Large void zones and occupation times for coalescing random walks

by

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Summary. The basic coalescing random walk is a system of interacting particles. These particles start from every site of $\mathbb{Z}^d$, and each moves independently as a continuous-time random walk. When two particles visit the same site, they coalesce into a single particle. We are interested in: (a) the radius $R_d(T)$ of the largest ball centered at the origin which does not contain any particle at time T; and (b) the amount of time $\Lambda_d(T)$ when the origin is occupied during $[0,T]$. We describe the almost sure asymptotic behaviours of $R_d(T)$ and $\Lambda_d(T)$ (when $T \to \infty$), in three different regimes depending on whether $d = 1$, $d = 2$ or $d \geq 3$.

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1 Introduction

We consider an elementary example of interacting particle systems: the basic model of coalescing random walk. Particles start at time $t = 0$ from every site of $\mathbb{Z}^d$, and execute independent continuous-time simple random walks. Each particle jumps at the times of a rate one Poisson process, and when it jumps from position $x \in \mathbb{Z}^d$, it jumps to any one of the $(2d)$ neighbour sites of x with equal probability $1/(2d)$. The only interaction between the particles is when a particle jumps to a site which is already occupied by another particle: in this case, the two particles coalesce into one, which goes on to move as a continuous-time simple random walk.

It is known that there is a duality between the coalescing random walk and the linear voter model (Griffeath [13]).

For each $x \in \mathbb{Z}^d$, let $(\xi_d^x(t),\ t \ge 0)$ denote the movement of the particle starting from position x. Let $\xi_d(t) := \{\xi_d^x(t),\ x \in \mathbb{Z}^d\}$; it is the set of sites in $\mathbb{Z}^d$ which are occupied by a particle at time t.

In the present paper, we are interested in $R_d(t)$, the radius of the largest ball centered at the origin which does not contain any site occupied by $\xi_d(t)$, i.e.,

$$R_d(t) := \inf_{x \in \xi_d(t)} \|x\|,$$

where $\|x\|$ denotes the Euclidean modulus of $x \in \mathbb{Z}^d$.

The distributional behaviour of $R_d(T)$ for large T is known. Indeed, Arratia [1] showed that $\xi_d(T)$, suitably normalized, converges in distribution to a non-Poissonian point process for $d = 1$, and to a Poisson point process when $d \ge 2$. Our aim is to study the almost sure asymptotic properties of $R_d(T)$. Of course, since the origin is occupied infinitely often (this is clear for $d = 1$ or 2, and is a consequence of Theorem 1.3 below for $d \ge 3$), it is meaningless to study the lim inf behaviour of $R_d(T)$.

Theorem 1.1 *We have*

$$\limsup_{T \to \infty} \frac{R_1(T)}{(T \log\log T)^{1/2}} = 1, \qquad \text{a.s.,} \tag{1.1}$$

$$c_1 \le \limsup_{T \to \infty} \frac{R_2(T)}{T^{1/2} (\log T)^{-1/2} (\log\log T)^{1/2}} \le c_2, \qquad \text{a.s.,} \tag{1.2}$$

$$c_3 \le \limsup_{T \to \infty} \frac{R_d(T)}{(T \log T)^{1/d}} \le c_4, \qquad \text{a.s.,} \qquad d \ge 3, \tag{1.3}$$

where c_1, c_2, $c_3 = c_3(d)$ and $c_4 = c_4(d)$ are finite positive constants.

In order to state our second theorem, we consider the occupation time defined by

$$\Lambda_d(T) := \int_0^T \mathbf{1}_{\{0\in\xi_d(t)\}}\,\mathrm{d}t, \tag{1.4}$$

($\mathbf{1}_A$ denoting the indicator of A). In words, $\Lambda_d(T)$ stands for the total amount of time before T during which the origin is occupied by the coalescing random walk. Sometimes $\Lambda_d(T)$ is also referred to as the local time at 0 of the random walk.

We recall what the "typical values" of $\Lambda_d(T)$ are. Let

$$p_d(t) := \mathbb{P}\{0 \in \xi_d(t)\}, \tag{1.5}$$

which denotes the probability that 0 (or any given site) is occupied by the coalescing walk at time t. By means of the duality with the voter model, Bramson and Griffeath [8] determined accurately the asymptotic behaviour of $p_d(t)$.

