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EXTREMAL HYPERGRAPH PROBLEMS AND CONVEX HULLS

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The *profile* of a hypergraph on n vertices is $(f_0, f_1, \ldots, f_n)$ where f_i denotes the number of i-element edges. The extreme points of the set of profiles is determined for certain hypergraph classes. The results contain many old theorems of extremal set theory as particular cases (Sperner, Erdős—Ko—Rado, Daykin—Frankl—Green—Hilton).

1. Introduction

Let X be a finite set of n elements and $\mathcal{F}$ be a family of its subsets ($\mathcal{F}\subset 2^X$). Then $\mathcal{F}_k$ denotes the subfamily of the k-element subsets in $\mathcal{F}$: $\mathcal{F}_k=\{A\colon A\in\mathcal{F}, |A|=k\}$. Its size $|\mathcal{F}_k|$ is denoted by f_k. The vector $(f_0, f_1, \ldots, f_n)$ in the $(n+1)$-dimensional Enclidean space $\mathbf{R}^{n+1}$ is called the *profile* of $\mathcal{F}$.

If α is a finite set in $\mathbf{R}^{n+1}$, the *convex hull* $\langle\alpha\rangle$ of α is the set of all convex linear combinations of the elements of α. We say that $e\in\alpha$ is an *extreme point* of α iff e is not a convex linear combination of elements of α different from e. It is well-known that $\langle\alpha\rangle$ is equal to the set of all convex linear combinations of its extreme points. That is, the determination of the convex hull of a set is equivalent to finding its extreme points.

$\mathcal{F}$ is a *Sperner-family* iff it contains no members A, B with $A\subset B$. In the previous paper we determined the extreme points of the set of profiles of all Sperner-families. This was an easy consequence of a well-known inequality. A family is *intersecting* if $A, B\in\mathcal{F}$ implies $A\cap B\neq\emptyset$. The main result of [5] determines the extreme points of the set of profiles of the intersecting Sperner-families.

On the other hand, the present paper starts a systematic treatment of the area. It tries to determine the extreme points of the set of profiles of the simplest known classes of families, using the methods of the previous paper. The effort is successful for 3 classes:

1. intersecting families,
2. *k-Sperner-families* (there are no $k+1$ different members satisfying $F_1\subset\ldots\subset F_{k+1}$),

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3. $\mathcal{F}_1, \ldots, \mathcal{F}_t$ are not necessarily disjoint families, where $G \in \mathcal{F}_i$, $H \in \mathcal{F}_j$, $i \neq j$, $G \neq H$ imply $G \not\subset H$.

Moreover, the method of the previous paper is analyzed here. One of the ideas of the proofs is the following. A cyclic ordering $\mathcal{C}$ is taken of the underlying set X and consider only the sets containg consecutive elements in $\mathcal{C}$. Any problem of the above type can be realized for these consecutive sets, as well. Their solution is easier but in some cases (in all the cases solved in these 2 papers) is sufficient. Theorem 4 describes the connection between the sets of extreme points of the original problem and of the "consecutive" variant. An example will be given ($F_1, F_2 \in \mathcal{F}$ implies $|F_1 \cap F_2| \geqq l$) when the original problem is hopeless while the "consecutive" variant can be solved. Theorem 4 is, of course, too weak in this case.

We also list some known extremal theorems which are consequences of our results.

For instance in Case 3 our method gives a unified proof of 3 different statements of [1].

2. General results (=tools)

2.1. *Essential extreme points.* Let $\mathbf{A}$ be a class of families of subsets of the n-element set X, that is, $\mathbf{A} \subset 2^{2^X}$. $\mu(\mathbf{A})$ denotes the set of profiles of the families belonging to $\mathbf{A}$:

$$\mu(\mathbf{A}) = \{(f_0, \ldots, f_n) \colon f_i = |\mathcal{F}_i|, \mathcal{F} \in \mathbf{A}\}. \tag{1}$$

The set of extreme points of $\mu(\mathbf{A})$ is denoted by $\varepsilon(\mathbf{A})$.

The $\mathbf{A}$'s considered in this paper are *hereditary,* that is, $\mathcal{G} \subset \mathcal{F} \in \mathbf{A}$ implies $\mathcal{G} \in \mathbf{A}$. For hereditary $\mathbf{A}$'s there is a way of reduction of the set of extreme points. Before stating the theorem we have to introduce some more notations. $\mu^*(\mathbf{A})$ is the set of *maximal profiles:* $\mu^*(\mathbf{A})$ contains those elements $(f_0, \ldots, f_n)$ of $\mu(\mathbf{A})$ for which $(g_0, \ldots, g_n) \in \mu(\mathbf{A})$ $(g_0, \ldots, g_n) \geqq (f_0, \ldots, f_n)$ (it denotes $g_0 \geqq f_0, \ldots, g_n \geqq f_n$) imply $(f_0, \ldots, f_n) = (g_0, \ldots, g_n)$. Furthermore let $\varepsilon^*(\mathbf{A}) = \varepsilon(\mathbf{A}) \cap \mu^*(\mathbf{A})$ be the set of the *essential extreme points.*

Theorem 1. *Suppose that* $\mathbf{A}$ *is hereditary. Then any element of* $\varepsilon(\mathbf{A})$ *can be obtained by changing some coordinates of an element of* $\varepsilon^*(\mathbf{A})$ *to zero.* ∎

This fact is obvious. The proof requires very simple technique, therefore it is omitted.

The significance of the theorem is that for a given $\mathbf{A}$ it is sufficient to determine the set $\varepsilon^*(\mathbf{A})$. Changing the components to zero we obtain a set of vectors, these should be individually checked if they are extreme points.

If we want to prove that a certain set of points is $\varepsilon(\mathbf{A})$ then we have to show that 1) any point of $\mu(\mathbf{A})$ can be expressed as a convex linear combination of the elements of $\varepsilon(\mathbf{A})$, and 2) the elements of $\varepsilon(\mathbf{A})$ are extreme points. To prove the first condition an equality should be proved. The next theorem reduces this equality for an inequality. If ε is a set of vectors, ε^0 denotes the set of vectors obtained by changing the components of the vectors of ε for zero in all possible ways.

