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Sperner Families Satisfying Additional Conditions and their Convex Hulls

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Abstract. The profile of a hypergraph on n vertices is $(f_0, \dots, f_n)$ where f_i denotes the number of i-element edges. The extreme points of the set of the profiles are determined for Sperner hypergraphs satisfying some additional conditions. The results contain some old theorems of extremal set theory as particular cases.

1. Introduction

Let X be a finite set of n elements and $\mathscr{F}$ be a family of its subsets ($\mathscr{F} \subset 2^X$). Then $\mathscr{F}_k$ denotes the subfamily of the k-element subsets in $\mathscr{F}$: $\mathscr{F}_k = \{A \in \mathscr{F}, |A| = k\}$. Its size $|\mathscr{F}_k|$ is denoted by f_k. The vector $(f_0, \dots, f_n)$ in the $(n+1)$-dimensional Euclidean space R^{n+1} is called the *profile* $p(\mathscr{F})$ of $\mathscr{F}$. Let $\mathbb{A}$ be a set of families of subsets of X, i.e. $\mathbb{A} \subset 2^{2^X}$. Denote by

$$\mu(\mathbb{A}) = \{p(\mathscr{F}) = (f_0, \dots, f_n) \colon \mathscr{F} \in \mathbb{A}\}$$

the set of the profiles of the families belonging to $\mathbb{A}$. Then $\mu(\mathbb{A})$ is a finite set of points in R^{n+1}. Let $\langle \mu(\mathbb{A}) \rangle$ be the convex hull of $\mu(\mathbb{A})$ and $\varepsilon(\mathbb{A})$ the set of all *extreme points* (i.e. vertices of $\langle \mu(\mathbb{A}) \rangle$). It is easy to see that $\varepsilon(\mathbb{A}) \subseteq \mu(\mathbb{A})$. So we may call the elements of $\varepsilon(\mathbb{A})$ extreme points of $\mu(\mathbb{A})$. It is well-known that any linear function over $\langle \mu(\mathbb{A}) \rangle$ attains its maximum/minimum at an extreme point. This means, if we want to maximize a linear expression of $(f_0, \dots, f_n)$ on the families belonging to $\mathbb{A}$ then it is enough to do it on the families with profiles equal to the extreme points of $\mu(\mathbb{A})$. In most cases this is simpler than to maximize on the whole set $\mathbb{A}$.

We say that $\mathscr{F}$ is a *Sperner family* if $F, G \in \mathscr{F}$, $F \neq G$ imply $F \not\subset G$. The set of Sperner families is denoted by $\mathbb{S}$. The extreme points of $\mu(\mathbb{S})$ are determined in [5], but the problem is equivalent to the well-known LYM-inequality.

Theorem.

$$\varepsilon(\mathbb{S}) = \left\{(0, \dots, 0), \Big(0, \dots, 0, \underset{i}{\binom{n}{i}}, 0, \dots, 0\Big), i = 0, 1, \dots, n\right\}.$$

We say that the family $\mathscr{F}$ is *intersecting*, *cointersecting*, *complement-free* and *complementary*, iff

$$F, G \in \mathscr{F} \quad \text{implies} \quad F \cap G \neq \emptyset,$$

$$F, G \in \mathscr{F} \quad \text{implies} \quad F \cup G \neq X,$$

$$F \in \mathscr{F} \quad \text{implies} \quad X - F \notin \mathscr{F}$$

and

$$F \in \mathscr{F} \quad \text{implies} \quad X - F \in \mathscr{F},$$

respectively. The aim of the present paper is to determine the extreme points of three subfamilies of Sperner families:
1) intersecting, cointersecting Sperner families,
2) complementary Sperner families and
3) complement-free Sperner families.

2. Tools (Preliminaries)

2.1. The method of [5] is applied in the present paper (see also [6]). The necessary definitions and results are repeated for sake of completeness.