Throughout the paper, we write $a(t) \sim b(t)$ $(t\to\infty)$ to denote $\lim_{t\to\infty} a(t)/b(t) = 1$.

Fact 1.2 (Bramson and Griffeath [8]) *Let $p_d(t)$ be as in* (1.5). *As $t\to\infty$,*

$$p_1(t) \sim \frac{1}{\sqrt{\pi t}}, \tag{1.6}$$

$$p_2(t) \sim \frac{\log t}{2\pi t}, \tag{1.7}$$

$$p_d(t) \sim \frac{1}{\gamma_d\, t}, \qquad d\ge 3, \tag{1.8}$$

where γ_d, $d\ge 3$, denotes the probability that a d-dimensional simple symmetric random walk never returns to its starting point.

By Fubini's theorem, $\mathbb{E}[\,\Lambda_d(T)\,] = \int_0^T p_d(t)\,\mathrm{d}t$, from which it follows that, when $T\to\infty$,

$$\mathbb{E}[\,\Lambda_1(T)\,] \sim \frac{2\sqrt{T}}{\sqrt{\pi}}, \tag{1.9}$$

$$\mathbb{E}[\,\Lambda_2(T)\,] \sim \frac{(\log T)^2}{4\pi}, \tag{1.10}$$

$$\mathbb{E}[\,\Lambda_d(T)\,] \sim \frac{\log T}{\gamma_d}, \qquad d\ge 3. \tag{1.11}$$

It is easy to determine the asymptotic behaviour of $\Lambda_1(T)$. We have, almost surely $\Lambda_1(T) = T^{(1/2)+o(1)}$ for $T\to\infty$. Indeed, the lower bounds follows immediately by considering only a single particle (say the one starting from the origin), whereas the upper bound is a simple consequence of (1.9), Chebyshev's inequality on taking a subsequence and applying the Borel–Cantelli lemma.

Our next result concerns the almost sure asymptotic behaviour of $\Lambda_d(T)$ (for $d \geq 2$), in the form of a law of large numbers.

Theorem 1.3 *For $d \geq 2$ we have*

$$\lim_{t\to\infty} \frac{\Lambda_d(T)}{\mathbb{E}[\,\Lambda_d(T)\,]} = 1, \qquad \text{a.s.} \tag{1.12}$$

For an account of general properties of coalescing random walk, we refer to the books of Liggett [16] and [17], formulated in terms of the voter model. Other asymptotic properties of the occupation times of the voter model can be found in Cox and Griffeath [10], Bramson *et al.* [6]. Let us also mention a few recent papers. In van den Berg and Kesten [3]–[4], the exact asymptotic density of general coalescing random walks in high dimensions was determined. There is also an interesting relationship between the voter model and super-Brownian motion, recently discovered by Cox *et al.* [9] and Bramson *et al.* [7].

The rest of the paper is organized as follows. In Section 2, we prove probability estimates for coalescing random walks in any dimensions. These probability estimates will be used in the proof of the higher-dimensional parts — (1.2) and (1.3) — of Theorem 1.1 and in the proof of Theorem 1.3. More precisely, we prove in Section 3 the upper bounds in (1.2)–(1.3), and in Section 4 the corresponding lower bounds. In Section 5, we exploit some special one-dimensional features to prove (1.1), and thus complete the proof of Theorem 1.1. Finally, Theorem 1.3 is proved in Section 6.

For any set A of $\mathbb{Z}^d$, $\#(A)$ denotes the cardinality of A. The letter c with subscript denotes unimportant constants which are finite and positive.

2 Probability estimates for coalescing random walks

The main aim of the present section is to prove probability estimates (Propositions 2.1 and 2.2 below) for coalescing random walks in any dimensions. These estimates will be used in the next sections in the proof of (1.2), (1.3) and (1.12).

As before, $\xi_d(t)$ denotes the set of all the sites which are occupied by the coalescing random walk at time t.

Here are the main probability estimates of the section.

Proposition 2.1 *Let $d \geq 1$, and let $(a(x),\ x \in \mathbb{Z}^d)$ be a collection of non-negative numbers such that $\sum_{x\in\mathbb{Z}^d} a(x) < \infty$. For $T > 0$, let*

$$\nu_T := \sum_{x\in\xi_d(T)} a(x).$$

Then for any integer $k \geq 1$,

$$\mathbb{E}\{[\,\nu_T - \mathbb{E}(\nu_T)\,]^{2k}\} \leq (2k)^{2k} \sum_{x\in\mathbb{Z}^d} a^{2k}(x) p_d(T) + \left(c_5\, k \sum_{x\in\mathbb{Z}^d} a^2(x) p_d(T) \right)^k, \tag{2.1}$$

where $c_5 = c_5(d) \in (0, \infty)$ is a numerical constant, and $p_d(T)$ is defined in (1.5).