Theorem 2. *Suppose that* $\mathbf{A}$ *is hereditary and a set* $\varepsilon = \{e_1, \ldots, e_m\} \subseteq \mu(\mathbf{A})$ *is given.*

If for any $\underline{f}\in\mu(\mathbf{A})$ *there are constants* $\lambda_1, \ldots, \lambda_m \geqq 0$, $\sum_{i=1}^{m} \lambda_i \leqq 1$ *satisfying*

$$\underline{f} \leqq \sum_{i=1}^{m} \lambda_i \underline{e}_i \tag{2}$$

then $\varepsilon^*(\mathbf{A}) \subseteq \varepsilon$. ∎

This claim is useful, but trivial. (If $\underline{g}\in\langle\mu(\mathbf{A})\rangle$ and $\underline{0}\leqq\underline{f}\leqq\underline{g}$ then $\underline{f}\in\langle\mu(\mathbf{A})\rangle$).

2.2. *Application of the duality theorem of linear programming.* Using the transposed forms $\underline{f}^T$ and $\underline{e}_i^T$ of the column vectors $\underline{f}$ and $\underline{e}_i$, resp., (2) can be written like

$$(\underline{e}_1^T \ldots \underline{e}_m^T) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} \geqq \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{3}$$

where $(\underline{e}_1^T \ldots \underline{e}_m^T)$ denotes the $(n+1)\times m$ matrix with columns $\underline{e}_1^T, \ldots, \underline{e}_m^T$. Its constraints are

$$\lambda_i \geqq 0 \quad (1 \leqq i \leqq m) \tag{4}$$

and

$$\sum_{i=1}^{m} \lambda_i \leqq 1.$$

Our aim is to find for $\underline{f}$ such λ_i's. This can be formulated in the way that

$$\min \sum_{i=1}^{m} \lambda_i \tag{5}$$

should be found under the conditions (3) and (4) and the solution (5) of this linear programming problem has to be $\leqq 1$. The dual of this problem is

$$\begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \\ \underline{e}_m \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \leqq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{6}$$

$$y_i \geqq 0 \quad (0 \leqq i \leqq n) \tag{7}$$

$$\max \sum_{i=0}^{n} f_i y_i. \tag{8}$$

By the duality theorem of linear programming (8) is equal to (5). (5)$\leqq 1$ iff (8)$\leqq 1$. This latter inequality can easily be formulated as

$$\sum_{i=0}^{n} f_i y_i \leqq 1$$

for any y_i's satisfying (6) and (7). It is worthwhile formulating this statement as a theorem:

Theorem 3. *Suppose that* **A** *is hereditary, a set* $\varepsilon=\{\underline{e}_1, \ldots, \underline{e}_m\} \subseteq \mu(\mathbf{A})$ *is given and*

$$\sum_{i=0}^{n} f_i y_i \leqq 1$$

holds for any $y_0, \ldots, y_n$ *satisfying* $y_i \geqq 0$ $(0 \leqq i \leqq n)$ *and*

$$\begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \\ \underline{e}_m \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \leqq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then $\varepsilon^*(\mathbf{A}) \subseteq \varepsilon$. ∎

2.3. *Reduction to the circle.* Take a cyclic permutation $\mathscr{C}$ of the underlying set X and consider only such subsets of X whose elements are consecutive in $\mathscr{C}$. These sets are called *consecutive sets in* $\mathscr{C}$. If $\mathscr{F}$ is a family of subsets of X, then $\mathscr{F}(\mathscr{C})$ is defined by $\mathscr{F}(\mathscr{C})=\{F\colon F\in\mathscr{F},\ F \text{ is consecutive in } \mathscr{C}\}$. Similarly, let $\mathbf{A}(\mathscr{C})=\{\mathscr{F}(\mathscr{C})\colon \mathscr{F}\in\mathbf{A}\}$. It is well-known (see e.g. [7]) that for some classes **A** it is enough to determine $\max\{|\mathscr{F}|\colon \mathscr{F}\in\mathbf{A}(\mathscr{C})\}$ and $\max\{|\mathscr{F}|\colon \mathscr{F}\in\mathbf{A}\}$ can be obtained from it by a simple counting argument. Of course, this extremal problem for $\mathbf{A}(\mathscr{C})$ is easier than for **A**. This method is sometimes called as the *permutation method.*

Before stating the result we have to introduce a notation. If $\underline{e}=(e_0, e_1, \ldots, e_n)$ then let

$$T(\underline{e}) = \left(e_0, e_1\binom{n}{1}\Big/n, e_2\binom{n}{2}\Big/n, \ldots, e_{n-1}\binom{n}{n-1}\Big/n, e_n\right).$$

Theorem 4. *(Blowing up the circle.) If* $\underline{e}_1, \ldots, \underline{e}_m$ *are the extreme points of* $\mu(\mathbf{A}(\mathscr{C}))$ *for any given cyclic permutation* $\mathscr{C}$ *then*

$$\mu(\mathbf{A}) \subseteq \langle\{T(\underline{e}_1), \ldots, T(\underline{e}_m)\}\rangle.$$

Proof. Let $\mathscr{F}$ be an element of **A**, with profile $(f_0, f_1, \ldots, f_n)$. Define the weight-function

$$\underline{w}(F) = \Big(\underset{0}{\widehat{0}}, \underset{1}{\widehat{0}}, \ldots, \underset{|F|}{\widehat{\tfrac{1}{(n-1)!}}}, \ldots, \underset{n}{\widehat{0}}\Big) \quad (F\subset X).$$

Consider the sum $\sum \underline{w}(F)$ for all pairs $(\mathscr{C}, F)$ where $\mathscr{C}$ is a cyclic permutation, $F\in\mathscr{F}$ and F is consecutive in $\mathscr{C}$.

For a fixed $\mathscr{C}$ we have

$$\sum_{F\in\mathscr{F}(\mathscr{C})} \underline{w}(F) = \frac{1}{(n-1)!}(\text{profile of } \mathscr{F}(\mathscr{C})).$$

Here the profile of $\mathscr{F}(\mathscr{C})$ is in $\mu(\mathbf{A}(\mathscr{C}))$, therefore it is a convex linear combination $\sum_{i=1}^{m} \lambda_i(\mathscr{C})\underline{e}_i$ of the extreme points $\underline{e}_1, \ldots, \underline{e}_m$ of $\mu(\mathbf{A}(\mathscr{C}))$ $\left(\lambda_i(\mathscr{C})\geqq 0,\ \sum_{i=1}^{m}\lambda_i(\mathscr{C})=1\right)$.

