Two-part set systems

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Abstract

The two part Sperner theorem of Katona and Kleitman states that if X is an n-element set with partition $X_1 \cup X_2$, and $\mathcal{F}$ is a family of subsets of X such that no two sets $A, B \in \mathcal{F}$ satisfy $A \subset B$ (or $B \subset A$) and $A \cap X_i = B \cap X_i$ for some i, then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. We consider variations of this problem by replacing the Sperner property with the intersection property and considering families that satisfiy various combinations of these properties on one or both parts X_1, X_2. Along the way, we prove the following new result which may be of independent interest: let $\mathcal{F}, \mathcal{G}$ be families of subsets of an n-element set such that $\mathcal{F}$ and $\mathcal{G}$ are both intersecting and cross-Sperner, meaning that if $A \in \mathcal{F}$ and $B \in \mathcal{G}$, then $A \not\subset B$ and $B \not\subset A$. Then $|\mathcal{F}| + |\mathcal{G}| < 2^{n-1}$ and there are exponentially many examples showing that this bound is tight.

Keywords: extremal set theory, Sperner, intersecting

1. Introduction

Let X be a finite set and let 2^X be the system of all subsets of X. The basic problem of the theory of extremal sets systems is to determine the maximum size that a set system $\mathcal{F} \subseteq 2^X$ can have provided $\mathcal{F}$ satisfies a prescribed property. The prototypes of investigated properties are the intersecting and Sperner properties. A set system $\mathcal{F}$ is *intersecting* if $F_1 \cap F_2 \neq \emptyset$ for any pair $F_1, F_2 \in \mathcal{F}$ and a set system $\mathcal{F}$ is *Sperner* if there do not exist two distinct sets $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subset F_2$. The celebrated theorems of Erdős, Ko, Rado

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[4] and of Sperner [13] determine the largest size that a uniform intersecting set system and Sperner system can have. Both theorems have many applications and generalizations.

One such generalization of the Sperner property is the so called *more part Sperner property*. In this case, the underlying set X is partitioned into m subsets $X_1, ..., X_m$ and the system $\mathcal{F} \subset 2^X$ is said to be m-part Sperner if for any pair $F_1, F_2 \in \mathcal{F}$ with $F_1 \subset F_2$ there exist at least two indices $1 \leq i_1 < i_2 \leq m$ such that $F_1 \cap X_{i_j} \subsetneq F_2 \cap X_{i_j}$ holds for $j = 1, 2$. Systems with this property were first considered in [9, 11]; for a survey of recent results see [2].

In this paper we will consider analogous problems for intersection properties and also some mixed more part properties in the case when m equals 2. All maximum size 2-part Sperner set systems were described by P.L. Erdős and G.O.H. Katona in [5, 6]. To rephrase the 2-part Sperner property it is convenient to introduce the following set systems of *traces*: for any $A \subseteq X_1$ and $B \subseteq X_2$ let $\mathcal{F}_A = \{F \cap X_2 : F \in \mathcal{F}, F \cap X_1 = A\}, \mathcal{F}_B = \{F \cap X_1 : F \in \mathcal{F}, F \cap X_2 = B\}$. Also, for any $F \in \mathcal{F}$ we will call $F \cap X_1$ and $F \cap X_2$ the *traces* of F on X_1 and X_2. One can easily see that a set system $\mathcal{F}$ is 2-part Sperner with respect to the partition $X = X_1 \cup X_2$ if and only if for any subset $A \subseteq X_1$ or $B \subseteq X_2$ the set systems $\mathcal{F}_A$ and $\mathcal{F}_B$ possess the Sperner property.

Having this equivalence in mind, it is natural to introduce the following three definitions where we always assume that the underlying set X is partitioned into two sets X_1 and X_2:

Definition 1. (i) a set system $\mathcal{F} \subseteq 2^X$ is 2-part intersecting (a 2I-system for short) if for any subset A of X_1 (and for any subset B of X_2) the trace system $\mathcal{F}_A$ on X_2 (and the trace system $\mathcal{F}_B$ on X_1) is intersecting,

(ii) a set system $\mathcal{F} \subseteq 2^X$ is 2-part intersecting, 2-part Sperner (a 2I2S-system for short) if for any subset A of X_1 and for any subset B of X_2 the trace systems $\mathcal{F}_A$ on X_2 and $\mathcal{F}_B$ on X_1 are intersecting and Sperner,

(iii) a set system $\mathcal{F} \subseteq 2^X$ is 1-part intersecting, 1-part Sperner (a 1I1S-system for short) if there exists no pair of distinct sets F_1, F_2 in $\mathcal{F}$ such that the traces of F_1, F_2 are disjoint at one of the parts and are in containment at the other.

