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All Maximum Size Two-Part Sperner Systems: In Short

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In this note we give a very short proof for the description of all maximum size two-part Sperner systems.

Katona [6] and Kleitman [8] independently observed that the statement of the Sperner theorem remains unchanged if the conditions are relaxed in the following way. Let $X = X_1 \cup X_2$ be a partition of the underlying set X, $|X_i| = n_i$, $n_1 + n_2 = n$ (with $n_1 \geqslant n_2$). We say that $\mathscr{F}$ is a *two-part Sperner family* if $E, F \in \mathscr{F}, E \subsetneq F \Rightarrow \forall i : (F \setminus E) \not\subseteq X_i$. It was proved that the size of a two-part Sperner family cannot exceed $\binom{n}{\lfloor n/2 \rfloor}$. It took 20 years to find all maximum size two-part Sperner families [3]. The description requires some more definitions.

The two-dimensional *profile matrix* $M(\mathscr{F})$ is defined by $M_{ij}(\mathscr{F}) = \#\{F \in \mathscr{F} : |X_1 \cap F| = i, |X_2 \cap F| = j\}$. This can be considered as a point in the real space $\mathbb{R}^{(n_1+1)(n_2+1)}$. The profile matrices of the two-part Sperner families determine a point set in this space, and it is known that the vertices of their polytope are the profile matrices of the homogeneous systems (see [2, Theorem 3.2]), where a family $\mathscr{F}$ is called *homogeneous* (with respect to the partition X_1, X_2) if $F \in \mathscr{F}$ implies $E \in \mathscr{F}$ for all sets E satisfying $\forall i : |E \cap X_i| = |F \cap X_i|$. A homogeneous family can be described by the set $I(\mathscr{F}) = \{(i_1, i_2) : \forall j : |F \cap X_j| = i_j$ for some $F \in \mathscr{F}\}$. If $\mathscr{F}$ is a homogeneous two-part Sperner family, then $I(\mathscr{F})$ cannot contain pairs with the same first or second components, respectively. Consequently we have $|I(\mathscr{F})| \leqslant n_2 + 1$. We say that a homogeneous family $\mathscr{F}$ is *full* if $|I(\mathscr{F})| = n_2 + 1$. Then, for

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every i_2 ($0 \leqslant i_2 \leqslant n_2$), there is a unique $f(i_2)$ such that $(f(i_2), i_2) \in I(\mathscr{F})$. A homogeneous family is called *well-paired* if it is full, $\{f(i) : i = 0, 1, \ldots, n_2\}$ is the (n_2+1)-element interval around $\lfloor n_1/2 \rfloor$ and

$$\binom{n_2}{i} < \binom{n_2}{j} \text{ implies } \binom{n_1}{f(i)} \leqslant \binom{n_1}{f(j)} \tag{1}$$

for every pair $1 \leqslant i, j \leqslant n_2$. It is clear that well-pairing is not unique.

Theorem 1 ([3]). *Let $\mathscr{F}$ be a two-part Sperner family with parts X_1, X_2. Then*

$$|\mathscr{F}| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

with equality if and only if $\mathscr{F}$ is a homogeneous well-paired family.

In [9] and [4] further proofs were given but none were really short or easy to understand. Here we give a very short proof of Theorem 1.

Proof. Let $F \subset X_2$. Then define $\mathscr{F}(F) := \{E \subset X_1 : E \cup F \in \mathscr{F}\}$. Now $\mathscr{F}(F)$ is a Sperner family, therefore, due to the well-known LYM (Lubell–Yamamoto–Meshalkin) inequality,

$$\sum_{E \in \mathscr{F}(F)} \frac{1}{\binom{n_1}{|E|}} \leqslant 1, \tag{2}$$

with equality if and only if $\mathscr{F}(F)$ consists of a full level in the subset-lattice of X_1. (For details see, for example, [7].) Summing this inequality over all i-element subsets of X_2, we have, for all $i = 0, \ldots, n_2$,

$$\sum_{F \in \binom{X_2}{i}} \sum_{E \in \mathscr{F}(F)} \frac{1}{\binom{n_1}{|E|}} \frac{1}{\binom{n_2}{i}} \leqslant 1. \tag{3}$$