Hence

$$\sum_{\mathscr{C},F} \underline{w}(F) = \sum_{\mathscr{C}} \sum_{F} \underline{w}(F) = \sum_{\mathscr{C}} \frac{1}{(n-1)!} \sum_{i=1}^{m} \lambda_i(\mathscr{C}) \underline{e}_i = \sum_{i=1}^{m} \frac{1}{(n-1)!} \Big(\sum_{\mathscr{C}} \lambda_i(\mathscr{C})\Big) \underline{e}$$

follows where $\sum_{i=1}^{m} \frac{1}{(n-1)!} \sum_{\mathscr{C}} \lambda_i(\mathscr{C}) = 1$. We have proved that

(9) $\sum_{\mathscr{C},F} \underline{w}(F)$ *is a convex linear combination of* $\underline{e}_1, \ldots, \underline{e}_m$.

On the other hand, summing in the other way around we obtain

(10)
$$\sum_{\mathscr{C},F} \underline{w}(F) = \sum_{F} \sum_{\mathscr{C}} \underline{w}(F) = \sum_{F}{}^{*} \left(0, 0, \ldots, \frac{|F|!(n-|F|)!}{(n-1)!}, \ldots, 0\right)$$
$$= \left(f_0, f_1, f_2 n \Big/ \binom{n}{2}, \ldots, f_i n \Big/ \binom{n}{i}, \ldots, f_{n-1}, f_n\right);$$

where $\sum^*$ denotes that $(1, 0, \ldots, 0)$ and $(0, 0, \ldots, 0, 1)$ are taken for $F=\emptyset$ and $F=X$, resp., as the number of cyclic permutations in which F is consecutive is $|F|!(n-|F|)!$ for $0<|F|<n$ but it is $(n-1)!$ for $|F|=0, n$. It follows by (9) that (10) is a convex linear combination of $\underline{e}_1, \ldots, \underline{e}_m$. This implies that $(f_0, f_1, \ldots, f_n)$ is a convex linear combination of $T(\underline{e}_1), \ldots, T(\underline{e}_m)$. ∎

This theorem is really useful if $T(\underline{e}_1), \ldots, T(\underline{e}_m) \in \mu(\mathbf{A})$ holds. (This can easily be checked.) Then $\langle\{T(\underline{e}_1), \ldots, T(\underline{e}_m)\}\rangle \subseteq \mu(\mathbf{A})$ and $\mu(\mathbf{A}) = \langle\{T(\underline{e}_1), \ldots, T(\underline{e}_m)\}\rangle$ obviously follow. $T(\underline{e}_1), \ldots, T(\underline{e}_m)$ are the extreme points of $\mathbf{A}$. Unfortunately, this is not true in general. An example will be given when $\langle\{T(\underline{e}_1), \ldots, T(\underline{e}_m)\}\rangle$ is much larger than $\langle\mu(\mathbf{A})\rangle$.

3. k-Sperner-families

Let $\mathbf{S}_k$ denote the class of k-Sperner-families on an n-element set.

Theorem 5. *The extreme points of* $(\mathbf{S}_k)$ *are the vectors whose* ith *components are either* $\binom{n}{i}$ *or* 0 *but have at most* k *non-zero components.*

Proof. It is trivial that these vectors are in $\mu(\mathbf{S}_k)$. To the vector $\left(0, \ldots, \binom{n}{i_1}, 0, \ldots, 0, \binom{n}{i_l}, \ldots, 0\right)$ $(l \leq k)$ one can find a k-sperner-family $\mathscr{F}$ with this profile: take all $i_1, \ldots, i_l$-element subsets of X.

Moreover, these points are extreme. Let $\mathscr{E} = \left(0, \ldots, 0, \binom{n}{i_1}, 0, \ldots, 0, \binom{n}{i_l}, 0, \ldots, 0\right)$; $(l \leq k)$. It is easy to check that no $\underline{u} \in \mathscr{E}$ is a convex linear combination of the other points of $\mu(\mathbf{S}_k)$.

On the other hand, we have to prove that any element of $\mu(\mathbf{S}_k)$ can be expressed as a convex linear combination of these vectors. Theorem 4 can be applied if we show that the extreme points of $\mu(\mathbf{S}_k(\mathscr{C}))$ are the vectors whose ith components are either n or 0 for $0<i<n$ and either 1 or 0 for $i=0, n$, but have at most k non-zero components. By Theorem 1 it is sufficient to prove that $\varepsilon^*(\mathbf{S}_k(\mathscr{C}))$ is the set of vec-

tors whose ith components are either n or 0 for $0<i<n$ and either 1 or 0 for $i=0, n$, but have exactly k non-zero components. To prove this we apply Theorem 3. The inequality

$$\sum_{i=0}^{n} f_i y_i \leqq 1 \tag{11}$$

has to be verified for any k-Sperner-family in $\mathscr{C}$ with profile $(f_0, \dots, f_n)$ and for any system of y's satisfying $y_i \geqq 0$ $(0 \leqq i \leqq n)$ and

$$\sum_{j=1}^{k} \varrho(i_j) n y_{i_j} \leqq 1 \tag{12}$$

for any choice $0 \leqq i_1 < \dots < i_k \leqq n$ where $\varrho(0) = \varrho(n) = 1/n$ $\varrho(i) = 1$ $(1 \leqq i \leqq n-1)$.

Let us first show that (11) holds for the following simple systems of values:

$$y_0 = \frac{1}{k}, \; y_1 = \dots = y_{n-1} = \frac{1}{nk}, \; y_n = \frac{1}{k},$$

$$y_0 = 1, \; y_1 = \dots = y_n = 0$$

$$y_0 = 0, \dots, y_i = \frac{1}{n}, \dots, y_n = 0 \quad (1 \leqq i \leqq n-1)$$

and

$$y_0 = \dots = y_{n-1} = 0, \quad y_n = 1.$$

In other words we have to prove the inequalities

$$\frac{f_0}{k} + \sum_{n=1}^{n-1} \frac{f_i}{nk} + \frac{f_n}{k} \leqq 1 \tag{13}$$

$$f_0 \leqq 1 \tag{14}$$

$$f_i \leqq n \quad (1 \leqq i \leqq n-1) \tag{15}$$

$$f_n \leqq 1 \tag{16}$$

for the profile $(f_0, \dots, f_n)$ of any k-Sperner-family. (14)—(16) are trivial. The real problem is (13). Suppose first that $f_0 = f_n = 0$ and consider a fixed $\mathscr{F}(\mathscr{C})$ with this profile. Any element of X can be the "starting" point of at most k members of $\mathscr{F}(\mathscr{C})$ because of the k-Sperner property. Thus $|\mathscr{F}(\mathscr{C})| = \sum_{i=1}^{n-1} f_i \leqq nk$. (13) follows. If exactly one of f_0 and f_n is 1 then the number of members $\mathscr{F}(\mathscr{C})$ "starting" with a fixed element is at most $k-1$. (13) follows like above. The case $f_0 = f_n = 1$ is analogous.

