K w_K.$$

Hence, a fortiori,

$$-u_{k+m}^{k-1} + w_i v_{k+m}^{k-1} < iw_i. \qquad \square$$

It might be interesting to remark at this point that, from an algebraic point of view, checking whether the inequalities in Lemma 2 hold amounts to asking whether the inequality $(c-a)w_c w_a \leqslant (c-b)w_c w_b + (b-a)w_b w_a$ is true for any $a \leqslant b \leqslant c$ with $w_a \leqslant \min\{w_b, w_c\}$. Hence Lemma 2 implies the following algebraic relation, which we state without going through the details.

Proposition 2. *Let $w_0, \ldots, w_n$ be a positive log-concave sequence of real numbers. Then for all $0 \leqslant a \leqslant b \leqslant c \leqslant n$ with $w_a \leqslant w_c$,*

$$(c-a)w_c w_a \leqslant (c-b)w_c w_b + (b-a)w_b w_a.$$

We are now in the position to formulate our main result, which offers the solution of the linear program (P_m) for the choice $W = w_k + \cdots + w_{k+m}$.

Theorem 2. *Let $w_0, \ldots, w_n$ be a positive log-concave sequence and k and m such that $k \leqslant h$ and $w_k \leqslant w_{k+m}$. Let furthermore $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ satisfy the conditions*

$$x_0 + x_1 + \cdots + x_n \leqslant m + 1,$$

$$w_0 x_0 + w_1 x_1 + \cdots + w_n x_n \geqslant w_k + \cdots + w_{k+m},$$

$$0 \leqslant x_i \leqslant 1.$$

Then

$$w_1 x_1 + 2w_2 x_2 + \cdots + n w_n x_n \geqslant k w_k + \cdots + (k+m) w_{k+m}.$$

Moreover, equality holds if and only if

$$x_i = \begin{cases} 1 & \text{if } k \leqslant i \leqslant k+m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the dual program (D_m) with restrictions

$$-u + w_i v - z_i \leqslant i w_i,$$

$$u, v, z_i \geqslant 0.$$

Choosing $(u^*, v^*) = (u_{k+m}^{k-1}, v_{k+m}^{k-1})$, we set

$$z_i^* = \begin{cases} i w_i + u^* - w_i v^* & \text{if } k \leqslant i \leqslant k+m, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2 implies that $(u^*, v^*, z_0^*, \ldots, z_n^*)$ satisfies the restrictions of (D_m). Moreover, the objective function value associated with that dual solution is

$$k w_k + \cdots + (k+m) w_{k+m}.$$

Hence we have found an optimal solution for (D_m). It remains to show that the minimal solution of (P_m) is uniquely determined. A closer look at the proof of Lemma 2 reveals that in fact strict inequality holds:

$$-u_{k+m}^{k-1} + w_i v_{k+m}^{k-1} \begin{cases} < i w_i & \text{if } i \leqslant k-2, \\ > i w_i & \text{if } k \leqslant i \leqslant k+m-1, \\ < i w_i & \text{if } i \geqslant k+m+1. \end{cases}$$

Applying the complementary slackness conditions to an arbitrary optimal solution $x^* = (x_0^*, \ldots, x_n^*)$ of (P_m), we, therefore, see

$$x_i^* = \begin{cases} 0 & \text{if } i \leqslant k-2, \\ 1 & \text{if } k \leqslant i \leqslant k+m-1, \\ 0 & \text{if } k \geqslant k+m+1. \end{cases}$$

Hence

$$x_{k-1}^* + x_{k+m}^* \leqslant 1,$$

$$w_{k-1}x_{k-1}^* + w_{k+m}x_{k+m}^* \geqslant w_{k+m}.$$

Because $w_{k-1} < w_{k+m}$, $x_{k+m}^* = 1$ follows. □

Note that the hypothesis $w_k \leqslant w_{k+m}$ in Theorem 2 can generally not be removed. This is easy to see already with the choice $w_i = \binom{n}{i}$ $(i = 0, \ldots, n)$.

3. LYM-sets

We now interpret the results of Section 2 for a finite set S. We assume that S is partitioned into $n + 1$ pairwise disjoint subsets P_i:

$$S = P_0 \cup P_1 \cup \cdots \cup P_n.$$

The partition induces a *rank function* r on S via

$$r(s) = i \quad \text{if } s \in P_i.$$

Conversely, each partition of S may be thought of as being induced by some 'rank function'. Fixing the partition $(P_0, \ldots, P_n)$ we define the associated *Whitney numbers* for $i = 0, 1, \ldots, n$:

$$w_i = |P_i|.$$

Given any subset $F \subseteq S$, the *profile* of F (relative to $(P_0, P_1, \ldots, P_n)$) is the vector $(f_0(F), \ldots, f_n(F))$, where

$$f_i(F) = |F \cap P_i| = |\{x \in F \mid r(x) = i\}|.$$

It is customary to *normalize* a profile vector $\boldsymbol{f} = (f_0, \ldots, f_n)$ to the vector $\boldsymbol{x}(\boldsymbol{f}) = (x_0(\boldsymbol{f}), \ldots, x_n(\boldsymbol{f}))$ satisfying

$$x_i(\boldsymbol{f}) = f_i / w_i.$$

We say that $F \subseteq S$ is an *LYM-set* if its normalized profile vector $\boldsymbol{x} = (x_0, \ldots, x_n)$ lies in the Sperner polytope S_0, i.e., if

$$x_0 + x_1 + \cdots + x_n \leqslant 1.$$

The *normalized matching property* (2.3) of Section 2 has now the following interpretation. Let $F_k \subseteq P_k$ be arbitrary and call $G_{k+l} \subseteq P_{k+l}$ a *shadow* of F_k if

$$F_k \cup (P_{k+l} \setminus G_{k+l})$$

is an LYM-set. Then the cardinality A of F_k and the cardinality A^* of the shadow G_{k+l} are related via (2.3), which is the usual normalized matching condition (see Graham and Harper [5] and Greene and Kleitman [6]).