Proposition 2.2 *We have,*

$$\mathrm{Var}[\Lambda_1(T)] \;\leq\; c_6\, T, \qquad (T \geq 1) \tag{2.2}$$

$$\mathrm{Var}[\Lambda_2(T)] \;\leq\; c_7\, (\log T)^3, \qquad (T \geq 2) \tag{2.3}$$

$$\mathrm{Var}[\Lambda_d(T)] \;\leq\; c_8\, \log T, \qquad d \geq 3, \qquad (T \geq 2) \tag{2.4}$$

where c_6, c_7 and $c_8 = c_8(d)$ are (finite) constants.

The estimate (2.2) for $\mathrm{Var}[\Lambda_1(T)]$ is not of any use. It is stated in Proposition 2.2 only for the sake of completeness.

The rest of the section is devoted to the proofs of Propositions 2.1 and 2.2, which rely on the van den Berg–Kesten–Reimer (BKR) correlation inequality. We however do not need in this paper the full strength of the BKR inequality, and state it here only for the special binary case.

Let V be a finite set and let $\Omega = \{0,1\}^V$. For $\omega \in \Omega$ and $K \subset V$, let $[\omega]_K$ denote the set of all ω' which agree with ω on K: $\omega'_i = \omega_i$, $i \in K$. For A, $B \subset \Omega$, $A \Box B$ is defined as the set of all $\omega \in \Omega$ for which there exist disjoint K, $L \subset V$ with $[\omega]_K \subset A$, and $[\omega]_L \subset B$. The BKR inequality is recalled as follows.

Fact 2.3 (van den Berg and Kesten [4], Reimer [19]) *Let μ be a product measure on Ω. For any A, $B \subset \Omega$,*

$$\mu(A \,\Box\, B) \le \mu(A)\mu(B).$$

The BKR correlation inequality allows us to overcome some dependence difficulty in the study of coalescing random walks. Here is an application. We say that two $\mathbb{R}$-valued random variables X and Y are negatively dependent, if $\mathbb{P}\{X \ge a,\ Y \ge b\} \le \mathbb{P}\{X \ge a\}\,\mathbb{P}\{Y \ge b\}$ for all real numbers a and b. In the literature, this negative dependence bears the more technical name of "negative upper orthant dependence" (Joag-Dev and Proschan [14]).

Lemma 2.4 *Let $d \ge 1$, and $n \ge 1$. Let $x_1, \cdots, x_n$ and y be distinct sites in $\mathbb{Z}^d$, and let $a_1, \cdots, a_n$ be non-negative numbers. Then for any $T > 0$, $\sum_{i=1}^n a_i \mathbf{1}_{\{x_i \in \xi_d(T)\}}$ and $\mathbf{1}_{\{y \in \xi_d(T)\}}$ are negatively dependent.*

Proof of Lemma 2.4. When $n = 1$, this was proved by Arratia [1] and is also a special case of Lemma 2.4 of van den Berg and Kesten [4], p. 8. In [4] the model of coalescing random walk is more general, than in the present paper. We outline the proof of our Lemma 2.4, following the same discretisation schema and proof of Lemma 2.4 of van den Berg and Kesten [4]. For each $x \in \mathbb{Z}^d$ and $v \in \mathbb{Z}^d$ with $\|x - v\| = 1$ consider a Poisson point process (in time) with intensity $1/(2d)$ and for each Poisson point draw an arrow from x to v. These Poisson processes are assumed to be independent of each other. Now the coalescing random walk can be described as follows: a particle starting from site $x \in \mathbb{Z}^d$ at time $t = 0$ stays in that position until there is an outgoing arrow from that position and jumps to the other endpoint of that arrow. It stays in this new position until there is an outgoing arrow from this new position and again jumps to the other endpoint of this arrow, etc. If two particles are in the same position at the same time, they stay together forever (they coalesce). Now for fixed T consider a partition $[\ell\delta,\ (\ell+1)\delta)\ \ell = 1, 2, \ldots, k$ of $[0, T]$ such that $k\delta = T$. Consider the slight modification of the coalescing random walk in discrete time setting as follows. The particles postpone their jumps until the end of the time interval in which the corresponding arrow is located. Moreover, if the time interval has more than one outgoing arrow from a site in which a particle is located, the particle will stay in that location forever. For a fixed positive integer N such that $\|x_i\| \le N$, $i = 1, \cdots, n$, let $X_N := \{x \in \mathbb{Z}^d : \|x\| \le 2N\}$. We define $V := X_N \times K \times Q$, where $K := \{1, 2, \ldots, k\}$ and $Q := \{v \in \mathbb{Z}^d, \|v\| = 1\}$. For