Let us prove now (11) under the general assumption (12). Consider a fixed system of y's and order $\varrho(i) y_i$: $\varrho(l_1) y_{l_1} \geqq \dots \geqq \varrho(l_{n+1}) y_{l_{n+1}}$ where $l_1, \dots, l_{n+1}$ is a permutation of $0, 1, \dots, n$. It follows by (12) that $\sum_{j=1}^{k} \varrho(l_j) y_{l_j} \leqq 1$. If there is a strict inequality here, then multiply all the y's with a constant (>1) to achieve

$$\sum_{j=1}^{k} \varrho(l_j) y_{l_j} = \frac{1}{n}. \tag{17}$$

It is easy to see that it is sufficient to prove (11) for such y's. (12) and (17) imply

$$\varrho(t)y_t \leqq \frac{1}{n} - \sum_{j=1}^{k-1} \varrho(l_j)y_{l_j} = \varrho(l_k)y_{l_k}$$

for any $t \neq l_1, \ldots, l_{k-1}$. Hence we have

$$\sum_{i=0}^{n} f_i y_i \leqq \sum_{j=1}^{k-1} f_{l_j} y_{l_j} + \sum_{t \neq l_1, \ldots, l_{k-1}} \frac{f_t}{\varrho(t)} \varrho(l_k) y_{l_k}$$

$$= \sum_{j=1}^{k-1} \frac{f_{l_j}}{\varrho(l_j)} \left(\varrho(l_j)y_{l_j} - \varrho(l_k)y_{l_k}\right) + \sum_{t=0}^{n} \frac{f_t}{\varrho(t)} \varrho(l_k) y_{l_k}.$$

For the latter row we obtain an upper estimate applying (13)—(16) and (17):

$$\leqq \sum_{j=1}^{k-1} n\left(\varrho(l_j)y_{l_j} - \varrho(l_k)y_{l_k}\right) + \varrho(l_k)y_{l_k} nk$$

$$= n\left(\frac{1}{n} - \varrho(l_k)y_{l_k}\right) - n(k-1)\varrho(l_k)y_{l_k} + nk\varrho(l_k)y_{l_k} = 1.$$

We have proved that (11) holds for y's satisfying $y_i \geqq 0$ $(0 \leqq i \leqq n)$ and (12). The application of Theorem 3 finishes the proof. ▮

The following theorem is an easy consequence of Theorem 5.

Theorem 5a. *The hyperplanes bordering* $\langle \mu(\mathrm{S}_k) \rangle$ *are*

$$f_i \geqq 0 \quad (0 \leqq i \leqq n)$$

$$f_i \Big/ \binom{n}{i} \leqq 1 \quad (0 \leqq i \leqq n)$$

$$\sum_{i=0}^{n} f_i \Big/ \binom{n}{i} \leqq k. \quad ▮$$

Theorem 5 makes it easy to maximize $|\mathscr{F}| = \sum_{i=0}^{n} f_i$ for families $\mathscr{F}$ belonging to S_k. It is sufficient to look for this maximum among the extreme points of $\mu(\mathrm{S}_k)$.

Theorem (Erdős [3])

$$\max_{\mathscr{F} \in \mathrm{S}_k} |\mathscr{F}| = \sum_{i=\lfloor (n-k+1)/2 \rfloor}^{\lfloor (n+k-1)/2 \rfloor} \binom{n}{i}. \quad ▮$$

For $k=1$ this is the old Sperner theorem [8].

4. Intersecting families

A family $\mathscr{F}$ is called *t-intersecting* $(1 \leqq t \leqq n)$ if $F_1, F_2 \in \mathscr{F}$ implies $|F_1 \cap F_2| \geqq t$. Let $\mathbf{I}_t$ denote the class of t-intersecting families on an n-element set. The 1-intersecting families are called simply *intersecting*. In case $t=1$, $\mathbf{I}$ is written rather than $\mathbf{I}_1$. It seems to be too hard to determine the extreme points of $\mu(\mathbf{I}_t)$. We are able to do this only for $t=1$. However, it can be done for $\mathbf{I}_t(\mathscr{C})$. Before formulating the result we prove some preliminary lemmas.

Lemma 1. *Suppose that $A_1, \ldots, A_u$ are v-element consecutive sets along a cyclic permutation $\mathscr{C}$ of an n-element set such that $|A_i \cap A_j| \geqq t \geqq 1$ for any $1 \leqq i < j \leqq u$ where $t \leqq v \leqq 1/2(n+t-1)$. Then $u \leqq v-t+1$ holds.*

Proof. Let $A_1 = \{x_1, \ldots, x_v\}$ and suppose that the elements are ordered in this way. Another A cannot meet A_1 in both ends by the conditions. Therefore the possible endpoints for A are $x_t, \ldots, x_{v-1}$, while the possible starting points are $x_2, \ldots, x_{v-t+1}$. However the set ending with $x_i (t \leqq i \leqq v-1)$ and the one starting with x_{i-t+2} meet in $t-1$ elements only. Hence at most one of them can be among the A's. Consequently there are at most $v-t$ such A's. ∎

Lemma 2. *If $A_1, \ldots, A_u$ are v-element consecutive sets along a cyclic permutation of an n-element set then*

$$\Big|\bigcup_{i=1}^{u} A_i\Big| \geqq \min(n, u+v-1).$$

Proof. Suppose first that have is an A_i containing no starting point of another A. Then the number of starting points is u while the number of other points of A_i is $v-1$, that is, $|\bigcup_{i=1}^{u} A_i| \geqq u+v-1$. On the other hand, if any A_i contains the starting point of another one then the union of them is the whole underlying set X, that is, $|\bigcup_{i=1}^{u} A_i| = n$. ∎

Lemma 3. *Let $(f_0, \ldots, f_n) \in \mu(\mathbf{I}_t(\mathscr{C}))$, $f_i \neq 0$ for some $i\left(t \leqq i \leqq \frac{n+t-1}{2}\right)$. Suppose that $t \leqq j \leqq n+t-1-i$ holds for some j. Then $f_j \leqq j+i-f_i-2(t-1)$ holds.*

