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# On two-part Sperner systems for regular posets

Extended Abstract

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**Abstract**

In the Boolean lattice the BLYM inequality holds with equality if and only if the Sperner family consists of one complete level of subsets. In this paper we extend this *strict* BLYM-property to a subclass of normal posets. On the basis of this result we prove a strict two-part Sperner theorem of the direct product of any two posets from the same subclass.

## 1 Introduction

One of the central theme in extremal set theory is Sperner's theorem [15] from 1928, and its generalizations. A *Sperner family* (or an *antichain*) $\mathcal{H}$ is a family of subsets on the underlying finite set $X$ such that none of them is a proper subset of another. Sperner's theorem states that a family consisting of all subsets with the same cardinality $\lfloor |X|/2 \rfloor$ is a largest Sperner family on $X$.

An early generalization of the Sperner theorem is the BLYM (Bollobás 1965 [3], Lubell 1966 [11], Yamamoto 1954 [16] and Meshalkin 1963 [12]) inequality.

**Theorem 1.1** *If $P_i(\mathcal{H})$ denotes the number of $i$-element sets in the Sperner family $\mathcal{H}$ (the array of these quantities are called the* profile vector *of the*

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*family), then*

$$\sum_{i=0}^{n} \frac{P_i(\mathcal{H})}{\binom{|X|}{i}} \le 1. \tag{1}$$

Paul Erdős found another generalization of the Sperner theorem:

**Theorem 1.2 (Paul Erdős [4])** *Assume that set systems $\mathcal{H}_i$ $(i = 1, 2, \ldots, k)$ are pairwise disjoint Sperner families on the underlying set $X$, then*

$$\sum_{i=1}^{k} |\mathcal{H}_i| \le \sum_{i=0}^{k-1} \binom{|X|}{\left\lfloor \frac{|X|+1-k}{2} \right\rfloor + i}, \tag{2}$$

*where the sum includes the $k$ largest binomial coefficients.*

For $0 < k \le |X|$, a family $\mathcal{H}$ is called an *$k$-Sperner family*, if no $k+1$ sets in $\mathcal{H}$ form a chain for inclusion. A dual Dilworth theorem for posets immediately implies that any $k$-Sperner family can be decomposed into the union of $k$ pairwise disjoint Sperner families, therefore Theorem 1.2 gives a sharp upper bound on the size of $k$-Sperner families.

The following common generalizations of the BLYM inequality and Theorem 1.2 has been folklore and was first in print in [6].

**Theorem 1.3 (Folklore)** *If $\mathcal{H}$ is a $k$-Sperner family on $X$, then*

$$\sum_{i=0}^{n} \frac{P_i(\mathcal{H})}{\binom{|X|}{i}} \le k,$$

*with equality if and only if $\mathcal{H}$ contains every subset of $k$ given distinct sizes.*

We say that the Boolean lattice satisfies the *strict* $k$-BLYM inequality.

In the mid sixties Katona and Kleitman discovered independently and almost simultaneously that one can relax the condition of the Sperner theorem while keeping its conclusion: Let $X = X_1 \uplus X_2$ be a fixed partition. A family $\mathcal{H} \subset 2^X$ is called a *two-part Sperner family* if $\forall E, F \in \mathcal{H} \quad E \subsetneqq F \Rightarrow \forall i :$ $F \setminus E \not\subseteq X_i$. Since Sperner families are also two-part Sperner families (for any two-partition), therefore no maximum two-part Sperner family can be smaller than a maximum Sperner family. But the next is also true:

**Theorem 1.4 (G.O.H. Katona [7] and D.J. Kleitman [9].)** *The size of a two-part Sperner family cannot exceed the size of a maximum Sperner family.*

A family $\mathcal{H}$ of subsets on a given underlying set $X$ with 2 partition classes is called *homogeneous*, if for any subset of $X$ the sizes of the subset's intersections with the partition classes already determine whether this particular subset belongs to the family $\mathcal{H}$. Since in the profile matrix (which is actually the two-dimensional profile vector) of a homogeneous two-part Sperner family no column/raw can contain more than one non-zero element therefore the profile matrix cannot contain more than $\min\{|X_1|+1; |X_2|+1\}$ non-zero elements.