The set $\mathbb{A} \subset 2^{2^X}$ is *hereditary* if $\mathscr{G} \subset \mathscr{F} \in \mathbb{A}$ implies $\mathscr{G} \in \mathbb{A}$. Denote by $\mu^*(\mathbb{A})$ the set of *maximal profiles* of $\mathbb{A}$: it contains those elements $(f_0, \ldots, f_n)$ of $\mu(\mathbb{A})$ for which $(g_0, \ldots, g_n) \in \mu(\mathbb{A})$ and $(g_0, \ldots, g_n) \geq (f_0, \ldots, f_n)$ (that is, $g_0 \geq f_0, \ldots, g_n \geq f_n$) imply $(f_0, \ldots, f_n) = (g_0, \ldots, g_n)$. Furthermore let $\varepsilon^*(\mathbb{A}) = \mu^*(\mathbb{A}) \cap \varepsilon(\mathbb{A})$ be the set of the *essential extreme points* of $\mu(\mathbb{A})$.

Theorem A. *Suppose that* $\mathbb{A}$ *is hereditary. Then any element of* $\varepsilon(\mathbb{A})$ *can be obtained by changing some coordinates of an essential extreme point of* $\mu(\mathbb{A})$ *to zero.*

The theorem states that, for a given $\mathbb{A}$, it is sufficient to determine the set $\varepsilon^*(\mathbb{A})$. Replacing the components by zero in every possible way we obtain a set of vectors, these should be individually checked if they are extreme points.

Theorem B. *Suppose that* $\mathbb{A}$ *is hereditary, a set* $\varepsilon = \{e_1, \ldots, e_m\} \subset \mu(\mathbb{A})$ *is given and*

$$\sum_{i=0}^{n} f_i y_i \leq 1$$

holds for any $f \in \mu(\mathbb{A})$ *and any* $y_0, \ldots, y_n$ *satisfying* $y_i \geq 0$ $(0 \leq i \leq n)$ *and*

$$\begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} \leq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then $\varepsilon^*(\mathbb{A}) \subseteq \varepsilon$.

2.2. Reduction to the circle. Take a cyclic permutation C of the underlying set X and consider only those subsets of X whose elements are consecutive in C. These sets are called *consecutive sets* in C. If $\mathscr{F}$ is a family of subsets of X then $\mathscr{F}(C)$ is defined by

$$\mathscr{F}(C) = \{F \in \mathscr{F}: F \text{ is consecutive in } C\}.$$

Similarly, let

$$\mathbb{A}(C) = \{\mathscr{F}(C): \mathscr{F} \in \mathbb{A}\}.$$

Assign the vector

$$T(e) = \left(e_0, e_1 \frac{1}{n}\binom{n}{1}, \ldots, e_{n-1}\frac{1}{n}\binom{n}{n-1}, e_n\right)$$

with $e = (e_0, \ldots, e_n)$.

Theorem C. (Blowing up the circle.) *If* $e_1, \ldots, e_m$ *are the extreme points of* $\mu(\mathbb{A}(C))$ *for any given cyclic permutation* C *then*

$$\mu(\mathbb{A}) \subseteq \langle T(e_1), \ldots, T(e_m)\rangle.$$

If $T(e_1), \ldots, T(e_m) \in \mu(\mathbb{A})$, *then these are the extreme points of* $\mathbb{A}$.

(This is true for the problems discussed in the present paper.)

Theorem D. *Theorem B remains true if* $\mathbb{A}$ *is replaced by* $\mathbb{A}(C)$ *where* C *is any cyclic permutation of* X.

3. Intersecting, Cointersecting Sperner Families

Theorem 1. *The set of essential extreme points of the set* $\mathbb{B}$ *of all intersecting, cointersecting Sperner families is*

$$\varepsilon^*(\mathbb{B}) = \{u_i, i = 1, \ldots, [n/2]; v_j, n/2 < j < n\}$$

where

$$u_i = \Big(0, \ldots, 0, \underset{i}{\binom{n-1}{i-1}}, 0, \ldots, 0\Big),$$

$$v_j = \Big(0, \ldots, 0, \underset{j}{\binom{n-1}{j}}, 0, \ldots, 0\Big).$$

Remark. Both Theorems 1 and 2 can be proved very easily, without the general theory. We chose the present way to unify the paper.