We will address the problem of finding the maximum possible size of a set system possessing the properties above. Some of our bounds will apply regardless of the sizes of the parts in the 2-partition and some will only apply to special cases. We will be mostly interested in the case when $|X_1| = |X_2|$. Clearly, for any 2-part set system $\mathcal{F}$ we have $|\mathcal{F}| = \sum_{A \subseteq X_1} |\mathcal{F}_A| = \sum_{B \subseteq X_2} |\mathcal{F}_B|$. As any intersecting system of subsets of X_1 has size at most $2^{|X_1|-1}$, it follows that any 2I-system has size at most $2^{|X_2|}2^{|X_1|-1} = 2^{|X|-1}$. In Section 2 we will prove the following theorem.

Theorem 2. *Let $\mathcal{F} \subseteq 2^X$ be a 2-part intersecting system of maximum size. If the 2-partition $X = X_1 \cup X_2$ is non-trivial (i.e. $X_1 \neq \emptyset$, $X_2 \neq \emptyset$), then the following inequality holds:*

$$|\mathcal{F}| \leq \frac{3}{8} 2^{|X|}.$$

The bound is best possible if X_1 or X_2 is a singleton. Moreover, if $|X_1| = |X_2|$, then there exists a 2-part intersecting system of size $\frac{1}{3}\left(2^{|X|} + 2\right)$.

The rest of Section 2 is devoted to 2I2S systems. We prove the following result.

Theorem 3. *Let $\mathcal{F} \subseteq 2^X$ be a 2-part intersecting, 2-part Sperner system of maximum size. Then $|\mathcal{F}| \leq \binom{|X|}{\lceil |X|/2 \rceil}$ holds. This bound is asymptotically sharp as long as $|X_1| = o(|X_2|^{1/2})$. If $|X_1| = |X_2|$ holds, then there exists a 2I2S system of size $c\binom{|X|}{\lceil |X|/2 \rceil}$ with $c > 2/3$.*

The main result of the paper is proved in Section 3. We determine the maximum size of a 1-part intersecting 1-part Sperner set system.

Theorem 4. *Let $\mathcal{F}$ be a maximum size 1-part intersecting, 1-part Sperner set system. Then $|\mathcal{F}| = 2^{|X|-2}$.*

2. 2I- and 2I2S-systems

In this section we consider two-part intersecting and two-part intersecting, two-part Sperner set systems. We first consider a general construction that produces large families with these properties. Let $\mathcal{A}_1, ..., \mathcal{A}_m$ and $\mathcal{B}_1, ..., \mathcal{B}_m$ be partitions of 2^{X_1} and 2^{X_2} into disjoint intersecting (or intersecting, Sperner) systems some of which may possibly be empty. Then the set system $\mathcal{F} := \cup_{i=1}^m \mathcal{A}_i \times \mathcal{B}_i = \{A \cup B : A \in \mathcal{A}_i, B \in \mathcal{B}_i \text{ for some } 1 \leq i \leq m\}$ is a 2I- (2I2S)-system by definition.

Fact 5. Let $0 \leq x_1 \leq ... \leq x_n$, $0 \leq y_1 \leq ... \leq y_n$ be real numbers and π be a permutation of the first n integers. Then we have the following inequalities:

$$\sum_{i=1}^{n} x_i y_{\pi(i)} \leq \sum_{i=1}^{n} x_i y_i \leq \max\left\{\sum_{i=1}^{n} x_i^2, \sum_{i=1}^{n} y_i^2\right\}.$$

Thus to maximize the size of a family obtained through the general construction one should enumerate the $\mathcal{A}_i$'s and the $\mathcal{B}_i$'s in decreasing order according to their size. Moreover, if $|X_1| = |X_2|$, then it is enough to consider partitions $\mathcal{A}_1, ..., \mathcal{A}_m$ of 2^{X_1} and the sum $\sum_{i=1}^m |\mathcal{A}_i|^2$.

