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# Quasi-kernels and quasi-sinks in infinite graphs ☆

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ABSTRACT

Given a directed graph $G = (V, E)$ an independent set $A \subset V$ is called *quasi-kernel (quasi-sink)* iff for each point $v$ there is a path of length at most 2 from some point of $A$ to $v$ (from $v$ to some point of $A$). Every finite directed graph has a quasi-kernel. The plain generalization for infinite graphs fails, even for tournaments. We study the following conjecture: for any digraph $G = (V, E)$ there is a a partition $(V_0, V_1)$ of the vertex set such that the induced subgraph $G[V_0]$ has a quasi-kernel and the induced subgraph $G[V_1]$ has a quasi-sink.


## 1. Introduction

Given a directed graph $G = (V, E)$ an independent set $A \subset V$ is called *quasi-kernel (quasi-sink)* iff for each point $v$ there is a path of length at most 2 from some point of $A$ to $v$ (from $v$ to some point of $A$). (The notions have a fairly extensive literature: see, for example, [2–4].)

The starting point of our investigation was the following theorem:

**Theorem 1.1** (*Chvátal–Lovász, [1]*). *Every finite digraph (directed graph) contains a quasi-kernel.*

Our aim is to find similar theorems for infinite digraphs. The plain generalization of Theorem 1.1 fails even for infinite tournaments, which is shown by $(\mathbb{Z}, <)$, where $\mathbb{Z}$ denotes the set of the integers, and $(x, y)$ is an edge iff $x < y$.

However, not just for $(\mathbb{Z}, <)$ but for each tournament $G = (V, E)$ either it has a quasi-kernel or there are two vertices $a$ and $b$ such that $V = \mathrm{Out}(a) \cup \mathrm{In}(b)$ (see Theorem 3.1). This situation is typical among the infinite digraphs as shown by Theorem 2.1: *Each directed graph $G = \langle V, E\rangle$ contains two disjoint, independent subsets $A$ and $B$ of $V$ such that for each vertex $v$ there is a path of length at most 2 either from some point of $A$ to $v$, or from $v$ to some point of $B$.*

Before finding the (easy) proof of the claim above we tried to disprove it. However, instead of finding counterexamples we obtained "positive" statements. In Section 2 we prove some easy results showing that digraphs "resembling" finite graphs have quasi-kernels.

In Section 3 we study tournament-like digraphs, and graphs which are built from simple blocks. Such a digraph $G$ may not have a quasi-kernel or quasi-sink but the vertices has a partition $(V_0, V_1)$ such that $G[V_0]$ has a quasi-kernel and $G[V_1]$ has a quasi-sink.

These observations led to formulate the following conjecture.

**Conjecture 1.2.** *Given any digraph $G = (V, E)$ one can find a partition $(V_0, V_1)$ of the vertex set such that the induced subgraph $G[V_0]$ has a quasi-kernel and $G[V_1]$ has a quasi-sink.*

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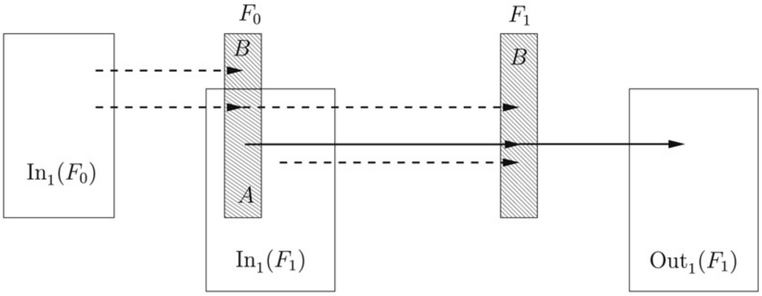


**Fig. 1.**

Section 4 studies the structure of infinite tournaments without quasi-kernels. For $n \in \mathbb{N}$ denote by $\mathfrak{Out}_n$ the family of digraphs $G = (V, E)$ which have an independent set $A \subset V$ such that for each point $v$ there is a path of length at most $n$ from some point of $A$ to $v$. Theorem 4.2 characterizes infinite tournaments in $\mathfrak{Out}_n$ for each $n \geq 3$. This characterization implies immediately that the classes $\mathfrak{Out}_3, \mathfrak{Out}_4, \ldots$ contain the same tournaments. To show that $\mathfrak{Out}_2$ and $\mathfrak{Out}_3$ contain different tournaments (see Theorem 5.1), we developed a recursive method to construct infinite digraphs from certain finite ones in Section 5. One might hope that this method may help to disprove our conjecture, but this is not the case, because Theorem 5.7 claims that all digraphs obtained by this method also satisfy Conjecture 1.2.

We will use standard combinatorial and set-theoretical notations. If $V$ is a set then $V^*$ denotes the family of finite sequences of elements of $V$. If $a, b \in V^*$ then $a^\frown b$ is the *concatenation* of the two sequences. If $A, B \subset V^*$ let $A^\frown B = \{a^\frown b : a \in A, b \in B\}$. Whenever $x \in V^*$ we write $A^\frown x$ for $A^\frown\{x\}$. The family of two element subsets of $V$ is denoted by $[V]^2$.

If $G = (V, E)$ is a digraph and $W \subset V$, the induced subgraph of $G$ on $W$ is denoted by $G[W]$, i.e. $G[W] = (W, E \cap (W \times W))$.

To simplify the formulation of our results we introduce some terminology. Assume that $G = (V, E)$ is a digraph and $A \subset V$. For $n \in \mathbb{N}$ let us define

$$\mathrm{In}_n^G(A) = \{v \in V : \text{ there is a path of length at most } n \text{ which leads from } v \text{ to some point of } A\}$$

and

$$\mathrm{Out}_n^G(A) = \{v \in V : \text{there is a path of length at most } n \text{ which leads from some point of } A \text{ to } v\}.$$

Put

$$\mathrm{Out}_\infty^G(A) = \bigcup\{\mathrm{Out}_n^G(A) : n \in \mathbb{N}\}$$

and

$$\mathrm{In}_\infty^G(A) = \bigcup\{\mathrm{In}_n^G(A) : n \in \mathbb{N}\}.$$

If $A = \{a\}$ we write $\mathrm{In}_n^G(a)$ for $\mathrm{In}_n^G(\{a\})$, and $\mathrm{Out}_n^G(a)$ for $\mathrm{Out}_n^G(\{a\})$. We will omit the superscript $G$ whenever the digraph is clear from the context.

Using this notation above the classes $\mathfrak{Out}_2, \mathfrak{Out}_3, \ldots, \mathfrak{Out}_\infty, \mathfrak{In}_2, \mathfrak{In}_3, \ldots$ and $\mathfrak{In}_\infty$ of digraphs are defined as follows. For $n \in \mathbb{N} \cup \{\infty\}$ the digraph $G = (V, E)$ is in $\mathfrak{In}_n$ iff there is an independent set $A \subset V$ such that $V = \mathrm{In}_n^G(A)$, and $G \in \mathfrak{Out}_n$ iff there is an independent set $B \subset V$ such that $V = \mathrm{Out}_n^G(B)$. We say that "*$A$ witnesses $G \in \mathfrak{In}_n$*" and "*$B$ witnesses $G \in \mathfrak{Out}_n$*".