Proof. Suppose that $\mathscr{F} \in \mathbf{I}_t(\mathscr{C})$ holds and its profile is $(f_0, \ldots, f_n)$. Let $\mathscr{F}_i = \{F_1, \ldots, F_{f_i}\}$. Consider the family $\mathscr{A} = \{A\colon |A| = n-j, |A \cap F_l| \geqq i-t+1$ for some $1 \leqq l \leqq i\}$. The starting points of the $(n-j)$-element consecutive sets satisfying $|A \cap F_l| \geqq i-t+1$ for a fixed l form a consecutive set of size $n-j-i+2t-1$. Applying Lemma 2 the total number of these starting points is at least $\min(n, n-j-i+2t-2+f_i)$. Therefore this is a lower bound for $|\mathscr{A}|$. $A \in \mathscr{A}$ implies that $|X-A| = j$ and $|(X-A) \cap F| \leqq t-1$. Hence we have at least $\min(n, n-j-i+2t-2+f_i)$ j-element consecutive subsets $X-A$ not belonging to $\mathscr{F}$. Therefore $f_j = |\mathscr{F}_j| \leqq \max(0, j+i-f_i-2(t-1))$. ∎

We remark that Lemma 1 implies $f_i \leqq i-t+1$ hence $j+i-f_i-2(t-1) > 0$.

Lemma 4. $(f_0, \ldots, f_n) \in \mu(\mathbf{I}_t(\mathscr{C}))$ *iff the following conditions are fulfille.*

$$f_i = 0 \quad (0 \leqq i < t), \tag{18}$$

$$f_i \leqq i-t+1 \quad (t \leqq i \leqq (n+t-1)/2), \tag{19}$$

$$f_j \leqq \min\{j+i-f_i-2(t-1)\}\big((n+t-1)/2 < j \leqq n\big) \tag{20}$$

where the minimum is taken on all i satisfying

$$t \leqq i \leqq n+t-1-j, \quad f_i \neq 0. \tag{21}$$

If this set is empty then (20) *has the form* $f_j \leqq n$ $(j<n)$. $f_n \leqq 1$.

Proof. (18) trivially follows from $(f_0, \ldots, f_n) \in \mu(\mathbf{I}_t(\mathscr{C}))$ by the definitions. (19) and (20) are consequenses of Lemmas 1 and 3, respectively.

Conversely we have to prove that if (18)—(20) hold than there is an $\mathscr{F} \in \mathbf{I}_t(\mathscr{C})$ with profile $(f_0, \ldots, f_n)$. This will be done by a construction. Let $x_1, \ldots, x_n$ be the elements of X according their order in $\mathscr{C}$. For $t \leqq i \leqq (n+t-1)/2$, choose the consecutive sets with endpoints $x_i, x_{i-1}, \ldots, x_{i-f_i+1}$. On the other hand, if $(n+t-1)/2 < i$, take the sets with endpoints x_t; $x_{t+1}, \ldots, x_{t+f_i-1}$. This family $\mathscr{F}$ is trivially t-intersecting. ∎

So we obtained a purely algebraic characterization of the polytope $\langle \mu(\mathbf{I}_t(\mathscr{C})) \rangle$. Now the description of its essential vertices (Lemma 5) requires only linear algebraic technique, so the proof of it will be sketched only.

Lemma 5. $\varepsilon^*(\mathbf{I}_t(\mathscr{C}))$ *consists of the following vectors*

(22)

$$\begin{array}{ccccccccc}
(0, \ldots, & 0, & k-t+1, & k-t+2, \ldots, & n-k, & n, \ldots, & n, & 1) & \\
\widehat{n} & \widehat{k-1} & \widehat{k} & \widehat{k+1} & \widehat{n+t-1-k} & \widehat{n+t-k} & \widehat{n-1} & \widehat{n} & \left(t \leqq k \leqq \dfrac{n+t-1}{2}\right)
\end{array}$$

$$\begin{array}{ccccc}
(0, & \ldots, & 0, & n, \ldots, n, & 1) \\
 & & \widehat{\frac{n+t}{2}} & & \widehat{n}
\end{array} \qquad (n+t \quad \textit{is even}).$$

Proof. (Sketch). It is clear that $(22) \subseteqq \langle \mu(\mathbf{I}_t(\mathscr{C})) \rangle$ and they are convex linearly independent.

If $f \in \langle \mu(\mathbf{I}_t(\mathscr{C})) \rangle$ a vertex then it can be obtained as an intersection of $(n+1)$ hyperplanes of the form (18)—(21). It is easy to check that if $f \in \langle \mu(\mathbf{I}_t(\mathscr{C})) \rangle$ and f satisfies $(n+1)$ inequalities of form (18)—(21) by equality then f can be obtained from an element of (22) changing some components for zero. So (22) are the essential vertices of $\langle \mu(\mathbf{I}_t(\mathscr{C})) \rangle$. ∎

If $t=1$ we may apply Theorem 4, Lemma 6 and Theorem 1 to determine all the extreme points e of $\mu(\mathbf{I}(\mathscr{C}))$. The vectors $T(\underline{e})$ are

(23)

$$\Big(\underset{\hat{0}}{0},\dots,0,\underset{\hat{k}}{\binom{n-1}{k-1}},\underset{\widehat{k+1}}{\binom{n-1}{k}},\dots,\underset{\widehat{n-k}}{\binom{n-1}{n-k-1}},\underset{\widehat{n-k+1}}{\binom{n}{n-k+1}},\dots,\binom{n}{n-1},\underset{\hat{n}}{1}\Big)\quad \left(1\le k\le \frac{n}{2}\right)$$

$$\left(\underset{\hat{0}}{0},\dots,0,\underset{\widehat{\frac{n+1}{2}}}{\binom{n}{\frac{n+1}{2}}},\dots,\binom{n}{n-1},\underset{\hat{n}}{1}\right)\quad (n \text{ is odd})$$

and the vectors obtained by substituting 0's into some components. The vectors listed in (23) are in $\mu(\mathbf{I})$ as the following construction shows. Fix an element x of the underlying set X and take all the k-element, $k+1$-element, ..., $(n-k)$-element subsets containing x and take all $(n-k+1)$-element, ..., n-element sets. It is easy to see that this is an intersecting family and its profile is the desired vector. The same construction works for the vectors with the zeros. This proves the following.