2.1. Two-part intersecting systems

In this subsection we prove Theorem 2. In the proof we use the following theorem of Kleitman [10].

Theorem 6 (Kleitman [10])**.** *Let $\mathcal{F}_1, ..., \mathcal{F}_m \subseteq 2^{[n]}$ be intersecting set systems. Then*

$$|\mathcal{F}_1 \cup ... \cup \mathcal{F}_m| \leq 2^n - 2^{n-m}.$$

Proof of Theorem 2. For any subset A of X_1 let $\bar{A}$ denote its complement $X_1 \setminus A$. By definition, both $\mathcal{F}_A$ and $\mathcal{F}_{\bar{A}}$ are intersecting. Also, these set systems are disjoint as $B \in \mathcal{F}_A \cap \mathcal{F}_{\bar{A}}$ implies $B \cup A, B \cup \bar{A} \in \mathcal{F}$ which contradicts the 2-part intersecting property of $\mathcal{F}$. Thus by Theorem 6 we have $|\mathcal{F}_A| + |\mathcal{F}_{\bar{A}}| \leq 2^{|X_2|-1} + 2^{|X_2|-2}$.

Altogether we obtain

$$|\mathcal{F}| \leq 2^{|X_1|-1}(2^{|X_2|-1} + 2^{|X_2|-2}) = \frac{3}{8}2^{|X|}.$$

Our best lower bounds arise from our general construction. If X_1 consists of a single element x_1, then let $\mathcal{A}_1 = \{\{x_1\}\}, \mathcal{A}_2 = \{\emptyset\}$ and $\mathcal{B}_1 = \{B \subset X_2 : x_2 \in B\}, \mathcal{B}_2 = \{B \subset X_2 : x_2 \notin B, x_2' \in B\}$ for two fixed elements $x_2, x_2' \in X_2$ and the other $\mathcal{B}_i$'s be arbitrary while the other $\mathcal{A}_i$'s be empty. For the set system $\mathcal{F}$ we obtain via the general construction, we have $|\mathcal{F}| = 2^{|X_2|-1} + 2^{|X_2|-2} = \frac{3}{8}2^{|X|}$.

Finally, let us suppose that $|X_1| = |X_2| = |X|/2$ and let the elements of X_1 and X_2 be $x_1^1, ..., x_m^1$ and $x_1^2, ..., x_m^2$. Let us define the partition of 2^{X_1} and 2^{X_2} in the following way: $\mathcal{A}_i := \{A \subset X_1 \setminus \{x_1^1, ..., x_{i-1}^1\} : x_i^1 \in A\}, \mathcal{B}_i := \{B \subset X_2 \setminus \{x_1^2, ..., x_{i-1}^2\} : x_i^2 \in B\}$ for all $1 \leq i \leq m+1$ (i.e. $\mathcal{A}_{m+1} = \mathcal{B}_{m+1} = \{\emptyset\}$). Then for the set system $\mathcal{F}$ arising from the general construction we have

$$|\mathcal{F}| = 1 + \sum_{i=1}^{m} 2^{|X|-2i} = \frac{2^{|X|}+2}{3}. \qquad \square$$

Remark 7. Theorem 6 shows that the above set system for the $|X_1| = |X_2|$ case is best possible among those that we can obtain via the general construction. Indeed, by Fact 5 we know that we have to consider partitions of 2^{X_1} to intersecting set systems with sizes $s_1, s_2, ..., s_m$ and maximize $\sum_{i=1}^m s_i^2$. But a partition maximizes this sum of squares if for all $1 \leq j \leq m$ the sums $\sum_{i=1}^j s_i$ are maximized. In the construction we use, the sums $\sum_{i=1}^j s_i$ match the upper bound of Theorem 6.

2.2. Two-part intersecting, two-part Sperner systems

In this subsection we consider 2I2S-systems and prove Theorem 3. To be able to use the general construction, we need to define a partition of the power set into intersecting Sperner set systems.