For $n, k \in \mathbb{N} \cup \{\infty\}$ define the class $\mathfrak{In}_n$-$\mathfrak{Out}_k$ of digraphs as follows: $G \in \mathfrak{In}_n$-$\mathfrak{Out}_k$ if and only if there is a partition $(V_1, V_2)$ of the vertex set $V$ such that $G[V_1] \in \mathfrak{In}_n$ and $G[V_2] \in \mathfrak{Out}_k$. We say that "*$(V_1, V_2)$ witnesses $G \in \mathfrak{In}_n$-$\mathfrak{Out}_k$*".

Using this new terminology we can reformulate the Theorem of Chvátal and Lovász and our Conjecture as follows:

**Theorem 1.1.** *Every finite digraph is in $\mathfrak{Out}_2$,*

**Conjecture 1.2.** *Every digraph is in $\mathfrak{In}_2$-$\mathfrak{Out}_2$.*

## 2. Stepping-up theorems

**Theorem 2.1.** *Each directed graph $G = \langle V, E\rangle$ contains two disjoint, independent subsets $A$ and $B$ of $V$ such that $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$.*

This result is a joint work with András Hajnal, and it is included with his kind permission.

**Proof.** Let $F_0$ be a maximal independent subset in $G$, and let $F_1$ be a maximal independent subset in $G[V \setminus \mathrm{In}_1(F_0)]$. Put $A = F_0 \cap \mathrm{In}_1(F_1)$ and $B = F_1 \cup (F_0 \setminus A)$, see Fig. 1.

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The sets $A$ and $B$ are clearly independent. Moreover,

$$\mathrm{In}_1(F_0) = \mathrm{In}_1(F_0 \cap \mathrm{In}_1(F_1)) \cup \mathrm{In}_1(F_0 \setminus \mathrm{In}_1(F_1)) \subset \mathrm{In}_2(F_1) \cup \mathrm{In}_1(B) \subset \mathrm{In}_2(B). \tag{1}$$

Since $F_1 \subset \mathrm{Out}_1(A)$ and so $\mathrm{Out}_1(F_1) \subset \mathrm{Out}_2(A)$ we have

$$V \setminus \mathrm{In}_1(F_0) \subset \mathrm{Out}_1(F_1) \cup \mathrm{In}_1(F_1) \subset \mathrm{Out}_2(A) \cup \mathrm{In}_1(B) \subset \mathrm{Out}_2(A) \cup \mathrm{In}_2(B). \tag{2}$$

(1) and (2) together yield $V = \mathrm{Out}_2(A) \cup \mathrm{In}_2(B)$. □

By a standard application of Gödel's Compactness Theorem one can get the following consequence of Theorem 1.1 for infinite graphs:

**Corollary 2.2.** *If in a digraph $G$ every vertex has finite in-degree then $G$ has a quasi-kernel.*

Next we prove two stepping-up theorems. The first will imply immediately that every finitely chromatic digraph has quasi-kernel. The second one will be applied in the next section.

**Definition 2.3.** A directed graph $G$ is *hereditary in $\mathfrak{Out}_n$* (*or hereditary in $\mathfrak{In}_m$-$\mathfrak{Out}_n$*) iff all induced subgraphs of $G$ are in $\mathfrak{Out}_n$ (or in $\mathfrak{In}_m$-$\mathfrak{Out}_n$, respectively).

**Theorem 2.4.** *Let $G = (V, E)$ be a directed graph and let $n \geq 1$. Assume that $V$ has a partition $(V_0, V_1, \ldots, V_k)$ such that*

(i) *$G[V_0]$ is hereditary in $\mathfrak{Out}_{n+1}$,*
(ii) *for $1 \leq i < k$ $G[V_i]$ is hereditary in $\mathfrak{Out}_n$,*
(iii) *either $k = 0$ or $G[V_k]$ is in $\mathfrak{Out}_n$.*

*Then $G$ is $\mathfrak{Out}_{n+1}$.*

**Proof.** By induction on $k$. For $k = 0$ the claim is trivial. Assume now that $k \geq 1$, the statement is true for $k-1$ and prove it for $k$.

By (iii) $V_k = \mathrm{Out}_n^{G[V_k]}(A_k)$ for some independent sets $A_k \subset V_k$. For $0 \leq i < k$ let $V_i' = V_i \setminus \mathrm{Out}_1^G(A_k)$ and put $V' = \bigcup\{V_i' : 0 \leq i < k\}$. Then we can apply the inductive hypothesis for $G' = G[V']$ because (i) and (ii) imply that the partition $(V_0', V_1', \ldots, V_{k-1}')$ satisfies (i)–(iii). Thus, $V'$ contains an independent set $A'$ such that $V' = \mathrm{Out}_{n+1}^{G[V']}(A')$.

Let $\bar{A} = A' \cup (A_k \setminus \mathrm{Out}_1^G(A'))$. Then $\bar{A}$ is independent because $\mathrm{Out}_1^G(A_k) \cap A' \subset \mathrm{Out}_1^G(A_k) \cap V' = \emptyset$, moreover $A_k \subset \mathrm{Out}_1^G(\bar{A})$ and so $\mathrm{Out}_1^G(A_k) \subset \mathrm{Out}_2^G(\bar{A})$. Since $n + 1 \geq 2$ it follows that $V = \mathrm{Out}_{n+1}^G(\bar{A})$. □

This result gives us the following generalization of the Chvátal–Lovász Theorem:

**Corollary 2.5.** *If $G$ has finite chromatic number then $G \in \mathfrak{Out}_2$.*

**Proof.** Indeed, the monochromatic classes are independent, so they are hereditary in $\mathfrak{Out}_1$. Thus, we can apply Theorem 2.4 to obtain $G \in \mathfrak{Out}_2$. □

The following generalization of Theorem 2.4 is mainly a technical tool to be used later.

**Theorem 2.6.** *Let $G = (V, E)$ be a directed graph and let $\ell, m \geq 1$. Assume that $V$ has a partition $(V_0, V_1, \ldots, V_k)$ such that*

(i) *$G[V_0]$ is hereditary in $\mathfrak{In}_{m+1}$-$\mathfrak{Out}_{\ell+1}$,*
(ii) *for $1 \leq i < k$ $G[V_i]$ is hereditary in $\mathfrak{In}_m$-$\mathfrak{Out}_\ell$,*
(iii) *either $k = 0$ or $G[V_k]$ is in $\mathfrak{In}_m$-$\mathfrak{Out}_\ell$.*

*Then $G$ is in $\mathfrak{In}_{m+1}$-$\mathfrak{Out}_{\ell+1}$.*

**Proof.** Similarly to the proof of Theorem 2.4, we use induction on $k$. For $k = 0$ the statement is trivial. Assume that $k \geq 1$, the claim is true for $k-1$ and prove it for $k$.