A two-part Sperner family is called *well-paired* if it is homogeneous and - roughly speaking - if larger levels from $X_1$ are paired with larger ones from $X_2$ and smaller levels with smaller ones. The well-pairing is clearly not unique, but actually all well-paired two-part Sperner families achieve maximum size. As it later turned out, the two-part Sperner theorem is *strict*:

**Theorem 1.5 (P.L. Erdős and G.O.H. Katona [5])** *A two-part Sperner family $\mathcal{H}$ has maximum size if and only if $\mathcal{H}$ is a well-paired family.*

In [14], [6] and [1] further proofs were given. One of the main tools in [1] was the following strict BLYM-type inequality:

**Theorem 1.6 (two-part BLYM-inequality, [1])** *Let $\mathcal{H}$ be a two-part Sperner family on the underlying set $X_1 \uplus X_2$ where $|X_2| \le |X_1|$. Then*

$$\sum_{i,j} \frac{M_{ij}(\mathcal{H})}{\binom{|X_1|}{i}\binom{|X_2|}{j}} \le |X_2| + 1. \tag{3}$$

*Furthermore, if $\mathcal{H}$ is a maximum size two-part Sperner family, then equality holds and $\mathcal{H}$ is homogenous. Consequently the two-part Sperner families satisfy the strict BLYM-inequality.*

The study of the Sperner problem for *partially ordered sets* (or *poset*, for short) emerged naturally. One of the early encounter is Baker's observation ([2], 1969), that every regular poset (see later its definition) satisfies the Sperner property. Somewhat later Schonheim [13] and Katona [8] generalized the two-part Sperner theorem for certain posets.

In 1974 Kleitman proved an important result (see Theorem 2.1) which is suitable to prove BLYM-type theorems for *normal* posets. What is interesting here, however, that normal posets generally do not satisfy the strict Sperner property.

In this extended abstract our main objective is to find the right setup for the strict BLYM inequality and strict two-part Sperner theorem for direct products of posets. We describe a subclass of normal posets which satisfy the strict $k$-BLYM inequality. We extend the two-part BLYM inequality to direct products of two such posets. This allows us to prove our main result (Theorem 2.3): a strict two-part Sperner theorem for direct products.

## 2 Definitions and results

In the remaining of this paper each poset $P$ has a *rank function* $r$. We denote the collection of elements of rank $i$ by $L_i(P)$ and call it the $i$th level of the poset, where $i = 0, 1, \ldots, r(P)$, and $r(P)$ is the maximum rank in $P$. Symbol $N_i(P)$ denotes the $i$th Whitney number of $P$, that is $N_i(P) = |L_i(P)|$. With some abuse of notation we will use simply $L_i$ and $N_i$ when this does not cause confusion.

Denote $G(P) = (V, E)$ the usual *Hasse diagram* of the ranked poset $P$, and denote $G_i(P)$ the induced bipartite subgraphs between the $i$th and $(i+1)$th levels (for $i = 0, \ldots, r(P) - 1$). The poset $P$ is called *rank-connected* if for each $i$ the bipartite graph $G_i(P)$ is connected.

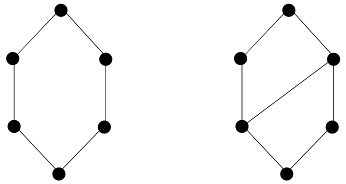
A poset $P$ is called *regular* if for every element $a \in P$ of a given rank, both the number of elements which cover $a$ and the number of elements covered by $a$ do not depend on the choice of $a$. For the lower (resp. upper) degree of an element $a \in P$ (vertex in $G(P)$) we use the notation $d^-(a)$ (resp. $d^+(a)$). For the notions of *normal* posets and *regular chain covering* we use the common definitions. Recall that every regular poset is normal.

The following important result was used for a wide range of posets to prove their strong Spernerity and $k$-BLYM inequalities.

**Theorem 2.1 (D.J. Kleitman [10])** *The following statements are equivalent:* (i) *$P$ is normal;* (ii) *$P$ satisfies BLYM inequality;* (iii) *$P$ satisfies $k$-BLYM inequality for $k = 1, 2, \ldots, r(P)$;* (iv) *there exists a regular chain covering of $P$.*

Note that the ealier Baker's result [2], which actually says that every regular poset satisfies BLYM inequality (and the proof is based on the regular chain covering), is a special case of Theorem 2.1. To prove our main result (Theorem 2.3) we actually use this weaker version of Kleitman's result.