$(z,\ell,v)\in V$ let $\omega_{(z,\ell,v)}=1$ if there is an outgoing arrow from z to $z+v$ in the time interval $[\ell\delta,(\ell+1)\delta)$ and $\omega_{z,\ell,v}=0$ otherwise. We clearly have

$$\mathbb{P}\{\omega_{z,\ell,v}=1\}=1-\exp\left(-\frac{\delta}{2d}\right),$$

and $\omega_{z,\ell,v}$, $z\in X_N$, $\ell\in K$, $v\in Q$ are i.i.d. random variables. Let $\widetilde{\xi}_d(T)$ be the set of occupied sites in this modified model. Fix $a>0$ and let

$$\begin{aligned} A_i &:= \{x_i\in\widetilde{\xi}_d(T)\}, \qquad i=1,\cdots,n,\\ B &:= \{y\in\widetilde{\xi}_d(T)\},\\ A &:= \left\{\sum_{i=1}^n a_i\mathbf{1}_{\{x_i\in\widetilde{\xi}_d(T)\}}\ge a\right\}, \end{aligned}$$

and let D be the event that for all $x\in X_N$ and $\ell\in K$ there is at most one outgoing arrow from x in the interval $[\ell\delta,(\ell+1)\delta)$. Since $a_1,\cdots,a_n$ are non-negative, there exists $J=J(a,a_1,\cdots,a_n)$, a set of subsets of $\{1,2,\cdots,n\}$ (depending on a and on a_i) such that

$$A=\bigcup_{\{j_1,\cdots,j_k\}\in J}(A_{j_1}\cap\cdots\cap A_{j_k}).$$

According to [4], pp. 9-10, there are sets $\widetilde{A}_i$, $i=1,\cdots,n$, $\widetilde{B}$, which are union of cylinders, such that $A_i\cap D=\widetilde{A}_i\cap D$ and $(A_i\cap B\cap D)\subset(\widetilde{A}_i\,\Box\,\widetilde{B})$, $i=1,\cdots,n$. Therefore,

$$\begin{aligned} A\cap B\cap D &= \bigcup_{\{j_1,\cdots,j_k\}\in J}[(A_{j_1}\cap B\cap D)\cap\cdots\cap(A_{j_k}\cap B\cap D)]\\ &\subset \bigcup_{\{j_1,\cdots,j_k\}\in J}\left[(\widetilde{A}_{j_1}\,\Box\,\widetilde{B})\cap\cdots\cap(\widetilde{A}_{j_k}\,\Box\,\widetilde{B})\right]. \end{aligned}$$

The sites $x_1,\cdots,x_n$ and y being distinct, we can choose a common $L\subset V$ in the definition of $\widetilde{A}_i\,\Box\,\widetilde{B}$ such that $[\omega]_{K_i}\subset\widetilde{A}_i$, and $[\omega]_L\subset\widetilde{B}$. Thus $\left[(\widetilde{A}_{j_1}\,\Box\,\widetilde{B})\cap\cdots\cap(\widetilde{A}_{j_k}\,\Box\,\widetilde{B})\right]\subset\left[(\widetilde{A}_{j_1}\cap\cdots\cap\widetilde{A}_{j_k})\,\Box\,\widetilde{B}\right]$. Accordingly, by Lemma 3.1(iv) of van den Berg and Fiebig [2],

$$\begin{aligned} A\cap B\cap D &\subset \bigcup_{\{j_1,\cdots,j_k\}\in J}\left[(\widetilde{A}_{j_1}\cap\cdots\cap\widetilde{A}_{j_k})\,\Box\,\widetilde{B}\right]\\ &\subset \left[\bigcup_{\{j_1,\cdots,j_k\}\in J}(\widetilde{A}_{j_1}\cap\cdots\cap\widetilde{A}_{j_k})\right]\Box\,\widetilde{B}\\ &= \widetilde{A}\,\Box\,\widetilde{B}, \end{aligned}$$

where

$$\widetilde{A} = \bigcup_{\{j_1,\cdots,j_k\}\in J} (\widetilde{A}_{j_1} \cap \cdots \cap \widetilde{A}_{j_k}).$$