Let $(X_k, Y_k)$ be an $\mathfrak{In}_m$-$\mathfrak{Out}_\ell$-partition of $G[V_k]$, i.e. $X_k = \mathrm{Out}_\ell^{G[X_k]}(A_k)$ and $Y_k = \mathrm{In}_m^{G[Y_k]}(B_k)$ for some independent sets $A_k$ and $B_k$.

Put $V^* = (\mathrm{Out}_1^G(A_k) \cup \mathrm{In}_1^G(B_k)) \setminus V_k$ and $V' = (V \setminus V_k) \setminus V^*$. For $0 \leq i' < k$ let $V_i' = V_i \cap V'$.

Then we can apply the inductive hypothesis for $G' = G[V']$ because the partition $(V_0', V_1', \ldots, V_{k-1}')$ satisfies (i)–(iii). Thus, $V'$ has a partition $(X', Y')$ and there are independent sets $A' \subset X'$ and $B' \subset Y'$ such that $X' = \mathrm{Out}_{\ell+1}^{G[X']}(A')$ and $Y' = \mathrm{In}_{m+1}^{G[Y']}(B')$.

Let $(X, Y)$ be a partition of $V$ such that $(X \setminus V^*, Y \setminus V^*) = (X' \cup X_k, Y' \cup Y_k)$, $X \cap V^* \subset \mathrm{Out}_1^G(A_k)$ and $Y \cap V^* \subset \mathrm{In}_1^G(B_k)$.

Then $A = A' \cup (A_k \setminus \mathrm{Out}_1^G(A'))$ and $B = B' \cup (B_k \setminus \mathrm{In}_1^G(B'))$ are independent subsets of $X$ and $Y$, respectively. Moreover, $X = \mathrm{Out}_{\ell+1}^{G[X]}(A)$ and $Y = \mathrm{In}_{m+1}^{G[Y]}(B)$. □

The next corollary proves our conjectures for graphs which are built from simple blocks.

**Corollary 2.7.** *Suppose $G$ has a partition $(A_1, \ldots, A_k)$ such that each $G[A_i]$ is hereditary in $\mathfrak{In}_1$-$\mathfrak{Out}_1$ (for example, isomorphic to one of $(\mathbb{Z}, <)$, $(\mathbb{N}, <)$, $(\mathbb{N}, >)$, or has no edges) then $G \in \mathfrak{In}_2$-$\mathfrak{Out}_2$.*

**Proof.** Since every $G[A_i]$ is hereditary in $\mathfrak{In}_1$-$\mathfrak{Out}_1$ apply Theorem 2.6 directly. □

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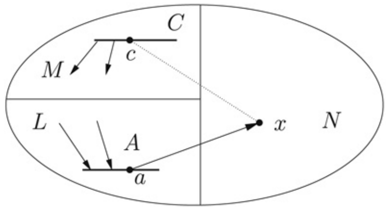


**Fig. 2.**

## 3. Tournament-like digraphs

Recall that $(\mathbb{Z}, <) \notin \mathfrak{Out}_2$ but it is in $\mathfrak{In}_1$-$\mathfrak{Out}_1$. We show that this remains true for arbitrary tournaments.

**Theorem 3.1.** *An infinite tournament is either in $\mathfrak{Out}_2$, or it is in $\mathfrak{In}_1$-$\mathfrak{Out}_1$.*

**Proof.** Let $G = (V, E)$ be a tournament, and $x \in V$ be arbitrary. If $y \notin \mathrm{Out}_2(x)$ then $V = \mathrm{In}_1(x) \cup \mathrm{Out}_1(y)$. Indeed, if $z \notin \mathrm{Out}_1^G(y)$ then $(z, y) \in E$ but $xzy$ is not a directed path of length two in $G$ by the choice of $y$, so $(x, z) \notin E$. Thus, $(z, x) \in E$, i.e. $z \in \mathrm{In}_1^G(x)$. Since $z$ was arbitrary, we obtain $G \in \mathfrak{In}_1$-$\mathfrak{Out}_1$. $\square$

If $G = (V, E)$ is a digraph define the *undirected complement* of the digraph $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$. The graph $\widetilde{G}$ can be used to measure the difference between $G$ and a tournament: the more edges are in $\widetilde{G}$, the larger the difference between $G$ and a tournament is. For example, $G$ is a tournament iff $\widetilde{G}$ does not have any edge.

**Theorem 3.2.** *Let $G = (V, E)$ be a directed graph. If $K_n \not\subset \widetilde{G}$ for some $n \geq 2$ then $G \in \mathfrak{In}_2$-$\mathfrak{Out}_2$. Moreover, if $\widetilde{G}$ is empty then $G \in \mathfrak{Out}_2 \cup \mathfrak{In}_1$-$\mathfrak{Out}_1$, and if $\widetilde{G}$ is triangle-free, then either $G \in \mathfrak{In}_1$-$\mathfrak{Out}_2$, or $G \in \mathfrak{In}_2$-$\mathfrak{Out}_1$.*

**Proof.** By induction on $n$. If $n = 2$ then $\widetilde{G}$ does not contain edges, i.e. $G$ is a tournament and so we are done by the previous theorem.

Assume now that the theorem is true for $n - 1$ and prove it for $n$. Let $A$ be a maximal independent set in $G$. If $V = \mathrm{Out}_2(A)$ then we are done.

If this is not the case, then let $C$ be a maximal independent set in $G[V \setminus \mathrm{Out}_2(A)]$. Let $L = \mathrm{In}_1(A) \setminus C$, $M = \mathrm{Out}_1(C) \setminus L$ and $N = V \setminus (L \cup M)$, see Fig. 2.

**Claim 1.** *There is no edge between $N$ and $C$.*

**Proof of the Claim.** Let $x \in N$. If $a \in A$ then $(x, a) \notin E$ because $x \notin \mathrm{In}_1(A)$ but $(a, x) \in E$ for some $a \in A$ because $A$ was maximal. Moreover, for each $c \in C$ we have $(c, x) \notin E$ because $x \notin \mathrm{Out}_1(C)$. But $(x, c) \notin E$ as well otherwise the path $(a, x, c)$ witnesses that $c \in \mathrm{Out}_2(A)$. $\square$

Since $C \neq \emptyset$ we have that $K_{n-1} \not\subset \widetilde{G[N]}$ (otherwise $\widetilde{G}$ would contain $K_n$). Hence we can apply the inductive hypothesis for $G[N]$.

*Case* 1. $n = 3$.