Theorem 6. *$\varepsilon^*(\mathbf{I})$ consists of the vectors listed under* (23). ∎

The number of extreme points is exponentially large. However, if $\sum_{i=0}^{n} C_i f_i$ should be maximized, where $C_i \ge 0$ then it is sufficient to consider $\varepsilon^*(\mathbf{I})$. The size of this set is linear. The most known consequence of the above theorem is the

Erdős—Ko—Rado theorem [4]. *If $\mathscr{F}$ is an intersecting family of k-element subsets of an n-element set and $k\le n/2$ then* $\max|\mathscr{F}|=\binom{n-1}{k-1}$. ∎

This follows from Theorem 6 since no extreme point has a larger kth component.

To determine $\max|\mathscr{F}|$ over any intersecting family $\mathscr{F}\subset 2^n$ is trivial. However it can also be deduced from Theorem 6. $|\mathscr{F}|=\sum_{i=0}^{n} f_i$ implies that we have to consider the sum of the components in the extreme points. It is easy to see that $f_i+f_{n-i}=\binom{n}{n-i}$ for any extreme point and $0\le i\le (n-1)/2$. Moreover, $f_{n/2}=\frac{1}{2}\binom{n}{n/2}$ holds. Hence $\sum_{i=0}^{n} f_i=2^{n-1}$. In the same way, it is easy to deduce $\max|\mathscr{F}|$ for intersecting families with any size constraint. $\max\sum_{i=0}^{n} i\cdot f_i$ can also be determined. For a further application see [2].

If we try to combine Lemma 5 and Theorem 4 for t-intersecting families, then the vectors $T(e)$ will not belong to $\mu(\mathbf{I}_t)$, therefore they are not extreme points, either. To determine the extreme points of $\mu(\mathbf{I}_t)$ seems to be very hard. It would imply the solution of many open problems. Such an open problem, raised by Erdős, Ko and Rado, is to maximize the size of a 2-intersecting family of $2n$-element subsets of a $4n$-element set [4]. (Lemma 5 answers the same question for the circle.) Let us note that one extreme point of $\mu(\mathbf{I}_t)$ is known, the one maximizing $|\mathcal{F}|=\sum_{i=0}^{n} f_i$ [6].

Finally we give a variant of Theorem 6. It can be proved by the duality theorem.

Theorem 6a. *If* $(f_0, \ldots, f_n) \in \mu(\mathbf{I})$ *and* $y_0, y_1, \ldots, y_n \geqq 0$ *satisfy the inequalities*

$$\binom{n-1}{k-1} y_k + \binom{n-1}{k} y_{k-1} + \cdots$$

$$\cdots + \binom{n-1}{n-k-1} y_{n-k} + \binom{n}{n-k+1} y_{n-k+1} + \cdots + \binom{n}{n} y_n \leqq 1 \quad \left(1 \leqq k \leqq \frac{n}{2}\right)$$

$$\binom{n}{\frac{n+1}{2}} y_{\frac{n+1}{2}} + \cdots + \binom{n}{n} y_n \leqq 1 \quad (\textit{if } n \textit{ is odd})$$

then

$$\sum_{i=0}^{n} f_i y_i \leqq 1. \quad \blacksquare$$

5. More families without inclusion among them

Daykin, Frankl, Greene and Hilton [1] investigated the families with the following properties. Let $t \geqq 2$ be an integer and let $\mathcal{F}^i (1 \leqq i \leqq t)$ be a family of distinct subsets of an n-element set X. The families are not necessarily disjoint but $A_i \in \mathcal{F}^i$, $A_j \in \mathcal{F}^j$, $i \neq j$, $A_i \neq A_j$ imply $A_i \not\subset A_j$. In notation: $(\mathcal{F}^1, \ldots, \mathcal{F}^t) \in \mathbf{W}_t$. The profile of an element of $\mathbf{W}_t$ is $(f_0, \ldots, f_n)$ where $f_i = \sum_{j=1}^{t} |\mathcal{F}_i^j|$. It can be considered as the profile of $\sum_{j=1}^{t} \mathcal{F}^j$ with multiplicities. The definitions and the results of Section 2 can be repeated for families with multiplicities. $\mathbf{W}_t$ is obviously hereditary, so it is enough to determine $\varepsilon^*(\mathbf{W}_t)$ instead of $\varepsilon(\mathbf{W}_t)$. Colour the sets occuring exactly ones or more times by green or red, resp. It is easy to see that a red set cannot be in inclusion with any other green or red set. Therefore a red set can be added to all $\mathcal{F}^j$ without violating the conditions. In this way we associated to any $(\mathcal{F}^1, \ldots, \mathcal{F}^t) \in \mathbf{W}_t$ two families $\mathcal{R}$ and $\mathcal{G}$ where no member of $\mathcal{R}$ is in inclusion with any member of $\mathcal{R} \cup \mathcal{G}$ and the members of $\mathcal{G}$ have multiplicity 1 while the multiplicity of any member of $\mathcal{R}$ is between 1 and t. The set of such pairs $(\mathcal{R}, \mathcal{G})$ is denoted by $\mathbf{B}_t$. It is easy to see that, conversely, the members of any $(\mathcal{R}, \mathcal{G}) \in \mathbf{B}_t$ can be distributed into sets $\mathcal{F}^1, \ldots, \mathcal{F}^t$. (Put all green sets into $\mathcal{F}^1$, the copies of the red sets into different $\mathcal{F}$'s.) This shows $\mu(\mathbf{W}_t) = \mu(\mathbf{B}_t)$.

Theorem 7. $\varepsilon^*(\mathbf{W}_t)=\varepsilon^*(\mathbf{B}_t)$ $(t\geqq 2)$ *consists of the vectors*

$$\left(0, \dots, 0,\ t\binom{n}{i},\ 0, \dots, 0\right) \quad (0 \leqq i \leqq n)$$

and additionally

$$\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right) \quad \text{if} \quad t < n+1.$$

The proof is based on the following lemmas.

Lemma 6. *If* $(f_0, \dots, f_n)\in\mu(\mathbf{B}_t(\mathscr{C}))$ *then*

$$\sum_{j=1}^{t} f_{i_j} \leqq tn$$

for any distinct $i_1, \dots, i_t$.