By the BKR inequality (Fact 2.3), this implies

$$\begin{aligned}\mathbb{P}(A\cap B) &\le \mathbb{P}(\widetilde{A}\,\Box\,\widetilde{B}) + \mathbb{P}(D^c) \le \mathbb{P}(\widetilde{A})\mathbb{P}(\widetilde{B}) + \mathbb{P}(D^c)\\ &\le (\mathbb{P}(A)+\mathbb{P}(D^c))(\mathbb{P}(B)+\mathbb{P}(D^c)) + \mathbb{P}(D^c).\end{aligned}$$

Now one can go back to infinitely many particles and continuous time case by letting $\delta \to 0$ and $N \to \infty$ as in van den Berg and Kesten [4], p. 10. $\square$

Lemma 2.5 *Let $d \ge 1$, $n \ge 1$ and $k \ge 1$. For any distinct sites $x_1, \cdots, x_n$ in $\mathbb{Z}^d$, and any non-negative numbers $a_1, \cdots, a_n$, we have*

$$\mathbb{E}\left\{ \left(\sum_{i=1}^n a_i \left[\mathbf{1}_{\{x_i \in \xi_d(T)\}} - p_d(T) \right] \right)^{2k} \right\} \le \mathbb{E}\left\{ \left(\sum_{i=1}^n a_i \left[Y_i(T) - p_d(T) \right] \right)^{2k} \right\}, \tag{2.5}$$

where $p_d(T)$ is defined in (1.5), *and $Y_1(T), \cdots, Y_n(T)$ are independent random variables such that for any i, $Y_i(T)$ is distributed as $\mathbf{1}_{\{x_i\in\xi_d(T)\}}$.*

Proof of Lemma 2.5. According to Theorem 2 of Shao [20], if X_1 and X_2 are negatively dependent such that $\mathbb{E}(|X_1|^{2k} + |X_2|^{2k}) < \infty$, then

$$\mathbb{E}\{(X_1+X_2)^{2k}\} \le \mathbb{E}\{(Y_1+Y_2)^{2k}\}, \tag{2.6}$$

where Y_1 and Y_2 are independent random variables such that Y_i is distributed as X_i (for $i=1$ and 2). We mention that Shao [20] proved (2.6) for negatively associated random variables, and that for a pair of random variables (which is the case here), the properties of negative association and negative dependence are equivalent (Joag-Dev and Proschan [14]).

If $n=1$, (2.5) is trivial. For $n \ge 2$, we note that according to Lemma 2.4, the random variables $\sum_{i=1}^{n-1} a_i[\,\mathbf{1}_{\{x_i\in\xi_d(T)\}} - p_d(T)]$ and $a_n[\,\mathbf{1}_{\{x_n\in\xi_d(T)\}} - p_d(T)]$ are negatively dependent. Therefore, Lemma 2.5 follows from (2.6) by induction. $\square$

We now recall the well-known Rosenthal's inequality (see for example Petrov [18], p. 59).

Fact 2.6 (Rosenthal's inequality) *Let $k \geq 1$ be an integer, and let $Z_1, \cdots, Z_N$ be independent mean-zero random variables such that $\mathbb{E}[\,Z_i^{2k}\,] < \infty$ for all $i \leq N$. Then*

$$\mathbb{E}\left[\left(\sum_{i=1}^N Z_i\right)^{2k}\right] \leq c_9 \sum_{i=1}^N \mathbb{E}[Z_i^{2k}] + c_{10}\left(\sum_{i=1}^N \mathbb{E}[Z_i^2]\right)^k, \tag{2.7}$$

where $c_9 = c_9(k)$ and $c_{10} = c_{10}(k)$ are finite constants whose values depend only on k.