Then $G[N]$ is a tournament. If $N = \mathrm{Out}_2^{G[N]}(d)$ for some $d \in N$ then $L = \mathrm{In}_1^{G[L]}(A)$ and $V \setminus L = \mathrm{Out}_2^{G[V \setminus L]}(C \cup \{d\})$. Thus, $G \in \mathfrak{In}_1$-$\mathfrak{Out}_2$.

Otherwise $N$ has a partition $P \cup R$ and there are $x \in P$ and $y \in R$ such that $P = \mathrm{Out}_1^{G[P]}(x)$ and $R = \mathrm{In}_1^{G[R]}(y)$. Then

$$M \cup P = \mathrm{Out}_1^{G[M \cup P]}(C \cup \{x\})$$

and

$$L \cup R = \mathrm{In}_2^{G[L \cup R]}(\{y\} \cup \{a \in A : (a, y) \notin E\}).$$

Thus, $G \in \mathfrak{In}_2$-$\mathfrak{Out}_1$.

*Case* 2. $n > 3$.

By the inductive hypothesis $G[N]$ is hereditary in $\mathfrak{In}_2$-$\mathfrak{Out}_2$ (since $K_n \not\subset \widetilde{G}$ is a hereditary property), moreover $G[L \cup M] \in \mathfrak{In}_1$-$\mathfrak{Out}_1$, hence we can apply Theorem 2.6 for $m = \ell = 1$, for the digraph $G$ and for the partition $(N, L \cup K)$ to yield $G \in \mathfrak{In}_2$-$\mathfrak{Out}_2$. $\square$

**Corollary 3.3.** *Let $G = (V, E)$ be a directed graph. If $\widetilde{G}$ has finite chromatic number then $G$ is $\mathfrak{In}_2$-$\mathfrak{Out}_2$.*

Indeed, if the chromatic number of $\widetilde{G}$ is $n$ then $\widetilde{G}$ does not contain $K_{n+1}$.

**Remark.** One can try to prove this corollary directly from Theorem 2.6. If $\widetilde{G}$ has finite chromatic number then the vertex set has a partition $(V_0, \dots, V_k)$ such that every $G[V_i]$ is a tournament and so $G[V_i]$ is hereditary in $\mathfrak{In}_1$-$\mathfrak{Out}_2$. Thus, applying directly Theorem 2.6 one gets only $G \in \mathfrak{In}_2$-$\mathfrak{Out}_3$.

An undirected graph is called *locally finite* iff every vertex has finite degree.

**Theorem 3.4.** *If $G = (V, E)$ is a digraph such that $\widetilde{G}$ is locally finite then $G \in \mathfrak{In}_2$-$\mathfrak{Out}_2$.*

**Proof.** We prove the claim by transfinite induction on $\lambda = |V|$. If $\lambda$ is finite then $G \in \mathfrak{Out}_2$ by Theorem 1.1. We can assume that $\lambda = |V|$ is infinite and the claim is true for graphs of cardinality $< \lambda$. We distinguish two cases.

*Case* 1: *There are $x, y \in V$ such that the set $U = \mathrm{Out}^G_1(x) \cap \mathrm{In}^G_1(y)$ has cardinality $\lambda$.*

We will find a partition $(X, Y)$ of $V$ such that $X = \mathrm{Out}^{G[X]}_2(x)$ and $Y = \mathrm{In}^{G[Y]}_2(y)$. To this end fix an enumeration of the vertices as $V = \langle v_\zeta : \zeta < \lambda \rangle$. By transfinite induction on $\zeta < \lambda$ we construct disjoint subsets $X_\zeta$ and $Y_\zeta$ of $V$ such that $|X_\zeta| + |Y_\zeta| \le \omega + |\zeta|$, $X_\zeta = \mathrm{Out}^{G[X_\zeta]}_2(x)$ and $Y_\zeta = \mathrm{In}^{G[Y_\zeta]}_2(y)$.

Put $X_0 = \{x\}$ and $Y_0 = \{y\}$. Assume that for all $\eta < \zeta$ we have already constructed $X_\eta, Y_\eta$. If $\zeta$ is a limit ordinal put $X_\zeta = \bigcup\{X_\xi : \xi < \zeta\}$ and $Y_\zeta = \bigcup\{Y_\xi : \xi < \zeta\}$

If $\zeta$ is not a limit ordinal, i.e. $\zeta = \eta + 1$, then we have $X_\eta$ and $Y_\eta$ in such a way that $X_\eta = \mathrm{Out}^{G[X_\eta]}_2(x)$ and $Y_\eta = \mathrm{In}^{G[Y_\eta]}_2(y)$. Let $i = \min\{i' : v_{i'} \notin X_\eta \cup Y_\eta\}$.

If $|\mathrm{In}^G_1(v_i) \cap U| = \lambda$ then let

$$j = \min\{j' : v_{j'} \in (\mathrm{In}^G_1(v_i) \cap \mathrm{Out}^G_1(x)) \setminus (X_\eta \cup Y_\eta)\},$$

and let $X_\zeta = X_\eta \cup \{v_i, v_j\}$ and $Y_\zeta = Y_\eta$.

If $|\mathrm{In}^G_1(v_i) \cap U| < \lambda$ then $|\mathrm{Out}^G_1(v_i) \cap U| = \lambda$ because $v_i$ has finite degree in $\widetilde{G}$. Let

$$j = \min\{j' : v_{j'} \in \left(\mathrm{Out}^G_1(v_i) \cap \mathrm{In}^G_1(y)\right) \setminus (X_\eta \cup Y_\eta)\},$$

and let $Y_\zeta = Y_\eta \cup \{v_i, v_j\}$ and $X_\zeta = X_\eta$. Put finally $X = X_\lambda$ and $Y = Y_\lambda$.

*Case* 2: $|\mathrm{Out}^G_1(x) \cap \mathrm{In}^G_1(y)| < \lambda$ *for each* $\{x, y\} \in [V]^2$.

Fix the vertices $x \ne y \in V$ arbitrarily, and put $W = V \setminus (\mathrm{Out}^G_1(x) \cup \mathrm{In}^G_1(y))$. Then $W \setminus (\mathrm{In}^G_1(x) \cap \mathrm{Out}^G_1(y)) = (W \setminus \mathrm{In}^G_1(x)) \cup (W \setminus \mathrm{Out}^G_1(y))$ is finite because $\widetilde{G}$ is locally finite. Thus, $|W| < \lambda$, hence $G[W]$ is hereditary in $\mathfrak{In}_2$-$\mathfrak{Out}_2$ by the inductive hypothesis. Moreover, $V \setminus W = \mathrm{Out}^G_1(x) \cup \mathrm{In}^G_1(y)$, hence $G[V \setminus W] \in \mathfrak{In}_1$-$\mathfrak{Out}_1$. Therefore, we can apply Theorem 2.6 for $m = \ell = 1$, the digraph $G$ and the partition $(W, V \setminus W)$ to yield $G \in \mathfrak{In}_2$-$\mathfrak{Out}_2$. □

## 4. Infinite tournaments

In this section we prove structural theorems for infinite tournaments.