Proof. The families

$$\mathscr{U}_i = \{A \subset X: |A| = i, x \in X,\}, \qquad i = 1, \ldots, [n/2],$$

$$\mathscr{V}_j = \{A \subset X: |A| = j, x \notin X,\}, \qquad n/2 < j < n$$

(where x is a fixed point of X) show that $u_i, u_j \in \mu(\mathbb{B})$. By Theorem C it is enough to verify

Lemma 1. *For any cyclic permutation C*

$$\varepsilon^*(\mathbb{B}(C)) = \{u_i', i = 1, \ldots, [n/2]; v_j', n/2 < j < n\}$$

where

$$u_i' = (0, \ldots, 0, \underset{i}{i}, 0, \ldots, 0)$$

$$v_j' = (0, \ldots, 0, \underset{j}{n-j}, 0, \ldots, 0).$$

Proof. By Theorem D it is sufficient to show that

$$\sum_{i=1}^{[n/2]} \frac{g_i}{i} + \sum_{n/2<j}^{n-1} \frac{g_j}{n-j} \leq 1 \tag{1}$$

holds for every intersecting, cointersecting Sperner family $\mathcal{G}$ of consecutive sets in a cyclic permutation C.

Let the cyclic permutation C be given by the ordering $X = \{x_1, \ldots, x_n\}$. By the Sperner property, at most one member of $\mathcal{G}$ can start or end, respectively, at any given element x_i. If they exist, they are denoted by S_i or E_i, respectively. As $\mathcal{G}$ is intersecting and cointersecting, we have

$$|\{E_i, S_{i+1}\} \cap \mathcal{G}| \leq 1 \tag{2}$$

for every $i = 1, \ldots, n$. Let $r = \min\{|A|: A \in \mathcal{G}\}$, $n - l = \max\{|A|: A \in \mathcal{G}\}$.

Case 1. $r \leq l$. Suppose $|A_0| = r$ and $A_0 = (x_1, \ldots, x_r)$. As $\mathcal{G}$ is intersecting, either the starting or the endpoint of every other member of $\mathcal{G}$ is in A_0, therefore

$$\mathcal{G} \subseteq \{A_0, E_1, S_2, \ldots, E_{r-1}, S_r\}.$$

(S_1 and E_r are excluded by the Sperner property.) (2) implies $|\mathcal{G}| \leq r$. Furthermore $r \leq l \leq n - |A|$ ($A \in \mathcal{G}$). Now the left hand side of (1) can be easily upperbounded:

$$\sum_{A \in \mathcal{G}, |A| \leq n/2} \frac{1}{|A|} + \sum_{A \in \mathcal{G}, |A| > n/2} \frac{1}{n - |A|} \leq \frac{\mathcal{G}}{r} \leq 1.$$

Case 2. $r > l$. Consider the family $\overline{\mathcal{G}} = \{\bar{A}: A \in \mathcal{G}\}$. The left hand side of (1) is the same for $\mathcal{G}$ or $\overline{\mathcal{G}}$. $\overline{\mathcal{G}}$ satisfies the conditions of Case 1. This proves (1), the lemma and the theorem. □

Remark. The unique essential bordering hyperplane of $\langle \mu(\mathbb{B}) \rangle$ is

$$\sum_{i=1}^{[n/2]} \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2<j}^{n-1} \frac{f_j}{\binom{n-1}{j}} \leq 1.$$

(See Corollary 1, later.) This hyperplane contains some points of $\mu(\mathbb{B})$ which are not extreme points:

$$\left(0,\ldots,0,\underset{i}{\binom{n-2}{i-1}},\ldots,0,\underset{n-i}{\binom{n-2}{n-i}},0,\ldots,0\right)$$

where $1 \le i \le n/2$. (If x and y are fixed distinct elements of X then the family

$$\{A \subset X: |A| = i, x \in A, y \notin A\} \cup \{B \subset X: |B| = n-i, x, y \notin B\}$$

has this profile.

4. Complementary Sperner Families

Denote by $\mathbb{D} \subset 2^{2^X}$ the set of all complementary Sperner families.

Theorem 2.

$$\varepsilon^*(\mathbb{D}) = \left\{z_i: i = 1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\},$$

where

$$z_i = \left(0,\ldots,0,\underset{i}{\binom{n-1}{i-1}},\ldots,0,\underset{n-i}{\binom{n-1}{n-i}},0,\ldots,0\right) \quad \left(1 \le i < \frac{n}{2}\right),$$

and

$$z_{n/2} = \left(0,\ldots,0,\underset{\frac{n}{2}}{\binom{n}{\frac{n}{2}}},0,\ldots,0\right).$$