Construction 8. Here we give a partition of the power set of Y into intersecting Sperner systems where all levels are partitioned into minimal number of (uniform) intersecting systems (we call this *canonical partition*). This partition is in the form of

$$\begin{aligned}
&\mathcal{Y}_k, \quad \text{for} \quad k = \left\lceil \frac{|Y|+1}{2} \right\rceil, \dots, |Y|;\\
&\mathcal{Y}_{i,j}, \quad \text{for} \quad i = 1, \dots, \left\lceil \frac{|Y|+1}{2} \right\rceil - 1, j = 1, \dots, |Y| - 2i + 1;\\
&\mathcal{Y}^*_\ell, \quad \text{for} \quad \ell = 0, \dots, \left\lceil \frac{|Y|+1}{2} \right\rceil - 1.
\end{aligned}$$

The systems $\mathcal{Y}_k$ are $\binom{Y}{k}$. Fix an enumeration $y_1, ..., y_{|Y|}$ of the elements of Y and define the systems $\mathcal{Y}_{i,j}$ as $\left\{Y' \in \binom{Y \setminus \{y_1,\dots,y_{j-1}\}}{i} : y_j \in Y'\right\}$. Finally let $\mathcal{Y}^*_\ell = \binom{Y}{\ell} \setminus \bigcup_{j=1}^{|Y|-2\ell+1} \mathcal{Y}_{\ell,j}$. We remark that the second and third types are identical to those in the corresponding Kneser construction. Note that the number of systems in the partition is quadratic in $|Y|$ but for any $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that

$$\left| \bigcup_{k=\lceil \frac{|Y|+1}{2} \rceil}^{|Y|} \mathcal{Y}_k \cup \bigcup_{i=|Y|/2-K|Y|^{1/2}}^{|Y|/2} \bigcup_{j=1}^{|Y|-2i+1} \mathcal{Y}_{i,j} \cup \bigcup_{\ell=|Y|/2-K|Y|^{1/2}}^{|Y|/2} \mathcal{Y}^*_\ell \right| \geq (1-\varepsilon)2^{|Y|}. \qquad (1)$$

Indeed, the sets in all the $\mathcal{Y}_k$ contain all subsets of Y of size greater than $|Y|/2$, and the remaining families $\mathcal{Y}_{i,j}, \mathcal{Y}^*_\ell$ contain all subsets of Y of size between $|Y|/2 - K|Y|^{1/2}$ and $|Y|/2$. Since the number of subsets of Y of size less than $|Y|/2 - K|Y|^{1/2}$ is less than $\varepsilon 2^{|Y|}$, the inequality in (1) follows. It is easy to see that the number of set systems in the union in (1) is at most $2K^2|Y|$.

Proof of Theorem 3. The upper bound of the theorem follows from the result of Katona [9] and Kleitman [11] stating that a 2-part Sperner system has size at most $\binom{|X|}{\lceil |X|/2 \rceil}$, since any 2I2S-system is 2-part Sperner.

We now prove the lower bound. For $i = 1, 2$ let $x_i = |X_i|$, and recall that $n = |X| = x_1 + x_2$. First we consider the case when the size of x_1 is negligible compared to the size of x_2. Let us assume that $x_1 = o(x_2^{1/2})$. As observed above, from the canonical partition of 2^{X_1} which has $\Theta(x_1^2)$ families, there are $m = O(x_1)$ families $\mathcal{F}^1_1, ..., \mathcal{F}^1_m \subset 2^{X_1}$ such that

$$\left| \bigcup_{i=1}^{m} \mathcal{F}^1_i \right| = (1 - o(1))2^{x_1}.$$

If $i = o(x_2^{1/2})$, then the system $\binom{X_2}{x_2/2+i}$ is intersecting Sperner and has size $(1-o(1))\binom{x_2}{x_2/2} = 1/2^{x_1}(1-o(1))\binom{n}{n/2}$. Thus, by the general constuction, we obtain the following 2I2S-system from these partitions:

$$\mathcal{F} = \bigcup_{i=1}^{m} \left\{ F \cup H : F \in \mathcal{F}_i^1, H \in \binom{X_2}{x_2/2 + i} \right\}.$$

By the above, $|\mathcal{F}|$ is equal to

$$\sum_{i=1}^{m} |\mathcal{F}_i^1| \binom{x_2}{\frac{x_2}{2} + i} \geq \frac{1}{2^{x_1}}(1 - o(1)) \binom{n}{\frac{n}{2}} \sum_{i=1}^{m} |\mathcal{F}_i^1| = (1 - o(1)) \binom{n}{\frac{n}{2}}.$$