Finally

$$\sum_{i=0}^{n_2} \sum_{F \in \binom{X_2}{i}} \sum_{E \in \mathscr{F}(F)} \frac{1}{\binom{n_1}{|E|}\binom{n_2}{i}} = \sum_{j,i} \frac{M_{ji}(\mathscr{F})}{\binom{n_1}{j}\binom{n_2}{i}} \leqslant n_2 + 1. \tag{4}$$

The middle term of inequality (4) is a linear function of the profile matrix, and it is easy to check that for all full homogeneous families (including well-paired homogeneous families) inequality (4) holds with equality. Therefore all maximum size two-part Sperner families must also satisfy it with equality.

Indeed, the cardinality of a family is the sum of the entries in its profile matrix. This is a positive linear function of the profile matrices, and therefore its maximum is attained only by points which are on the facets of the polytope spanned by the profile matrices of maximum size homogeneous two-part Sperner families. So the profile matrix of any maximum size two-part Sperner family is a convex linear combination of profile matrices of maximum size homogeneous families, and hence it satisfies any linear equality which is also satisfied by these vertices.

Consequently, for maximum size families inequalities (2) and (3) must hold with equality. Therefore, due to (3), each $F \subset X_2$ must be a trace on X_2 of some element of $\mathscr{F}$. Furthermore, for all $F \subset X_2$ the family $\mathscr{F}(F)$ must be a full level in the subset-lattice of X_1.

We consider first the case $n_1 = n_2$. In this case we can repeat the reasoning for all $E \subset X_1$, which proves that in this case each maximum size two-part Sperner system is homogeneous.

In the case of $n_1 > n_2$, denote by $\mathscr{F}(j) \subset 2^{X_2}$ the set of all $F \subset X_2$ such that $E \in \mathscr{F}(F)$ with a fixed $E \subset X_1, |E| = j$. (By the previous statement all j-element subsets of X_1 would define the same set.) Therefore, for all $j \in \{1, \ldots, n_1\}$, $\mathscr{F}(j)$ is a Sperner family, and $\mathscr{F}(j) \cap \mathscr{F}(j') = \emptyset$ holds for all $j \neq j'$. Furthermore, any t families among them form a t-Sperner family (their union does not contain a $(t+1)$-chain), and therefore, by Paul Erdős's theorem [1],

$$\sum_{k=1}^{t} |\mathscr{F}(j_k)|$$

is at most the sum of the t largest binomial coefficients. Now, the cardinality of our maximum size two-part Sperner system is

$$|\mathscr{F}| = \sum_{j=0}^{n_1} \binom{n_1}{j} |\mathscr{F}(j)|. \tag{5}$$

We get the largest value $\binom{n}{\lfloor n/2 \lfloor}$ in (5) with the following greedy algorithm. Let $B_0 \geqslant B_1 \geqslant \cdots \geqslant B_{n_1}$ and $A_0 \geqslant \cdots \geqslant A_{n_2}$ be enumerations of the binomial coefficients $\binom{n_1}{j}$ and $\binom{n_2}{i}$, respectively. If $A_0 > A_1$ then we have to match B_0 with A_0. We have already matched the pairs up to $k-1$. Now $A_k = A_{k+1}$. Then B_k and B_{k+1} must be matched with these two items. Recall that the corresponding $\mathscr{F}(j)$ and $\mathscr{F}(j')$ families are Sperner families, and so, by the LYM inequality, they may have that many elements if and only if each of the two families is equal to one full level in the subset lattice of X_2. So we get that every maximum size two-part Sperner family must be well-paired. □

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