In order to prove Proposition 2.1, we will need to know in (2.7) the dependence on k of the constants c_9 and c_{10}. So let us recall a refined version of (2.7) in Petrov [18], p. 62, which states that (2.7) holds with $c_9 := r^{2k}$ and $c_{10} := 2kr^k \mathrm{e}^r B(k, r-k)$, for any $r > k$, where B is the beta function. Taking $r = 2k$, and (2.7) becomes:

$$\mathbb{E}\left[\left(\sum_{i=1}^N Z_i\right)^{2k}\right] \leq (2k)^{2k} \sum_{i=1}^N \mathbb{E}[\,Z_i^{2k}\,] + (c_{11}\,k)^k\left(\sum_{i=1}^N \mathbb{E}[\,Z_i^2\,]\right)^k, \tag{2.8}$$

where $c_{11} \in (0,\infty)$ is an absolute constant.

We have now all the ingredients to prove Proposition 2.1.

Proof of Proposition 2.1. Let A be a finite subset of $\mathbb{Z}^d$, and consider the random variable $\nu_T^A := \sum_{x\in A} a(x)\mathbf{1}_{\{x\in\xi_d(T)\}}$. By Lemma 2.5 and (2.8), we have

$$\begin{aligned}\mathbb{E}\{[\,\nu_T^A - \mathbb{E}(\nu_T^A)\,]^{2k}\} \;&\leq\; (2k)^{2k}\sum_{x\in A} a^{2k}(x)\,\mathbb{E}\left\{\left[\mathbf{1}_{\{x\in\xi_d(T)\}} - p_d(T)\right]^{2k}\right\}\\ &\quad + (c_{11}\,k)^k\left(\sum_{x\in A} a^2(x)\,\mathbb{E}\left\{\left[\mathbf{1}_{\{x\in\xi_d(T)\}} - p_d(T)\right]^2\right\}\right)^k.\end{aligned}$$

Since $\mathbb{E}\{\,[\,\mathbf{1}_{\{x\in\xi_d(T)\}} - p_d(T)\,]^{2k}\,\} \leq \mathbb{E}\{\,\mathbf{1}_{\{x\in\xi_d(T)\}}\,\} = p_d(T)$, this implies

$$\mathbb{E}\{[\,\nu_T^A - \mathbb{E}(\nu_T^A)\,]^{2k}\} \leq (2k)^{2k}\sum_{x\in\mathbb{Z}^d} a^{2k}(x)p_d(T) + \left(c_{11}\,k\sum_{x\in\mathbb{Z}^d} a^2(x)\,p_d(T)\right)^k.$$

Take $A = A_n := \{x \in \mathbb{Z}^d :\ \|x\| \leq n\}$ and let $n \to \infty$. The monotone convergence theorem ensures that $\mathbb{E}(\nu_T^{A_n}) \to \mathbb{E}(\nu_T)$. Therefore, we obtain the proposition by an application of Fatou's lemma, with $c_5 := c_{11}$. $\square$

We now turn to the proof of Proposition 2.2. We first need a simple correlation result. Recall that $\xi_d^x(t)$ denotes the position at time t of the particle starting from x.

Lemma 2.7 *Let $d \geq 1$. For $t > s \geq 0$,*

$$\mathbb{P}\left(\bigcup_{x\in\mathbb{Z}^d} \{\xi_d^x(s) = 0,\ \xi_d^x(t) \neq 0,\ 0 \in \xi_d(t)\} \right) \leq p_d(s)\, p_d(t), \tag{2.9}$$

where $p_d(\cdot)$ is defined in (1.5).

Proof of Lemma 2.7. Recall that in case $n = 1$ our Lemma 2.4 is equivalent to the inequality

$$\mathbb{P}\left(\bigcup_{x\in\mathbb{Z}^d} \{\xi_d^x(t) = y,\ 0 \in \xi_d(t)\} \right) \leq \mathbb{P}\left(\bigcup_{x\in\mathbb{Z}^d} \{\xi_d^x(t) = y\} \right) \mathbb{P}\{0 \in \xi_d(t)\}$$

for $y \neq 0$, the proof of which was given in Lemma 2.4 of van den Berg and Kesten [4], p. 8. The very same proof shows also that for $s < t$ and $y \neq 0$,

$$\begin{aligned} &\mathbb{P}\left(\bigcup_{x\in\mathbb{Z}^d} \{\xi_d^x(s) = 0,\ \xi_d^x(t) = y,\ 0 \in \xi_d(t)\} \right) \\ \leq\ &\mathbb{P}\left(\bigcup_{x\in\mathbb{Z}^d} \{\xi_d^x(s) = 0,\ \xi_d^x(t) = y\} \right) \mathbb{P}\{0 \in \xi_d(t)\}. \end{aligned}$$

Now (2.9) follows by summing over $y \in \mathbb{Z}^d \backslash \{0\}$. $\square$

The section ends with the proof of Proposition 2.2.