Let us consider the case $x_1 = x_2$. We first show that the 2I2S-system $\mathcal{F}$ we derive from the canonical partition using our general construction has size $(2/3 - o(1))\binom{n}{\lceil n/2 \rceil}$. We then use Frankl and Füredi's construction [7] to improve this bound by a constant factor. For sake of simplicity, assume n is divisible by 4. Then our system has size

$$\sum_{i=n/4+1}^{n/2} \binom{n/2}{i}^2 + \sum_{i=1}^{n/4} \sum_{k=0}^{n/2-2i} \binom{n-1-k}{i-1}^2 + \sum_{i=1}^{n/4} \binom{2i-1}{i}^2,$$

where the sums belong to the three different system types in the canonical partition. We can write our system as $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where the first subsystem corresponds to the sets listed in the first summation, and the second one consists of the other sets. Then

$$\begin{aligned} |\mathcal{F}_1| = \sum_{i=n/4+1}^{n/2} \binom{n/2}{i}^2 &= \sum_{i=n/4+1}^{n/2} \binom{n/2}{i}\binom{n/2}{n/2-i} \\ &= 1/2\binom{n}{n/2} - \binom{n/2}{n/4}^2 = (1/2 - o(1))\binom{n}{n/2} \end{aligned}$$

as $\binom{n/2}{i}\binom{n/2}{n/2-i}$ is the number of those $n/2$-subsets of X that intersect X_1 in i elements.

Next we prove that $|\mathcal{F}_2| \geq (1/3 - o(1))|\mathcal{F}_1|$ which implies that $|\mathcal{F}_2| \geq \frac{1/2-o(1)}{3}\binom{n}{n/2}$ and thus $|\mathcal{F}| \geq (\frac{2}{3} - o(1)\binom{n}{n/2})$. We consider those members of $\mathcal{F}_2$ which intersect X_1 in i elements (and then intersect X_2 in i elements too). We will show that, for most values of i, the number of these sets is roughly a third of the number of those members of $\mathcal{F}_1$, which intersect X_1 (and then X_2 as well) in $n/2 - i$ elements. We have to compare

$$S_i = \binom{2i-1}{i}^2 + \sum_{k=0}^{n/2-2i} \binom{n/2-1-k}{i-1}^2 \quad \text{to} \quad \binom{n/2}{n/2-i}^2 = \binom{n/2}{i}^2.$$

We will be done, if we establish $S_i/(\binom{n/2}{i})^2 = 1/3 + o(1)$ for all $n/4 - n^{2/3} \le i \le n/4 - \log n$ as

$$\sum_{i<n/4-n^{2/3}} \binom{n/2}{n/2-i}^2 + \sum_{n/4-\log n<i\le n/2} \binom{n/2}{n/2-i}^2 = o\left(\binom{n}{n/2}\right).$$

To deduce $S_i/(\binom{n/2}{i})^2 = 1/3 + o(1)$ we need the following fact.

Fact 9. Let $a_1 \ge a_2 \ge ... \ge a_k > 0$ positive reals with $\sum_{\ell=1}^k a_\ell = 1$. If for some $j < k$ we have $a_\ell = 2^{-\ell} + o(1)$ for all $\ell < j$ and $\sum_{\ell=j}^k a_\ell = o(1)$, then $\sum_{\ell=1}^k a_\ell^2 = 1/3 + o(1)$.

All we have to do is to verify the conditions of Fact 9 to the numbers

$$r_\ell = \frac{\binom{n/2-\ell}{i-1}}{\binom{n/2}{i}} \text{ for } \ell = 1, ..., n/2 - 2i + 1 \text{ and } r_{n/2-2i+2} = \frac{\binom{2i-1}{i}}{\binom{n/2}{i}}$$

with $j = \min\{n/4 - i, n^{1/4}\}$ and $k = n/2 - 2i + 2$. First of all $\sum_\ell r_\ell = 1$ as these numbers correspond to the ratios of set systems in a partition. Next we show that $r_\ell = 2^{-\ell} + o(1)$ for all $\ell < j$. Writing $d_\ell = \frac{r_\ell}{r_{\ell-1}}$ for $2 \le \ell \le j-1$ and $i = n/4 - m$ we obtain