Proof. Let $\mathscr{G}$ be a complementary Sperner family. Its profile has the form

$$P(\mathscr{G}) = (f_0,\ldots,f_i,\ldots,f_{n-i},\ldots,f_n)$$

where $f_i = f_{n-i}$ $(0 \le i \le [n/2])$. The family

$$\mathscr{F} = \left\{F: F \in \mathscr{G}, \left(|F| < \frac{n}{2}\right) \text{ or } \left(|F| = \frac{n}{2}, x \in F\right)\right\}$$

is obviously an intersecting, cointersecting Sperner family with profile

$$(f_0,\ldots,f_{(n-1)/2},0,\ldots,0) \quad \text{or} \quad (f_0,\ldots,f_{(n/2)-1},\tfrac{1}{2}f_{n/2},0,\ldots,0).$$

By Theorem 1, this is a convex linear combination of the zero vector and u_i $(1 \le i \le n/2)$. Then $(f_0,\ldots,f_i,\ldots,f_{n-i},\ldots,f_n)$ is a linear combination of the zero vector and z_i $(1 \le i \le n/2)$. □

Corollary 1. (Bollobás, [1]) *If $\mathscr{F}$ is a complementary Sperner family then its profile satisfies the following inequality*:

$$\sum_{i=1}^{[n/2]} \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2<j}^{n-1} \frac{f_j}{\binom{n-1}{j}} \leq 2.$$

Proof. The left hand side of the inequality is a linear expression of the profile $p(\mathscr{F})$. So it attains its maximum for one of the extreme points. It is easy to check that the value of the expression is ≤ 2 for all the extreme points. □

We saw in the proof of Theorem 2 that the complementary Sperner families are in fact variants of the intersecting, cointersecting ones. There is, however, one more variant. The subsets, $F, G \subset X$ are called (following Marczewski, [10]) *qualitatively independent* if they divide X into 4 non-empty parts ($F - G, G - F, F \cap G, X - F - G$ are all non-empty). It is obvious that $\mathscr{F}$ is an intersecting, cointersecting, Sperner family iff any two members of $\mathscr{F}$ are qualitatively independent. This is the reason of the fact that the maximum size of such families, under different names, has been independently determined by several authors.

Corollary 2. ([1], [2], [8], [9]) *If $\mathscr{F}$ is an intersecting, cointersecting, Sperner family or a family of qualitatively independent sets then*

$$|\mathscr{F}| \leq \binom{n-1}{\left\lfloor \frac{n}{2} \right\rfloor - 1}.$$

If $\mathscr{F}$ is a complementary Sperner family then

$$|\mathscr{F}| \leq 2\binom{n-1}{\left\lfloor \frac{n}{2} \right\rfloor - 1}.$$

Proof. Check the inequalities for the extreme points in Theorems 1 and 2, resp. □

5. Complement-Free Sperner Families

Let $\mathbb{E}$ denote the set of all complement-free Sperner families on n elements.

Theorem 3. *If n is odd then*

$$\varepsilon(\mathbb{E}) = \varepsilon(\mathbb{S}),$$

if $n = 2k$ is even then

$$\varepsilon^*(\mathbb{E}) = \{u_i(i = 0, \ldots, k-1, k+1, \ldots, n), v_i(i = 1, \ldots, k-1), w_i(i = k+1, \ldots, n-1)\}$$

where

$$u_i = \Bigl(0,\ldots,0,\underset{i}{\binom{n}{i}},0,\ldots,0\Bigr),$$

$$v_i = \Bigl(0,\ldots,0,\underset{i}{\binom{n-1}{i-1}},0,\ldots,0,\underset{k}{\binom{n-1}{k}},0,\ldots,0\Bigr),$$

$$w_i = \Bigl(0,\ldots,0,\underset{k}{\binom{n-1}{k}},0,\ldots,0,\underset{i}{\binom{n-1}{i}},0,\ldots,0\Bigr).$$

Proof. $\mathbb{E} \subseteq \mathbb{S}$ is obvious. If n is odd then the elements of $\varepsilon(\mathbb{S})$ are in $\mu(\mathbb{E})$. Hence we obtain $\varepsilon(\mathbb{E}) = \varepsilon(\mathbb{S})$.