Proof of Proposition 2.2. Observe that

$$\mathbb{E}\{[\Lambda_d(T)]^2\} = 2 \iint_{0\leq t_1 < t_2 \leq T} \mathbb{P}\{0 \in \xi_d(t_1),\ 0 \in \xi_d(t_2)\}\, \mathrm{d}t_1\, \mathrm{d}t_2.$$

Moreover, for $t_1 < t_2$,

$$\begin{aligned} \mathbb{P}\{0 \in \xi_d(t_1),\ 0 \in \xi_d(t_2)\} &= \mathbb{P}\left\{\xi_d^x(t_1) = \xi_d^x(t_2) = 0 \text{ for some } x \in \mathbb{Z}^d\right\} \\ &\quad + \mathbb{P}\left\{\xi_d^x(t_1) = 0 \neq \xi_d^x(t_2) \text{ for some } x \in \mathbb{Z}^d,\ 0 \in \xi_d(t_2)\right\}. \end{aligned}$$

Let $q_t(0,0) := \mathbb{P}\{S_d^0(t) = 0\}$, where S_d^0 denotes a continuous-time rate one random walk starting from $0 \in \mathbb{Z}^d$. (We have ignored the dependence of $q_t(0,0)$ in d.) Then

$$\mathbb{P}\left\{\xi_d^x(t_1) = \xi_d^x(t_2) = 0 \text{ for some } x \in \mathbb{Z}^d\right\} = p_d(t_1)\, q_{t_2 - t_1}(0,0),$$

whereas by Lemma 2.7,

$$\mathbb{P}\left\{\xi_d^x(t_1) = 0 \neq \xi_d^x(t_2) \text{ for some } x \in \mathbb{Z}^d,\ 0 \in \xi_d(t_2)\right\} \leq p_d(t_1)\, p_d(t_2).$$

Therefore,

$$\mathbb{E}\{[\Lambda_d(T)]^2\} \leq 2 \iint_{0 \leq t_1 < t_2 \leq T} p_d(t_1) q_{t_2 - t_1}(0,0)\, \mathrm{d}t_1\, \mathrm{d}t_2 + \left(\int_0^T p_d(t)\, \mathrm{d}t \right)^2,$$

or, equivalently,

$$\mathrm{Var}[\Lambda_d(T)] \leq 2 \iint_{0 \leq t_1 < t_2 \leq T} p_d(t_1) q_{t_2 - t_1}(0,0)\, \mathrm{d}t_1\, \mathrm{d}t_2.$$

Now the proposition follows from Fact 1.2 and the well-known estimate $q_t(0,0) \leq \frac{c_{12}}{(t+1)^{d/2}}$ (for any $d \geq 1$ and some $c_{12} = c_{12}(d)$). $\square$

3 Proof of (1.2) and (1.3): upper bounds

This section is devoted to the proof of the upper bounds in the higher-dimensional parts ($d = 2$ and $d \geq 3$) of Theorem 1.1. We start with a preliminary estimate which holds in any dimension.

Lemma 3.1 *Let $d \geq 1$, and let $A \in \mathbb{Z}^d$ be a finite non-empty set. For any $T > 0$ and any integer $k \geq 1$,*

$$\mathbb{P}\{\, A \cap \xi_d(T) = \emptyset \,\} \leq \frac{(2k)^{2k}}{[\, p_d(T)\, \#(A)\,]^{2k-1}} + \frac{(c_5\, k)^k}{[\, p_d(T)\, \#(A)\,]^k},$$

where c_5 is the numerical constant in (2.1)*, and $p_d(T)$ is defined in* (1.5).