$$d_\ell = \frac{r_\ell}{r_{\ell-1}} = \frac{\binom{n/2-\ell}{i-1}}{\binom{n/2-\ell+1}{i-1}} = \frac{n/2-\ell+i+2}{n/2-\ell+1} = \frac{1}{2} + \frac{m-\ell/2+3/2}{n/2-\ell+1} = \frac{1}{2} + O(n^{-1/3})$$

and thus for $\ell < j \le n^{1/4}$

$$r_1 = \frac{i}{n/2} = \frac{1}{2} + o(1) \text{ and } r_\ell = r_1 \prod_{t=2}^{\ell} d_t = 2^{-\ell}(1 + O(jn^{-1/3})) = 2^{-\ell}(1 + O(n^{-1/12})).$$

Finally, from $m > \log n$ it follows that j tends to infinity and thus $\sum_{\ell=1}^j r_\ell = 1 - o(1)$. Consequently, $\sum_{\ell=j}^k r_\ell = o(1)$.

It remains to show that we can modify our construction so that it has size $(2/3+\varepsilon)\binom{n}{n/2}$ for some fixed $\varepsilon > 0$. In order to do so we replace some of the set systems in the canonical partition. First note that for any $\beta > 0$ the sum $\sum_{i=n/4-\beta n^{1/2}}^{n/4} \binom{n/2}{i}^2$ is a positive fraction of $\sum_{i=0}^{n/4} \binom{n/2}{i}^2$. Thus we will be done if for each i with $n/4-\beta n^{1/2} \le i \le n/4$ we can replace the set systems of the canonical partition that contain i-sets with other i-uniform set systems $\mathcal{H}_1^i, \mathcal{H}_2^i, ..., \mathcal{H}_{s_i}^i$ such that $\sum_{t=1}^{s_i} |\mathcal{H}_t^i|^2$ is at least $(1/3+\varepsilon)\binom{n/2}{i}^2$ for some positive ε.

Frankl and Füredi considered in [7] the following pair of i-uniform intersecting set systems on a base set Y: let Y be equipartitioned into $Y_1 \cup Y_2$ and define

$$\mathcal{G}_1^i = \left\{ G \in \binom{Y}{i} : |Y_1 \cap G| > |Y_1|/2 \right\},$$

$$\mathcal{G}_2^i = \left\{ G \in \binom{Y}{i} \setminus \mathcal{G}_1 : |Y_2 \cap G| > |Y_2|/2 \right\}.$$

They observed that if $|Y| = 2i + o(i^{1/2})$, then $|\mathcal{G}_1^i \cup \mathcal{G}_2^i| = (1 - o(1))\binom{|Y|}{i}$ and that for any $\alpha > 0$ there exists $\beta > 0$ such that if $|Y| \leq 2i + \beta i^{1/2}$, then $|\mathcal{G}_1^i \cup \mathcal{G}_2^i| \geq (1-\alpha)\binom{|Y|}{i}$.

Let us fix $0 < \alpha < 1/6$ and consider β as above. We define a modified version of the canonical partition for a given set Y. We replace the set systems $\mathcal{Y}_{i,j}$ for all $\frac{|Y|}{2} - \frac{\beta}{2\sqrt{2}}|Y|^{1/2} \leq i \leq \frac{|Y|}{2}$ and $j = 1, ...|Y| - 2i + 1$ with $\mathcal{G}_1^i$ and $\mathcal{G}_2^i$. As $|\mathcal{G}_1^i| + |\mathcal{G}_2^i| \geq (1-\alpha)\binom{n/2}{i}$, the ratio of $|\mathcal{G}_1^i|^2 + |\mathcal{G}_2^i|^2$ and $\binom{n}{i}^2$ is at least $2(\frac{1-\alpha}{2})^2 = 1/2 - \alpha + \alpha^2/2$ which is strictly larger than $1/3$ by choice of α. $\square$

Katona's proof that a 2-part Sperner set system can contain at most $\binom{n}{\lceil n/2 \rceil}$ sets used a theorem of Erdős [3] on the number of sets contained in the union of k Sperner set systems. Our proofs of Theorem 2 and Remark 7 used Theorem 6, Kleitman's result on the size of the union of k intersecting families. It seems natural to ask how large can the union of k intersecting Sperner set systems be as the problem seems to be interesting on its own right and it might help establishing bounds on 2S2I-systems. Unfortunately, we were only able to determine the exact result in the very special case when $k = 2$ and n is odd. The result follows easily from the following theorem of Greene, Katona and Kleitman.