The examples given in Figure 1 show that the condition of Theorem 2.1 is not sufficient for proving strict BLYM (or $k$-BLYM) inequality: These posets are normal but they do not posses the strict Sperner property (and therefore the strict BLYM inequality). The poset at Figure 1b is not regular, and the poset at Figure 1a, while it is regular, it is not rank-connected.



(a) (b)

Fig. 1.

Now we are ready to report our findings. We start with the generalization of the strict $k$-BLYM inequality:

**Theorem 2.2** *Let $P$ be a regular unimodal poset satisfying rank-connectivity and let $F \subset P$ be a $k$-Sperner system Then*

$$\sum_{a \in F} \frac{1}{N_{r(a)}} \leq k, \tag{4}$$

*and equality holds if and only if $F$ is homogeneous. Furthermore - as a consequence - the strict $k$-Sperner theorem also holds.*

Using this fact, similarly to paper [1] one can prove the following strict 2-part Sperner theorem:

**Theorem 2.3** *Let $P_1$ and $P_2$ be regular, unimodal posets, both satisfying rank-connectivity. Let $F \subset P_1 \times P_2$ be a maximum size two-part Sperner system. Then $F$ is a well paired homogeneous system.*

The following result plays a key role in the proof of Theorem 2.3:

**Lemma 2.4** *Let $P_1, P_2$ be regular, unimodal, rank-connected posets. Let $F$ be a maximum size two-part Sperner system. Assume that $n_2 \leq n_1$. Then*

$$\sum_{(a,b) \in F} \frac{1}{N^{(1)}_{r(a)} N^{(2)}_{r(b)}} = n_2 + 1. \tag{5}$$

## References

[1] H. Aydinian and P.L. Erdös, All maximum size two-part Sperner systems—in short, *Comb. Prob. Comp.* **16** (4) (2007), 553–555.

[2] K.A. Baker, A Generalization of Sperner's Lemma, *J. Comb. Theory* **6** (1969), 224–225.

[3] B. Bollobás, On generalized graphs, *Acta Mathematica Academiae Scientiarum Hungaricae* **16** (1965) (34), 447–452.

[4] Paul Erdős, On a lemma of Littlewood and Offord, *Bull. of the Amer. Math. Soc.* **51** (1945), 898–902.

[5] P.L. Erdős and G.O.H. Katona, All maximum two-part Sperner families, *J. Comb. Theory* (A) **43** (1986), 58–69.

[6] P.L. Erdős - Z. Füredi - G.O.H. Katona: Two-part and $k$-Sperner families – new proofs using permutations, *SIAM J. Discrete Math.* **19** (2005), 489–500.

[7] G.O.H. Katona: On a conjecture of Erdős and a stronger form of Sperner's theorem, *Studia Sci. Math. Hungar.* **1** (1966), 59–63.

[8] G.O.H. Katona, A generalization of some generalizations of Sperner's Theorem, *J. Combin. Theory* (B) (12) (1972), 72–81.

[9] D.J. Kleitman, On a lemma of Littlewood and Offord on the distribution of certain sums, *Math. Zeit.* **90** (1965), 251–259.

[10] D.J. Kleitman, On an extremal property of antichains in partial orders. The LYM property and some of its implications and applications, in *Combinatorics, Part 2: Graph theory; foundations, partitions and combinatorial geometry*, Math. Centre Tracts, No. 56, Math. Centrum, Amsterdam, (1974), 77–90.

[11] D. Lubell, A short proof of Sperner's lemma, *J. Comb. Theory* **1** (1966) (2), 299.

[12] L.D. Meshalkin, Generalization of Sperner's theorem on the number of subsets of a finite set, *Theory of Probability and its Applications* **8** (1963) (2), 203–204.

[13] J. Schonheim, A generalization of results of P. Erdős, G. Katona, and D.J. Kleitman concerning Sperner's theorem, *J. Combin Theory Ser. A* **11** (1971), 111–117.

[14] S. Shahriari: On the structure of maximum two-part Sperner families, *Discrete Math.* **162** (1996), 229–238.

[15] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27** (1928), 544–548.

[16] K. Yamamoto, Logarithmic order of free distributive lattice, *J. Math. Soc. Japan* **6** (1954), 343–353.