Proof. Let $(\mathscr{R}, \mathscr{G})\in\mathbf{B}_t(\mathscr{C})$ and let $(f_0, \dots, f_n)$ be its profile. Denote by r_j and g_j the number of i_j-element red and green members in $\mathscr{R}\cup\mathscr{G}$ resp. Hence

$$f_{i_j} \leqq tr_j + g_j \tag{24}$$

holds. The i_j-element green members and all the red ones in $\mathscr{R}\cup\mathscr{G}$ form a Sperner-family, therefore

$$g_j + \sum_{k=1}^{t} r_k \leqq n \quad (1 \leqq j \leqq t)$$

follows. Summing these inequalities we obtain

$$\sum_{j=1}^{t} (g_j + tr_j) \leqq tn.$$

Hence (24) implies the validity of the lemma. ▮

Lemma 7. *Suppose that* $c_0, \dots, c_n$ *are non-negative reals. Then, under the conditions*

$$z_i \leqq \frac{1}{t} \quad (0 \leqq i \leqq n,\ t \text{ is an integer}), \tag{25}$$

$$\sum_{i=0}^{n} z_i \leqq 1, \tag{26}$$

$\max \sum_{i=0}^{n} c_i z_i$ *is attained for*

$$z_0 = \dots = z_n = \frac{1}{t} \quad \text{if} \quad n+1 \leqq t,$$

(27)

$$(z_{i_1} = \dots = z_{i_t} = \frac{1}{t}, \quad z_j = 0 \quad (j = i_k) \quad \text{if} \quad n+1 > t.$$

Proof. It is trivial. ▮

Lemma 8. *Suppose that* $y_0, \ldots, y_n \geqq 0$ *satisfy the following inequalities:*

$$y_0 \leqq \frac{1}{t}, \quad y_i \leqq \frac{1}{tn} \quad (1 \leqq i < n), \quad y_n \leqq \frac{1}{t}, \tag{28}$$

$$y_0 + n \sum_{i=1}^{n-1} y_i + y_n \leqq 1. \tag{29}$$

Then $(f_0, \ldots, f_n) \in \mu(\mathbf{B}_t(\mathscr{C}))$ *implies*

$$\sum_{i=0}^{n} f_i y_i \leqq 1. \tag{30}$$

Proof. If $f_0 \neq 0$ then the empty set is either a red or a green member of $\mathscr{R} \cup \mathscr{G}$. If $\emptyset \in \mathscr{R}$ then there is no other member: $f_i = 0$ $(1 \leqq i \leqq n)$. $f_0 \leqq t$ and $y_0 \leqq 1/t$ imply the statement. If $\emptyset \in \mathscr{G}$ then $\mathscr{R}$ is empty, therefore $f_1 \leqq 1$, $f_i \leqq n$ $(1 \leqq i < n)$, $f_n \leqq 1$. (29) implies (30). If $f_n \neq 0$, the situation is analogous. We may suppose that $f_0 = f_n = 0$.

Introduce the notations $z_i = ny_i$, $c_i = f_i/n$ $(1 \leqq i < n)$. (28), (29) and $\sum_{i=0}^{n} f_i y_i$ give rise to $z_i \leqq 1/t (1 \leqq i < n)$, $\sum_{i=1}^{n-1} z_i \leqq 1$ and $\sum_{i=1}^{n-1} c_i z_i$. We may apply Lemma 7:

$$\sum_{i=1}^{n-1} f_i y_i = \sum_{i=1}^{n-1} c_i z_i \leqq \begin{cases} \dfrac{1}{t} \sum_{i=1}^{n-1} c_i = \dfrac{1}{nt} \sum_{i=1}^{n-1} f_i & \text{if} \quad n+1 \leqq t, \\ \dfrac{1}{t} \sum_{j=1}^{t} c_{i_j} = \dfrac{1}{nt} \sum_{j=1}^{t} f_{i_j} & \text{if} \quad n+1 > t. \end{cases}$$

This is at most 1, in the first case trivially, in the second case by Lemma 6. (30) is proved. ∎

Proof of Theorem 7. The vectors $(t, 0, \ldots, 0), (0, \ldots, 0, tn, 0, \ldots, 0)$, $(0, \ldots, 0, t)$ and $(1, n, \ldots, n, 1)$ are obviously in $\mu(\mathbf{B}_t(\mathscr{C}))$. Consequently, Lemma 8 and Theorem 3 imply that there vectors are the only candidates to be in $\varepsilon^*(\mathbf{B}_t(\mathscr{C}))$. Hence Theorem 1 gives the candidates for $\varepsilon(\mathbf{B}_t(\mathscr{C}))$.

If $t \geqq n+1$ then $(1, n, \ldots, n, 1) = t^{-1}(t, 0, \ldots, 0) + \sum t^{-1}(0, \ldots, 0, tn, 0, \ldots, 0) + t^{-1}(0, \ldots, 0, t) + (1-(n+1)t^{-1})$ $(0, \ldots, 0)$ shows that $(1, n, \ldots, n, 1)$ is a convex linear combination of the other ones. The extreme points of $\mu(\mathbf{B}_t(\mathscr{C}))$ are $(0, \ldots, 0)$, $(t, 0, \ldots, 0)$, $(0, \ldots, 0, tn, 0, \ldots, 0)$ and $(0, \ldots, 0, t)$.

Suppose now that $t < n+1$. The set of possible extreme points of $\mu(\mathbf{B}_t(\mathscr{C}))$ is completed with $(1, n, \ldots, n, 1)$ and with the vectors obtained by writing zeros in the place of some components of $(1, n, \ldots, n, 1)$. However, if the number of non-zero components is $\leqq t$ then it is a convex linear combination of $(0, \ldots, 0)$, $(t, 0, \ldots, 0)$, $(0, \ldots, 0, tn, 0, \ldots, 0)$ and $(0, \ldots, 0, t)$. It is easy to see that the remaining ones are all extreme points of $\mu(\mathbf{B}_t(\mathscr{C}))$. Applying Theorem 4 the obtained vectors are all element of $\mu(\mathbf{B}_t)$. Moreover they are all extreme points. This proves the theorem. ∎

Theorem 7a. *The hyperplanes bordering* $\langle\mu(\mathbf{B}_t)\rangle$ *are*

$$f_i \geqq 0 \quad (0 \leqq i \leqq n)$$

and

(31) $$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i} t} \leqq 1 \quad \textit{if} \quad t \geqq n+1,$$

(32) $$\sum_{j=1}^{t} \frac{f_{i_j}}{\binom{n}{i_j} t} \leqq 1 \quad (0 \leqq i_1 < i_2 \ldots < i_t \leqq n) \quad \textit{if} \quad t < n+1.$$