Theorem 10 (Greene, Katona, Kleitman [8]). *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting and Sperner set system, then the following inequality holds*

$$\sum_{F \in \mathcal{F},\ |F| \leq n/2} \frac{1}{\binom{n}{|F|-1}} + \sum_{F \in \mathcal{F},\ |F| > n/2} \frac{1}{\binom{n}{|F|}} \leq 1.$$

Corollary 11. *Let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ be intersecting Sperner set systems and $n = 2l + 1$ an odd integer. Then we have $|\mathcal{F} \cup \mathcal{G}| \leq \binom{n}{l+1} + \binom{n}{l+2}$ and the inequality is sharp as shown by $\mathcal{F} = \binom{[n]}{l+1}, \mathcal{G} = \binom{[n]}{l+2}$.*

Proof. We may assume that $\mathcal{F}$ and $\mathcal{G}$ are disjoint. Let us add the inequality of Theorem 10 for both systems $\mathcal{F}$ and $\mathcal{G}$. The bigger the number of the summands, the greater the cardinality of $\mathcal{F}$, therefore we need to keep the summands as small as possible to obtain the greatest number of summands. The set size for which the summand is the smallest is $l + 1$ and the second smallest summand is for set sizes l and $l + 2$. As by the disjointness of the systems the number of smallest summands is at most $\binom{n}{l+1}$, the result follows. $\square$

3. 1-part intersecting, 1-part Sperner systems

In this section we study 1-part Sperner 1-part intersecting set systems and prove Theorem 4. In order to prove the result we need a further definition. We say that the set systems $\mathcal{F}$ and $\mathcal{G}$ are *intersecting, cross-Sperner* if both $\mathcal{F}$ and $\mathcal{G}$ are intersecting and there is no $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \subset G$ or $G \subset F$. We will prove the following theorem which can be of independent interest.

Theorem 12. *Let $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ be a pair of cross-Sperner, intersecting set systems. Then we have*

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^{n-1}$$

and this bound is best possible.

One of our main tools will be the following special case of the Four Functions Theorem of Ahlswede and Daykin [1]. Let us write $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

Theorem 13 (Ahlswede-Daykin, [1])**.** *For any pair $\mathcal{A}, \mathcal{B}$ of set systems we have*

$$|\mathcal{A}||\mathcal{B}| \leq |\mathcal{A} \wedge \mathcal{B}||\mathcal{A} \vee \mathcal{B}|.$$

The other result we will use in our argument is due to Marica and Schönheim [12] and involves the difference set system $\Delta(\mathcal{F}) = \{F \setminus F' : F, F' \in \mathcal{F}\}$.

Theorem 14 (Marica – Schönheim [12])**.** *For any set system $\mathcal{F}$ we have $|\Delta(\mathcal{F})| \geq |\mathcal{F}|$.*

Corollary 15. *Let $\mathcal{D}$ be a downward closed set system and let $\mathcal{F}$ be an intersecting subsystem of $\mathcal{D}$. Then the inequality $2|\mathcal{F}| \leq |\mathcal{D}|$ holds.*

Proof. As $\mathcal{D}$ is downward closed and $\mathcal{F} \subset \mathcal{D}$, it follows that $\Delta(\mathcal{F}) \subset \mathcal{D}$. Furthermore, as $\mathcal{F}$ is intersecting, we have $\mathcal{F} \cap \Delta(\mathcal{F}) = \emptyset$ and thus we are done by Theorem 14. □

Proof of Theorem 12. Let us begin with defining the following four set systems

$$\mathcal{U} = \{U \subseteq [n] : \exists H \in \mathcal{F} \cup \mathcal{G} \text{ such that } H \subseteq U\}, \quad \mathcal{U}' = \mathcal{U} \setminus (\mathcal{F} \cup \mathcal{G}),$$

$$\mathcal{D} = \{D \subseteq [n] : \exists H \in \mathcal{F} \cup \mathcal{G} \text{ such that } D \subseteq H\}, \quad \mathcal{D}' = \mathcal{D} \setminus (\mathcal{F} \cup \mathcal{G}).$$