For any cardinal $\kappa$ let the digraph $\mathbb{T}_\kappa = (\kappa, \ge)$, i.e. $(x, y)$ is an edge if and only if $x \ge y$.

**Theorem 4.1.** *For an infinite tournament $G = (V, E)$ the following are equivalent:*

(i) $G \notin \mathfrak{Out}_\infty$,
(ii) *for some regular cardinal $\kappa$ there is a surjective homomorphism $\varphi : G \to \mathbb{T}_\kappa$.*

**Proof.** (ii) clearly implies (i): if $\varphi(x) = k$ then $\varphi(y) \le k$ for each $y \in \mathrm{Out}^G_\infty(x)$, and so $\mathrm{Out}^G_\infty(x) \ne V$ because $\varphi$ is surjective.

Assume now that (i) holds, i.e. $G \notin \mathfrak{Out}_\infty$. By transfinite recursion construct a sequence $\langle x_\eta : \eta < \xi \rangle$ of vertices such that

(a) $x_\zeta \notin \mathrm{Out}^G_\infty(\{x_\eta : \eta < \zeta\})$ for $\zeta < \xi$,
(b) $V = \mathrm{Out}^G_\infty(\{x_\eta : \eta < \xi\})$.

Since $(x_\zeta, x_\eta) \in E$ for $\eta < \zeta < \xi$ we have $\mathrm{Out}^G_\infty(\{x_\eta : \eta \le \zeta\}) = \mathrm{Out}^G_\infty(x_\zeta)$ for $\zeta < \xi$. So if $\xi = \zeta + 1$ then $V = \mathrm{Out}^G_\infty(x_\zeta)$ which contradicts $G \notin \mathfrak{Out}_\infty$. Thus, $\xi$ is a limit ordinal. Let $\kappa = \mathrm{cf}(\xi)$ and let $\langle \xi_\eta : \eta < \kappa \rangle$ be a strictly increasing cofinal sequence in $\xi$.

Define $\varphi : V \to \kappa$ by the formula $\varphi(v) = \min\{\eta : v \in \mathrm{Out}^G_\infty(x_{\xi_\eta})\}$. The map $\varphi$ is clearly a homomorphism onto $\mathbb{T}_\kappa$ because $\varphi(x_{\xi_\eta}) = \eta$. □

Define the digraph $\mathbb{T}^{(3)} = \langle \omega, E \rangle$ as follows

$$E = \{(x, y) : x \ge y\} \cup \{(x, x + 1) : x \in \omega\}. \tag{3}$$

$\mathbb{T}^{(3)}$ can be obtained from $\mathbb{T}_\omega$ by adding the edges $\{(n, n + 1) : n \in \omega\}$, see Fig. 3.

**Theorem 4.2.** *For an infinite tournament $G \in \mathfrak{Out}_\infty$ the following are equivalent:*

(i) $G \notin \mathfrak{Out}_3$,
(ii) $G \notin \mathfrak{Out}_n$ *for any* $n \ge 3$,
(iii) *there is a surjective homomorphism $\varphi : G \to \mathbb{T}^{(3)}$.*

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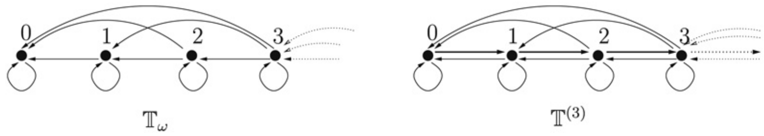


**Fig. 3.**

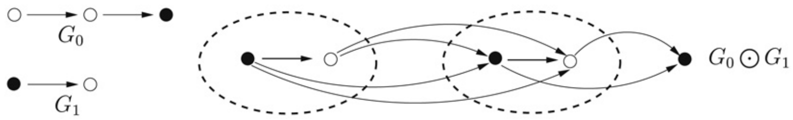


**Fig. 4.**

**Proof.** (iii) clearly implies (ii): if $\varphi(x) = k$ then $\varphi(y) \le k + n$ for each $y \in \mathrm{Out}_n^G(x)$.

To prove that (i) implies (ii) assume that $G \in \mathfrak{Out}_n$ for some $n \ge 3$. Fix $x \in V$ such that $V = \mathrm{Out}_n^G(x)$. If $V \ne \mathrm{Out}_3^G(x)$ then there is a $k > 3$ such that $V = \mathrm{Out}_k^G(x)$ but $V \ne \mathrm{Out}_{k-1}^G(x)$. Pick $y \in \mathrm{Out}_k^G(x) \setminus \mathrm{Out}_{k-1}^G(x)$. We claim that $V = \mathrm{Out}_3^G(y)$. Indeed, $\mathrm{Out}_{k-2}^G(x) \subset \mathrm{Out}_1^G(y)$ because $y \notin \mathrm{Out}_{k-1}^G(x)$. Hence $\mathrm{Out}_{k-1}^G(x) = \mathrm{Out}_1^G(\mathrm{Out}_{k-2}^G(x)) \subset \mathrm{Out}_2^G(y)$ and so finally we obtain that $V = \mathrm{Out}_k^G(x) = \mathrm{Out}_1^G(\mathrm{Out}_{k-1}^G(x)) \subset \mathrm{Out}_3^G(y)$.

Finally assume that (ii) holds. Since $G \in \mathfrak{Out}_\infty$ there is an $x \in V$ with $V = \mathrm{Out}_\infty^G(x)$. Define $\varphi : V \to \mathbb{N}$ as follows: $\varphi(y) = \min\{n : y \in \mathrm{Out}_n^G(x)\}$. $\varphi$ is clearly a homomorphism and it is onto because $\mathrm{Out}_n^G(x) \ne V$ for $n \in \mathbb{N}$. □

**Problem 4.3.** *Find a characterization of $G \notin \mathfrak{Out}_2$ a la Theorem 4.2.*

## 5. Infinite digraphs generated by a finite structure

**Theorem 5.1.** *There is an infinite tournament in $\mathfrak{Out}_3 \setminus \mathfrak{Out}_2$.*

To prove this claim we develop a recursive method to construct infinite digraphs from certain finite ones and we investigate the properties of the graphs which can be obtained in this way.

**Definition 5.2.** A terminated digraph is a triplet $G = (N, E, T)$, where $\vec{G} = (N \cup T, E)$ is a digraph, $N \cap T = \emptyset$ and $T \ne \emptyset$. The elements of $T$ are the *terminal vertices of* $G$, the elements of $N$ are the *nonterminal vertices of* $G$. For a terminated digraph $G = (N, E, T)$ write $V_G = N \cup T$, $E_G = E$, $T_G = T$ and $N_G = N$.