Proof. Theorem 7 implies that for any $(f_0, \ldots, f_n) \in \mu(\mathbf{B}_t)$ there are $\lambda_0, \ldots, \lambda_n, \lambda_{n+1} \geqq 0$ satisfying $\sum_{i=0}^{n+1} \lambda_i \leqq 1$ and

$$\begin{pmatrix} f_0 \\ \vdots \\ \vdots \\ f_n \end{pmatrix} \leqq \lambda_0 \begin{pmatrix} t\binom{n}{0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ t\binom{n}{1} \\ \vdots \\ 0 \end{pmatrix} + \ldots + \lambda_n \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ t\binom{n}{n} \end{pmatrix} + \lambda_{n+1} \begin{pmatrix} \binom{n}{0} \\ \binom{n}{1} \\ \vdots \\ \binom{n}{n} \end{pmatrix}$$

where $\lambda_{n+1}=0$ in the case $t \geqq n+1$. This can be considered as a linear programming problem with the result $\min \sum_{i=0}^{n+1} \lambda_i \leqq 1$. The dual problem maximizes $\sum_{i=0}^{n} f_i y_i$ under

(33) $$y_i \leqq \frac{1}{t\binom{n}{i}} \quad (0 \leqq i \leqq n)$$

(34) $$\sum_{i=0}^{n} \binom{n}{i} y_i \leqq 1 \quad \text{if} \quad t < n+1.$$

that is, $\sum_{i=0}^{n} f_i y_i \leqq 1$ holds under the conditions (33) an (34). Let us choose $y_i = \left(t\binom{n}{i}\right)^{-1}$ $(0 \leqq i \leqq n)$ if $t \geqq n+1$. $\sum_{i=0}^{n} f_i y_i \leqq 1$ becomes (31). Suppose now $t < n+1$ and choose $y_{i_1} = \ldots = y_{i_t} = \left(t\binom{n}{i}\right)^{-1}$ for some $0 \leqq i_1 < i_2 < \ldots < i_t \leqq n$. (33) and (34) are statisfied. This implies (32). Applying Lemma 8 with $z_i = y_i\binom{n}{i}$ and $c_i = f_i\binom{n}{i}^{-1}$ we obtain that if $\sum_{i=0}^{n} f_i y_i \leqq 1$ holds for the above special values of y's (that is, if (31) and (32) holds) then it holds for any system of non-negative y's statifying (33) and (34). The hyperplanes $\sum f_i y_i \leqq 1$ different from (31) and (32) are superfluous. ∎

Theorem 7 easily implies the first part of the theorem of [1]

(35) $$\sum_{i=0}^{n} f_i \leqq \max\left(t\binom{n}{\left[\frac{n}{2}\right]},\ 2^n\right) \quad \text{for} \quad (f_0, \ldots, f_n) \in \mathbf{B}_t.$$

The same theorem allows us to maximize $\sum_{i=0}^{n} f_i \binom{n}{i}^{-1}$ for $(f_0, \ldots, f_n) \in \mathbf{B}_t$:

(36) $$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leqq \max(t,\ n+1).$$

This is the third part of the result in [1]. It is somewhat disturbing that (36) does not imply (35). The reason is that $\langle \mu(\mathbf{B}_t) \rangle$ cannot be well characterized by an arbitrarily chosen hyperplane.

To obtain the second part of the theorem of [1] the red and the green members of $(\mathcal{R}, \mathcal{G}) \in \mathbf{B}_t$ should be separated in the profile. The *colour profile* $(r_0, \ldots, r_n, g_0, \ldots, g_n)$ of $(\mathcal{R}, \mathcal{G})$ is defined by $r_i = |\mathcal{R}_i|$, $g_i = |\mathcal{G}_i|$ $(0 \leqq i \leqq n)$. $\chi(\mathbf{B}_t)$ denotes the set of colour profiles of all members of $\mathbf{B}_t$. The proof of the next theorem is left to the reader.

Theorem 8. *The essential extreme points of* $\chi(\mathbf{B}_t)$ *are*

$$\Big(\underset{\frown\atop 0}{0}, \ldots, \underset{\frown\atop i}{\binom{n}{i}}, \ldots, \underset{\frown\atop n}{0},\ \underset{\frown\atop n+1}{0}, \quad \ldots, \quad \underset{\frown\atop 2n+1}{0}\Big) \quad (0 \leqq i \leqq n)$$

$$\Big(0, \ldots, \quad 0,\ \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\Big). \quad \blacksquare$$

In other words, for any profile $(r_0, \ldots, r_n, g_0, \ldots, g_n)$ there are $\lambda_0, \ldots, \lambda_n$, $\lambda_{n+1} \geqq 0$ satisfying $\sum_{i=0}^{n+1} \lambda_i \leqq 1$

(37) $$\begin{pmatrix} r_0 \\ \vdots \\ \vdots \\ r_n \\ g_0 \\ \vdots \\ g_n \end{pmatrix} \leqq \lambda_0 \begin{pmatrix} \binom{n}{0} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ \binom{n}{1} \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \ldots + \lambda_n \begin{pmatrix} 0 \\ \vdots \\ \binom{n}{n} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \lambda_{n+1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \binom{n}{0} \\ \binom{n}{1} \\ \vdots \\ \binom{n}{n} \end{pmatrix}.$$

Summing up the inequalities $r_i \leqq \lambda_i \binom{n}{i}$ $(0 \leqq i \leqq n)$, we obtain

$$r = \sum_{i=0}^{n} r_i \leqq \sum_{i=0}^{n} \lambda_i \binom{n}{i} \leqq \binom{n}{\left[\frac{n}{2}\right]} \sum_{i=0}^{n} \lambda_i .$$

Hence

$$\lambda_{n+1} \leqq 1 - \sum_{i=0}^{n} \lambda_i \leqq 1 - \frac{r}{\binom{n}{\left[\frac{n}{2}\right]}}$$

follows. Substituting this into (37), it is easy to see that

$$\sum_{i=0}^{n} g_i \leqq \lambda_{n+1} \sum_{i=0}^{n} \binom{n}{i} \leqq \left(1 - \frac{r}{\binom{n}{\left[\frac{n}{2}\right]}}\right) 2^n .$$

As the number of red sets with multiplicity is rt, the middle part of the theorem of [1] is proved: *If the number of* sets occuring at least twice in an $(\mathcal{F}^1, \dots, \mathcal{F}^t) \in \mathbf{W}_t$ *is* r, *then*

$$\sum_{i=0}^{n} f_i \leqq rt + \left(1 - \frac{r}{\binom{n}{\left[\frac{n}{2}\right]}}\right) 2^n .$$

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