The case $n = 2k$ is not so trivial. The following constructions show that u_i, v_i, $w_i \in \mu(\mathbb{E})$:

$$\mathcal{U}_i = \{A: A \subseteq X, |A| = i\} \qquad (i = 0,\ldots,k-1,k+1,\ldots,n),$$

$$\mathcal{V}_i = \{A: A \subseteq X, |A| = i, x \in A\} \cup \{A: A \subseteq X, |A| = k, x \notin A\} \quad (i = 1,\ldots,k-1),$$

$$\mathcal{W}_i = \{A: A \subseteq X, |A| = k, x \in A\} \cup \{A: A \subseteq X, |A| = i, x \notin A\} \; (i = k+1,\ldots,n-1).$$

By Theorem C it is enough to prove the following lemma.

Lemma 2. *Let C be a fixed cyclic permutation of X. Then $\varepsilon^*(\mathbb{E}(C)) = \{u_i'(i = 0,\ldots, k-1,k+1,\ldots,n), v_i'(i = 1,\ldots,k-1), w_i'(i = k+1,\ldots,n-1)\}$ where*

$$u_0' = (1,\ldots,0), \qquad u_n' = (0,\ldots,1),$$

$$u_i' = (0,\ldots,0,\underset{i}{n},0,\ldots,0) \qquad (i \neq 0,k,n),$$

$$v_i' = (0,\ldots,0,\underset{i}{i},0,\ldots,0,\underset{k}{k},0,\ldots,0),$$

$$w_i' = (0,\ldots,0,\underset{k}{k},0,\ldots,0,\underset{i}{i},0,\ldots,0).$$

The families realizing these profiles are analogous to the families $\mathcal{U}_i$, $\mathcal{V}_i$ and $\mathcal{W}_i$. By Theorem D, the following lemma implies the previous one.

Lemma 3. *$\sum_{i=0}^{n} f_i y_i \leq 1$ holds for any $f \in \mu(\mathbb{E}(C))$ and any $y_0, \ldots, y_n \geq 0$ satisfying*

$$e\begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} \leq 1. \tag{3}$$

with all vectors e listed in Lemma 2.

Proof. Write (3) component-wise:

$$
\begin{aligned}
&y_0 \le 1,\\
&y_n \le 1,\\
&ny_i \le 1 \qquad (i = 1,\dots,k-1,k+1,\dots,n-1),\\
&iy_i + ky_k \le 1 \qquad (i = 1,\dots,k-1),\\
&ky_k + (n-i)y_i \le 1 \qquad (i = k+1,\dots,n-1).
\end{aligned} \tag{4}
$$

(4) results in

$$
\begin{aligned}
&y_k \le \frac{1}{k},\\
&y_i \le \min\left(\frac{1}{n},\frac{1-ky_k}{i}\right) \qquad (i = 1,\dots,k-1),\\
&y_i \le \min\left(\frac{1}{n},\frac{1-ky_k}{n-i}\right) \qquad (i = k+1,\dots,n-1).
\end{aligned} \tag{5}
$$

If f_0 is non-zero then it is 1. By the Sperner property all other f_s are zero, the lemma trivially holds. Thus in the rest of the proof we suppose that $f_0 = f_n = 0$.

Note that the inequality in question is only slightly modified if f is supposed to be a Sperner family along C:

$$\sum_{i=0}^{n} f_i y_i \le \sum_{i=0}^{n} \frac{f_i}{n} \le 1.$$

This inequality is known (see [5]), it is in strong connection with the theorem in the introduction. (Actually, it is a version of the LYM-inequality for cyclic permutations.) Our inequality is almost a consequence of this one, the only difference being that we have the condition $y_k \le \frac{1}{k}$ instead of $y_k \le \frac{1}{n}$. So, we are done if either $f_k = 0$ or $y_k \le \frac{1}{n}$ holds. In what follows, we will suppose that

$$0 < f_k \quad \text{and} \quad \frac{1}{n} < y_k \le \frac{1}{k}. \tag{6}$$

(5) and (6) imply

$$y_i < y_k \qquad (i \ne k). \tag{7}$$

Fix a system of y_s satisfying (6) and choose a family $\mathscr{F} \in \mathbb{E}(C)$ maximizing

$$\sum_{i=1}^{n-1} f_i y_i.$$

As $f_k > 0$, there is an $A \in \mathscr{F}$ with k elements, say $A = \{x_1,\dots,x_k\}$.