To simplify our notation we write $\mathrm{Out}_n^G(A)$ (or $\mathrm{In}_k^G(B)$) for $\mathrm{Out}_n^{\vec{G}}(A)$ (or for $\mathrm{In}_k^{\vec{G}}(B)$, respectively).

Assume that we have two terminated digraphs $G_0 = (N_0, E_0, T_0)$ and $G_1 = (N_1, E_1, T_1)$. Construct a new terminated digraph $G_0 \odot G_1 = (N, E, T)$ from $G_0$ and $G_1$ as follows: keep the terminal vertices of $G_0$ and blow up each nonterminal vertex $v$ of $G_0$ to a (disjoint) copy of $G_1$. So we set

$$N = N_0 \times N_1 \quad \text{and} \quad T = T_0 \cup (N_0 \times T_1).$$

The edges will be "inherited" from $G$ and $H$ in a natural way.

If $x$ is a finite sequence of length $n$, then for $i < n$ denote by $x_{(i)}$ the $i$th member of the sequence, i.e. $x = \langle x_{(0)}, x_{(1)}, \ldots, x_{(n-1)} \rangle$.

If $x$ and $y$ are finite sequences, none of them is an initial segment of the other, then let $\Delta(x, y)$ be the minimal $i$ such that $x_{(i)} \ne y_{(i)}$. For example, if $a \ne b$ then $ab_{(0)} = a$, $ab_{(1)} = b$, $a_{(0)} = a$, $\Delta(aa, ab) = 1$ and $\Delta(b, ab) = 0$.

The elements of $N \cup T$ are just finite sequences of length $\le 2$, moreover none of them is an initial segment of some other. Using this notation, let

$$E = \{(x, y) \in (N \cup T) \times (N \cup T) : (x_{(\Delta(x,y))}, y_{(\Delta(x,y))}) \in E_{\Delta(x,y)}\}.$$

See Fig. 4.

Observe that

$$G_0[T_0] \text{ is a induced subgraph of } (G_0 \odot G_1)[T]. \tag{4}$$

Fix a terminated digraph $G = (N, E, T)$. Define the sequence $\langle G_n : n \in \mathbb{N} \rangle$ of terminated digraphs as follows: $G_0 = G$, $G_{n+1} = G_n \odot G$. Write $G_n = \langle N_n, E_n, T_n \rangle$ for $n < \omega$. Then we have

$$G_0[T_0] \subset G_1[T_1] \subset G_2[T_2] \subset \cdots \tag{5}$$

Take

$$G^\infty = \bigcup \{G_n[T_n] : n \in \mathbb{N}\}. \tag{6}$$

This was the informal definition of $G^\infty$. The formal definition is much shorter:

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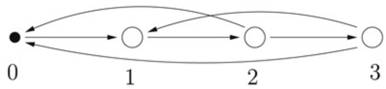


**Fig. 5.**

**Definition 5.3.** If $G = (N, E, T)$ is terminated digraph, then define the digraph $G^\infty = (N^* \frown T, F)$ as follows:

$$F = \{(x, y) : (x_{(\Delta(x,y))}, y_{(\Delta(x,y))}) \in E\},$$

where $\Delta(x, y) = \min\{i : x_{(i)} \neq y_{(i)}\}$.

We will write $V^\infty$ instead of $V(G^\infty)$ and $E^\infty$ instead of $E(G^\infty)$.

We will use the following convention: if $G = (N, E, T)$ is a terminated digraph, then $V$ denotes $N \cup T$.

First we prove two theorems which will give the example needed in Theorem 5.1.

**Proposition 5.4.** *Let $G = (N, E, T)$ be a finite terminated digraph. Then the following are equivalent:*

(i) $G^\infty \in \mathfrak{Out}_3$,
(ii) $G^\infty \in \mathfrak{Out}_\infty$,
(iii) $\mathrm{In}_1(v) \neq \{v\}$ *for each* $v \in N$.

**Proof.** Clearly (i) implies (ii).

Assume that (iii) fails: i.e. $\mathrm{In}_1^G(v) = \{v\}$ for some $v \in N$. Define $\varphi : V(G^\infty) \to \omega$ as follows: $\varphi(s) = \min\{n : s_{(n)} \neq v\}$. Then $\varphi$ is a surjective homomorphism from $G^\infty$ onto $\mathbb{T}_\omega$, so $G^\infty \notin \mathfrak{Out}_\infty$ (See Theorem 4.1). Thus, (ii) implies (iii).

Assume (iii). Write $V = N \cup T$. For $v \in V$ define $v' \in V^\infty$ as follows: $v' = v$ for $v \in T$ and $v' = v \frown t$ for $v \in N$, where $t \in T$ is arbitrary.

Let $A \subset V$ be an independent subset such that $V = \mathrm{Out}_2^G(A)$. Put $K = \{a' : a \in A\}$. Then $K$ is clearly independent in $G^\infty$. We claim that $V^\infty = \mathrm{Out}_3^{G^\infty}(K)$.

Let $x \in V^\infty$. Then there is $a \in A$ and a directed path $\langle s_0, \ldots, s_{n-1}\rangle$ from $a$ to $x(0)$ for some $n \leq 2$. If $n > 0$ then $\left\langle s'_0, \ldots, s'_{n-2}, x\right\rangle$ is a directed path in $G^\infty$, and so we are done because $s'_0 = a' \in K$.

Assume now that $n = 0$, i.e. $x(0) \in A$. If $x(0) \in T$ then $x = x(0)$ by the definition of $V^\infty$, and so $x = x(0) = x(0)' \in K$.

If $x(0) \in N$ then there is $w \in V$ with $(w, x(0)) \in E$ by (iii). Then there is an $a \in A$ and a directed path $\langle s_0, \ldots, s_{n-1}\rangle$ from $a$ to $w$ for some $n \leq 2$. Then $\left\langle s'_0, \ldots, s'_{n-1}, x\right\rangle$ is a path from $a' \in K$ to $x$ of length at most 3. $\square$

**Fact 5.5.** *If $(N \cup T, E)$ is a tournament for some terminated digraph $G = (N, E, T)$, then $G^\infty$ is also a tournament.*

**Proposition 5.6.** *If $G = (N, E, T)$ is a finite terminated digraph, then following are also equivalent:*

(i) $V^\infty = \mathrm{Out}_2^{G^\infty}(s)$ *for some* $s \in V^\infty$,
(ii) *there is an* $a \in T$ *with* $V = \mathrm{Out}_2^G(a)$.

**Proof.** First of all observe that (ii) clearly implies (i): if $V = \mathrm{Out}_2^G(a)$ for some $a \in T$ then $V^\infty = \mathrm{Out}_2^{G^\infty}(a)$.