'Typical' examples of LYM-sets arise as follows. Take S to be a Boolean algebra with (lattice) rank function r (and hence $w_i = \binom{n}{i}$). Then every antichain of S is an LYM-set [12, 10, 9]. Similar examples are provided by subspace lattices of finite vector spaces equipped with the dimension function or by divisor lattices of integers, where the rank of an element is the number of factors in a prime factor decomposition. For more examples of ordered sets whose antichains are LYM, see, e.g., [2]. The examples above are implied by a general construction that goes back to Kleitman [7].

Assume that $\mathscr{C}$ is a family of subsets ('chains') of S such that for all $x, y \in S$ and $i = 0, \ldots, n$,

(C_1) $|C \cap P_i| = 1$ for all $C \in \mathscr{C}$;

(C_2) $|\{C \in \mathscr{C} \mid x \in C\}| = |\{C \in \mathscr{C} \mid y \in C\}| = c_{r(x)}$ whenever $r(x) = r(y)$.

Proposition 3. *If $\mathscr{C}$ satisfies the conditions* (C_1) and (C_2) *and $F \subseteq S$ is such that $|F \cap C| \leqslant 1$ for all $C \in \mathscr{C}$, then F is LYM.*

Proof. By (C_1) and (C_2), we must have $c_i = |\mathscr{C}|/w_i$. Because each $C \in \mathscr{C}$ contains at most one element of F,

$$|\mathscr{C}| \geqslant \sum_{x \in F} c_{r(x)} = \sum_{i=0}^{n} f_i c_i,$$

where $(f_0, f, \ldots, f_n)$ is the profile vector of F. Hence

$$1 \geqslant \sum_{i=0}^{n} f_i \frac{1}{w_i}. \qquad \square$$

As an illustration, we give a standard construction for families $\mathscr{C}$ with properties (C_1) and (C_2). Let G be some group of permutations acting on the set S. Assume that $P_0, P_1, \ldots, P_n$ are the orbits of G. We now choose an arbitrary (but fixed) subset $C \subseteq S$ such that for $i = 0, 1, \ldots, n$,

$$|C \cap P_i| = 1.$$

Then the family

$$\mathscr{C} = \{C^g \mid g \in G\}$$

will have the desired properties. We say that the subset $F \subseteq S$ is an (LYM)$(m+1)$-*set* if its profile $(f_0, \ldots, f_n)$ satisfies

$$\frac{f_0}{w_0} + \frac{f_1}{w_1} + \cdots + \frac{f_n}{w_n} \leqslant m + 1.$$

Equivalently, F is an $(m+1)$-set if its normalized profile vector lies in the Sperner polytope S_m. Thus every union of (at most) $m+1$ LYM-sets is an $(m+1)$-set. We remark, however, that there may exist $(m+1)$-sets which cannot be expressed as unions of 1-sets.

On the other hand, Proposition 1 shows that extremal $(m+1)$-sets are very canonical: an $(m+1)$-set F corresponds to a vertex of S_m if and only if F is a union of at most $m+1$ blocks of the partition $(P_0, P_1, \ldots, P_n)$. (In the case of a Boolean algebra $S = \mathscr{B}_n$, this result was obtained by Erdős et al. [4]. It generalizes a result of Erdős [3], which in turn is a generalization of the original result, i.e. $m = 0$, of Sperner [11] formulated for antichains. For so-called *regular* ordered sets, whose antichains are known to be LYM, the analogous statement is deduced as Corollary 8.55 in Aigner [1]). For the subset $F \subseteq S$, consider the average rank

$$\bar{r}(F) = \frac{1}{|F|} \sum_{x \in F} r(x).$$

With the notation $w_i = |P_i|$, we may write $\bar{r}(F)$ in terms of the normalized profile vector $\boldsymbol{x} = (x_0, \ldots, x_n)$ of F:

$$\bar{r}(F) = \frac{w_1 x_1 + 2w_2 x_2 + \cdots + n w_n x_n}{w_0 x_0 + w_1 x_1 + \cdots + w_n x_n}.$$

Thus Theorem 2 implies

Corollary 2. *Let $r: S \to \{0, \ldots, n\}$ be a (surjective) rank function such that the Whitney numbers*

$$w_i = |\{x \in S \mid r(x) = i\}|$$

are logarithmically concave. Let k be such that $w_{k-1} < w_k \leqslant w_{k+m}$ and define $W = w_k + \cdots + w_{k+m}$. Then every $(m+1)$-set $F \subseteq S$ with $|F| \geqslant W$ has average rank

$$\bar{r}(F) \geqslant \frac{k w_k + \cdots + (k+m) w_{k+m}}{w_k + \cdots + w_{k+m}}.$$

Moreover, equality holds if and only if

$$F = P_k \cup \cdots \cup P_{k+m}. \qquad \square$$

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