Claim 1. *Either* $\{x_2, \ldots, x_{k+1}\}$ *or* $\{x_{k+2}, \ldots, x_n, x_1\}$ *is in* $\mathscr{F}$.

Suppose that $\{x_{k+2}, \ldots, x_n, x_1\} \notin \mathscr{F}$. We try to add $\{x_2, \ldots, x_{k+1}\}$ to $\mathscr{F}$. The only obstacle could be a member B of $\mathscr{F}$ which is either contained in $\{x_2, \ldots, x_{k+1}\}$ or contains $\{x_2, \ldots, x_{k+1}\}$. In the first case, B has to contain x_{k+1}, otherwise it would be a subset of A. Thus $B = \{x_j, \ldots, x_{k+1}\}$ for some $2 \le j \le k+1$. Similarly, if B contains $\{x_2, \ldots, x_{k+1}\}$ then $B = \{x_2, \ldots, x_l\}$ holds for some $k+1 \le l \le n$. It is clear, by the Sperner property, that at most one of these possible sets B can be in $\mathscr{F}$. Delete this B and add $\{x_2, \ldots, x_{k+1}\}$ to $\mathscr{F}$. By (7), this change increases $\sum_{i=1}^{n-1} f_i y_i$. This contradiction proves the statement.

A pair of complementing, consecutive k-element subsets is called an *equipartition.* We say that an equipartition is *represented* in $\mathscr{F}$ iff one of the parts is a member of $\mathscr{F}$. Claim 1 states that if an equipartition is represented in $\mathscr{F}$ then the neighbouring equipartition is also represented. By induction, this results in

Claim 2. *All equipartitions are represented in* $\mathscr{F}$. *That is,* $f_k = k$.

This leads to a new property of $\mathscr{F}$.

Claim 3. *Let F and G be members of* $\mathscr{F}$ *with sizes different from k. Then* $F \cap G \neq \emptyset$.

Let $|F| \le |G|$ and $G = \{x_1, \ldots, x_j\}$. First suppose $j < k$. By Claim 2 and the Sperner property, we have $\{x_{j+1}, \ldots, x_{j+k}\}$, $\{x_{k+1}, \ldots, x_n\} \in \mathscr{F}$. If F is disjoint to G then it must be contained in the union of $\{x_{j+1}, \ldots, x_{j+k}\}$ and $\{x_{k+1}, \ldots, x_n\}$, but neither one can contain it alone. Hence x_k, $x_{j+k+1} \in F$ follows. The size of F is $>j$, a contradiction. The case $j > k$ is easier. The set $\{x_{k+1}, \ldots, x_n\}$ covers the complement of G, therefore F can not entirely be there, by the Sperner property, again.

Claim 4. *Let F and G be members of* $\mathscr{F}$ *with sizes different from k. Then* $F \cup G \neq X$.

$\bar{\mathscr{F}}$ satisfies all the conditions of Claim 3. However, Claim 3 with $\bar{\mathscr{F}}$ is equivalent to Claim 4 with $\mathscr{F}$.

The last two claims result in

Claim 5. $\mathscr{F} - \mathscr{F}_k$ *is an intersecting, cointersecting Sperner family.*

Now we are able to prove the desired inequality. (5), Claim 2, Claim 5 and (1) will be used.

$$\sum_{i=1}^{n-1} f_i y_i \le k y_k + \sum_{i=1}^{k-1} f_i \frac{1 - k y_k}{i} + \sum_{i=k+1}^{n-1} f_i \frac{1 - k y_k}{n - i}$$

$$= k y_k + (1 - k y_k)\left(\sum_{i=1}^{h-1} \frac{f_i}{i} + \sum_{i=k+1}^{n-1} \frac{f_i}{n - i}\right) \le k y_k + (1 - k y_k) = 1.$$

This proves Lemma 3, Lemma 2 and Theorem 3. □

Corollary 3. [11] *If $\mathscr{F}$ is a complement-free family on an n-element set then*

$$|\mathscr{F}| \leq \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor + 1}.$$

Proof. $|\mathscr{F}| = \sum_{i=0}^{n} f_i$ is a linear expression of the profile, thus it is sufficient to check this sum for the extreme points in Theorem 3, what is easy. □

Remark. In [7] it is proved that Corollaries 2 and 3 are, in fact, equivalent. Corollary 3 is proved for integer sequences in [3].

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