Assume now that (ii) fails and let $s \in V^\infty$ be arbitrary, $s = r \frown a$, where $r \in N^*$ and $a \in T$. Since (ii) fails we can pick $b \in V \setminus \mathrm{Out}_2^G(a)$. Let $u = r \frown b \in V^\infty$ if $b \in T$, and let $u = r \frown b \frown c \in V^\infty$ for some $c \in T$ if $b \in N$. We claim that $u \notin \mathrm{Out}_2^{G^\infty}(s)$. Clearly $s \neq u$ and $(s, u) \neq E^\infty$ because $a \neq b$ and $(a, b) \notin E$, respectively.

Assume on the contrary that $\langle s, y, u\rangle$ is a directed path of length 2 in $G^\infty$. Since $r$ is a common initial segment of $s$ and $u$ we have that $r$ should be an initial segment of $y$, as well. Write $y = r \frown d \frown z$, where $d \in V$. Since $(s, y) \in E^\infty$ we have $d \neq b$. Since $(y, u) \in E^\infty$ we have $d \neq a$. Thus, $a$, $b$ and $d$ are pairwise different vertices and so $\langle a, d, b\rangle$ should be a directed path of length 2 in $G$ which contradicts $b \notin \mathrm{Out}_2^G(a)$. $\square$

**Proof of Theorem 5.1.** After this preparation we are ready to construct an infinite tournament in $\mathfrak{Out}_3 \setminus \mathfrak{Out}_2$. Consider the following terminated digraph: $G = (\{1, 2, 3\}, E, \{0\})$, where

$$E = \{(0, 1), (1, 2), (2, 3), (3, 1), (3, 0), (2, 0)\}.$$

See Fig. 5.

$\vec{G}$ is a finite tournament, so $G^\infty$ is a tournament by Fact 5.5, and by Propositions 5.4 and 5.6 we have $G^\infty \in \mathfrak{Out}_3 \setminus \mathfrak{Out}_2$. $\square$

**Theorem 5.7.** *If $G = \langle N, E, T\rangle$ is a finite terminated digraph, then $G^\infty \in \mathfrak{In}_2$-$\mathfrak{Out}_2$. Moreover, if $N$ is independent, then $G^\infty$ either has a quasi-kernel or a quasi-sink.*

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**Proof.** We distinguish two cases depending on whether $N$ is independent or not.

*Case* I: *There is an edge* $(x, y) \in E \cap (N \times N)$.

Let $A = \mathrm{In}_1^G(\{x, y\}) \cup \mathrm{Out}_1^G(\{x, y\})$ and choose an independent set $B \subset V \setminus A$ such that $\mathrm{Out}_2^{G[V\setminus A]}(B) = V \setminus A$. Write $B_N = B \cap N$ and $B_T = B \cap T$. Fix an element $t \in T$. Let

$$K = B_N^* {}^\frown xt, \qquad K' = B_N^* {}^\frown B_T, \qquad L = K \cup K' \quad \text{and} \quad M = B_N^* {}^\frown yt.$$

Clearly $L$ and $M$ are independent subsets in $V^\infty$.

We want to find a partition $(P, S)$ of $V^\infty$ such that $L$ is a quasi-kernel in $G^\infty[P]$, and $M$ is quasi-sink in $G^\infty[S]$.

Fix a partition $(X, Y)$ of $A$ such that

$$x \in X \subset \mathrm{In}_1^G(x) \cup \left(\mathrm{In}_1^G(y) \setminus \{y\}\right) \quad \text{and} \quad y \in Y \subset \left(\mathrm{Out}_1^G(x) \setminus \{x\}\right) \cup \mathrm{Out}_1^G(y). \tag{7}$$

Let $W = (B_N^* {}^\frown A {}^\frown V^*) \cap V^\infty$ and define the partition $(R, S)$ of $W$ as follows:

$$R = (B_n^* {}^\frown xt) \cup \left((B_n^* {}^\frown Y {}^\frown V^*) \cap V^\infty\right) \setminus (B_N^* {}^\frown yt) \tag{8}$$

and

$$S = (B_n^* {}^\frown yt) \cup \left((B_n^* {}^\frown X {}^\frown V^*) \cap V^\infty\right) \setminus (B_N^* {}^\frown xt). \tag{9}$$

**Claim 2.** *$K$ is a quasi-kernel in $G[R]$, and $M$ is a quasi-sink in $G[S]$.*

**Proof.** Let $b \in B_N^*$. Then

$$\mathrm{Out}_1^{G[R]}(b {}^\frown xt) \supset \left(b {}^\frown Y \cap \mathrm{Out}_1^G(x)\right) {}^\frown V^*. \tag{10}$$

Since $b {}^\frown yxt \in R$, $(b {}^\frown xt, b {}^\frown yxt) \in E^\infty$ and

$$\mathrm{Out}_1^{G[R]}(b {}^\frown yxt) \supset \left(b {}^\frown Y \cap \mathrm{Out}_1^G(y) \setminus \{y\}\right) {}^\frown V^* \tag{11}$$

we have

$$\mathrm{Out}_2^{G[R]}(b {}^\frown xt) \supset (b {}^\frown Y \cap \mathrm{Out}_1^G(y) \setminus \{y\}) {}^\frown V^* \tag{12}$$

(10) and (12) together give $R = \mathrm{Out}_2^{G[R]}(K)$. $S = \mathrm{In}_2^{G[S]}(M)$ can be proved similarly. □

Now let $Z = (V^\infty \setminus W) \cup K$.

**Claim 3.** $Z = \mathrm{Out}_2^{G^\infty[Z]}(L)$.

**Proof.** Let $s \in V^\infty \setminus W$. Write $s = s' {}^\frown p {}^\frown s''$, where $s' \in B_N^*$, $p \in V \setminus B_N$ and $s'' \in V^*$. Since $s \notin W$ we have $p \notin A$. If $p \in B_T$ then $s = s' {}^\frown p \in K$.

Hence we can assume that $p \in V \setminus (A \cup B)$. Thus, there is directed path $\langle x_0, \ldots, x_n\rangle$ in $G[V \setminus A]$ such that $1 \le n \le 2$, $x_0 \in B$ and $x_n = p$. Let $\overline{x_0} = x_0$ if $x_0 \in T$, and let $\overline{x_0} = x_0 {}^\frown xt$ if $x_0 \in N$. Then $s' {}^\frown \overline{x_0} \in L$ and $\left\langle s' {}^\frown \bar{x}_0, s\right\rangle \in E^\infty$ if $n = 1$, or $\left\langle s' {}^\frown \bar{x}_0, s' {}^\frown x_1, s\right\rangle$ is a directed path in $G^\infty$ if $n = 2$. Thus, $s \in \mathrm{Out}_2^{G^\infty[Z]}(L)$. □

Let $P = R \cup Z$. Then $(P, S)$ is a partition $V^\infty$ and it witnesses that the digraph $G^\infty$ is in $\mathfrak{In}_2$-$\mathfrak{Out}_2$: $L$ is a quasi-kernel in $G^\infty[R \cup Z]$, and $M$ is a quasi-sink in $G^\infty[S]$ by Claims 2 and 3. This concludes Case I.