Clearly, $\mathcal{D}'' = \{D' : \exists F \in \mathcal{F} \text{ such that } D' \subset F\}$ is downward closed (and, by definition, $\mathcal{F} \subset \mathcal{D}''$), hence by Corollary 15 we have $2|\mathcal{F}| \leq |\mathcal{D}''|$. Moreover by the cross-Sperner property, we have $(\mathcal{D}'' \setminus \mathcal{F}) \cap \mathcal{G} = \emptyset$, and therefore we have $\mathcal{D}'' \setminus \mathcal{F} \subset \mathcal{D}'$. Consequently $|\mathcal{F}| \leq |\mathcal{D}'|$ and, by symmetry, $|\mathcal{G}| \leq |\mathcal{D}'|$ also holds.

Note that $\mathcal{F} \wedge \mathcal{G} \subset \mathcal{D}'$. Indeed, $F \cap G \in \mathcal{D}$ by definition and $F \cap G \in \mathcal{F}$ (or $F \cap G \in \mathcal{G}$) would contradict the cross-Sperner property. Similarly, we obtain that $\mathcal{F} \vee \mathcal{G} \subset \mathcal{U}'$ and it is easy to see that the cross-Sperner property implies that $\mathcal{U}' \cap \mathcal{D}' = \emptyset$ and thus the four systems $\mathcal{F}, \mathcal{G}, \mathcal{U}', \mathcal{D}'$ are pairwise disjoint.

Now suppose as a contradiction that $|\mathcal{F}| + |\mathcal{G}| > 2^{n-1}$ and thus $|\mathcal{U}'| + |\mathcal{D}'| < 2^{n-1}$. By $|\mathcal{F}|, |\mathcal{G}| \leq |\mathcal{D}'|$ we obtain that $|\mathcal{U}'| < |\mathcal{F}|, |\mathcal{G}|$ and thus using Theorem 13 we have

$$|\mathcal{U}'||\mathcal{D}'| < |\mathcal{F}||\mathcal{G}| \leq |\mathcal{F} \wedge \mathcal{G}||\mathcal{F} \vee \mathcal{G}| \leq |\mathcal{U}'||\mathcal{D}'|,$$

a contradiction.

Finally, let us mention some pairs of set systems for which the sum of their sizes equals 2^{n-1}. Any maximum intersecting system $\mathcal{F}$ with $\mathcal{G}$ the empty set system is extremal, just as the pair $\mathcal{F}_1 = \{F \subset [n] : 1 \in F, 2 \notin F\}$, $\mathcal{G}_1 = \{G \subset [n] : 1 \notin G, 2 \in G\}$. Furthermore, for any $k \geq n/2$ the pair $\mathcal{F}_k = \{F \subset [n] : 1 \in F, |F| \leq k\}$, $\mathcal{G}_k = \{G \subset [n] : 1 \notin G, |G| \geq k\}$ has the required property, too. □

Proof of Theorem 4. First let us consider any pair of maximal intersecting systems $\mathcal{A} \subseteq 2^{X_1}$, $\mathcal{B} \subseteq 2^{X_2}$. Clearly, the set system $\mathcal{F} = \mathcal{A} \times \mathcal{B}$ is a 1I1S-system as any pair of sets $F_1, F_2 \in \mathcal{F}$ intersect both in X_1 and in X_2. This shows that a maximum 1I1S-system contains at least $2^{|X|-2}$ sets.

To obtain the upper bound of the theorem let $\mathcal{F}$ be any 1I1S-system. For any $A \subseteq X_1$ let $\bar{A}$ denote $X_1 \setminus A$. By definition, both $\mathcal{F}_A$ and $\mathcal{F}_{\bar{A}}$ are intersecting systems, and no element of the first can contain any element of the second (and vice versa). In other words they form a pair of intersecting, cross-Sperner systems. Due to Theorem 12 we have $|\mathcal{F}_A| + |\mathcal{F}_{\bar{A}}| \leq 2^{|X_2|-1}$. The number of pairs $A, \bar{A}$ is $2^{|X_1|-1}$ therefore we have $|\mathcal{F}| \leq 2^{|X|-2}$. □

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