*Case* II: *$N$ is independent.*

We show that $G^\infty \in \mathfrak{In}_2 \cup \mathfrak{Out}_2$.

**Lemma 5.8.** *If there is an independent set $A \subset V$ with $T \cap A \ne \emptyset$ such that $\mathrm{Out}_2^G(A) = V$ (or $\mathrm{In}_2^G(A) = V$) then $G^\infty \in \mathfrak{Out}_2$ (or $G^\infty \in \mathfrak{In}_2$, respectively).*

**Proof of Lemma 5.8.** We show that $K = (A \cap N)^* {}^\frown (A \cap T)$ is a quasi-kernel in $G^\infty$. The set $K$ is clearly independent in $G^\infty$ because $A$ was independent.

Fix an element $t \in T \cap A$. For $x \in T$ let $\bar{x} = x$ and for $x \in N$ let $\bar{x} = x {}^\frown t$.

Let $s \in V^\infty$. If $s \in A^*$ then $s \in K$, so we can assume that $s = s' {}^\frown p {}^\frown s''$, where $p \in V \setminus A$. Then there is an $a \in A$ such that either $(a, p) \in E$ or there is an $x \in V$ such that $(a, x) \in E$ and $(x, p) \in E$. Then $s' {}^\frown \bar{a} \in K$ and in the first case $(s' {}^\frown \bar{a}, s)$ is an edge in $G^\infty$, and in the second case $(s' {}^\frown \bar{a}, s' {}^\frown \bar{d}, s)$ is a directed path of length 2 in $G^\infty$. Therefore, $s \in \mathrm{Out}_2^{G^\infty}(s' {}^\frown \bar{a}) \subset \mathrm{Out}_2^{G^\infty}(K)$, as we claimed. □

**Lemma 5.9.** *If $T \not\subset \mathrm{Out}_1^G(N)$, then $G^\infty \in \mathfrak{Out}_2$, and if $T \not\subset \mathrm{In}_1^G(N)$, then $G^\infty \in \mathfrak{In}_2$.*

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**Proof of Lemma 5.9.** Let $t \in T \setminus \mathrm{Out}_1^G(N)$, $B = \mathrm{Out}_1^G(t)$, and $A' \subset V \setminus B$ be independent such that $V \setminus B = \mathrm{Out}_2^{G[V\setminus B]}(A')$. If $A = A' \cup \{t\}$ is independent, then by Lemma 5.8 we are done.

Otherwise there is an $a \in A'$ with $(a, t) \in E$ because $\mathrm{Out}_1^G(t) \cap A = \emptyset$. Then $t \in \mathrm{Out}_1^G(a)$ and so $a \notin N$, i.e. $a \in T$. Hence $A'$ satisfies the assumptions of Lemma 5.8, and so $G^\infty \in \mathfrak{Out}_2$. □

Thus, we may assume that

$$T \subset \mathrm{Out}_1^G(N) \cap \mathrm{In}_1^G(N). \tag{13}$$

With this assumption,

$$\text{if } N \subset \mathrm{Out}_1^G(T), \text{ then } G^\infty \in \mathfrak{Out}_2. \tag{14}$$

Indeed, we show that $K = \{y ^\frown t : y \in N\}$ is a quasi-kernel in $G^\infty$, where $t$ is an arbitrary element of $T$. $K$ is independent, as $N$ is so.

Let $s \in V^\infty$. If $s \in T$, then by (13) we have $(y, s) \in E$ for some $y \in N$ and so $s \in \mathrm{Out}_1^{G^\infty}(y ^\frown t) \subset \mathrm{Out}_1^{G^\infty}(K)$.

If $s = x ^\frown s'$ for some $x \in N$ then there is a $u \in T$ with $(u, x) \in E$ by the assumption $N \subset \mathrm{Out}_1^G(T)$. Then, by (13), there is a $y \in N$ with $(y, u) \in E$. Thus, $\langle y, u, x\rangle$ is a directed path of length 2 in $G$ and so $\langle y ^\frown t, u, x ^\frown s'\rangle$ is a directed path of length 2 in $G^\infty$. Hence $s \in \mathrm{Out}_2^{G^\infty}(K)$, as we claimed in (14).

Thus, we may assume that

$$N \not\subset \mathrm{Out}_1^G(T) \quad \text{and} \quad N \not\subset \mathrm{In}_1^G(T). \tag{15}$$

Let $A = N \setminus \mathrm{Out}_1^G(T)$ and $B = N \setminus \mathrm{In}_1^G(T)$. Hence $A \neq \emptyset$ and $B \neq \emptyset$ by (15).

Since $T \neq \emptyset$ and $T \subset \mathrm{In}_1^G(N)$ we have $N \cap \mathrm{Out}_1^G(T) \neq \emptyset$ and so $A \neq N$. Similarly, $B \neq N$.

Let $t \in T$ be fixed, and put $K = A^* {}^\frown (N \setminus A) {}^\frown t$. We claim that $K$ is a semi-kernel in $G^\infty$.

If $p \neq q \in K$ then we have $\{p(\Delta(p, q)), q(\Delta(p, q))\} \in [N]^2$ and so there is no edge between $p$ and $q$ in $G^\infty$. Hence $K$ is independent.

Let $L = A^* {}^\frown T$. Now we have

$$\mathrm{Out}_1^{G^\infty}(K) \supset L. \tag{16}$$

Indeed, if $x \in A^*$ and $s \in T$ then there is $c \in N$ with $(c, s) \in E$ because of (13). If $c \in N \setminus A$ then $x ^\frown c ^\frown t \in K$ and $(x ^\frown c ^\frown t, x ^\frown s) \in E^\infty$. If $c \in A$ then $x ^\frown c ^\frown b ^\frown t \in K$ and $(x ^\frown c ^\frown b ^\frown t, x ^\frown s) \in E^\infty$ for any $b \in N \setminus A \neq \emptyset$.

Moreover, we claim that

$$\mathrm{Out}_1^{G^\infty}(L) = V^\infty. \tag{17}$$

Indeed, let $x \in V^\infty \setminus L$. Let $n$ be maximal such that $x \restriction n \in A^n$. Since $x \notin L$ we have $c = x(n) \in N \setminus A$. Hence $c \in \mathrm{Out}_1^G(T)$, so we can pick $s \in T$ with $(s, c) \in E$. Then $(x \restriction n) ^\frown s \in L$ and $(x \restriction n ^\frown s, x) \in E^\infty$.

Hence $\mathrm{Out}_2^{G^\infty}(K) \supseteq \mathrm{Out}_1^{G^\infty}(L) = V^\infty$, i.e. $K$ is a quasi-kernel $G^\infty$, as we claimed. This concludes Case II, so Theorem 5.7 is proved. □

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