Proof of Lemma 3.1. Write $\nu_T := \#\{A \cap \xi_d(T)\} = \sum_{x \in \xi_d(T)} \mathbf{1}_{\{x \in A\}}$. Then $\{\, A \cap \xi_d(T) = \emptyset \,\} = \{\, \nu_T = 0 \,\}$. Note that

$$\mathbb{E}(\nu_T) = \mathbb{E}\left(\sum_{x \in \mathbb{Z}^d} \mathbf{1}_{\{x \in A\}}\, \mathbf{1}_{\{x \in \xi_d(T)\}} \right) = p_d(T)\, \#(A).$$

By Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}\{\nu_T = 0\} &\le \mathbb{P}\{\,|\nu_T - \mathbb{E}(\nu_T)| \ge \mathbb{E}(\nu_T)\} \\
&\le \frac{\mathbb{E}\{\,[\,\nu_T - \mathbb{E}(\nu_T)\,]^{2k}\}}{[\,\mathbb{E}(\nu_T)\,]^{2k}} \\
&= \frac{\mathbb{E}\{\,[\,\nu_T - \mathbb{E}(\nu_T)\,]^{2k}\}}{[\,p_d(T)\,\#(A)\,]^{2k}}.
\end{aligned}$$

It suffices now to apply Proposition 2.1 to $a(x) := \mathbf{1}_{\{x\in A\}}$. □

Proof of (1.3): upper bound. Assume $d \ge 3$. Let $\lambda \ge 1$ be a constant whose value will be determined later on, and let

$$C_d(T) := \left\{ x \in \mathbb{Z}^d : \ |x| \le (\lambda T \log T)^{1/d} \right\},$$

where $|x| := \max_{1\le i\le d} |x^{(i)}|$ denotes the L^∞-norm of $x := (x^{(1)}, \cdots, x^{(d)}) \in \mathbb{Z}^d$. In words, $C(T)$ denotes the set of lattice points in the cube centered at the origin with side length $2\lfloor(\lambda T \log T)^{1/d}\rfloor$.

Note that $\#(C_d(T)) \sim 2^d \lambda T \log T$ (for $T \to \infty$). On the other hand, according to (1.8), $p_d(T) \sim 1/(\gamma_d T)$, $T \to \infty$. Thus, for all sufficiently large T (how large depending on d and λ),

$$p_d(T)\#(C_d(T)) \ge \frac{2^{d-1}\lambda}{\gamma_d} \log T := c_{13}\,\lambda \log T,$$

where $c_{13} = c_{13}(d) := 2^{d-1}/\gamma_d$. Applying Lemma 3.1 to $A := C_d(T)$ and $k := \lfloor \log T \rfloor$, we obtain that for large T,

$$\begin{aligned}
\mathbb{P}\{\,C_d(T) \cap \xi_d(T) = \emptyset\,\} &\le \frac{(2k)^{2k}}{[\,c_{13}\,\lambda \log T\,]^{2k-1}} + \frac{(c_5\,k)^k}{[\,c_{13}\,\lambda \log T\,]^k} \\
&= c_{13}\,\lambda(\log T)\left(\frac{2k}{c_{13}\,\lambda \log T}\right)^{2k} + \left(\frac{c_5\,k}{c_{13}\,\lambda \log T}\right)^{k} \\
&\le c_{13}\,\lambda(\log T)\left(\frac{2}{c_{13}\,\lambda}\right)^{2k} + \left(\frac{c_5}{c_{13}\,\lambda}\right)^{k}
\end{aligned}$$

We now choose the constant λ so large that $\frac{2}{c_{13}\,\lambda} \le \mathrm{e}^{-1}$ and $\frac{c_5}{c_{13}\,\lambda} \le \mathrm{e}^{-2}$. Then for large T,

$$\mathbb{P}\{\,C_d(T) \cap \xi_d(T) = \emptyset\,\} \le \frac{c_{13}\,\lambda \log T + 1}{\mathrm{e}^{2k}} \le \frac{(c_{13}\,\lambda \log T + 1)\mathrm{e}^2}{T^2},$$

the last inequality following from the fact that $k \ge \log T - 1$.

By the Borel–Cantelli lemma, almost surely for all large integer n, $C_d(n) \cap \xi_d(n) \neq \emptyset$. There exists thus at least a particle, say $\xi_d^{x_n}(n)$, such that $\xi_d^{x_n}(n) \in C_d(n)$. Since $(\xi_d^{x_n}(n+t) - \xi_d^{x_n}(n),\ t \geq 0)$ is a continuous-time random walk, we have

$$
\begin{aligned}
& \mathbb{P}\left\{ \sup_{t\in[0,1]} \|\xi_d^{x_n}(n+t) - \xi_d^{x_n}(n)\| \geq (3\log n)^{1/2} \right\} \\
= \ & \mathbb{P}\left\{ \sup_{t\in[0,1]} \|\xi_d^{0}(t)\| \geq (3\log n)^{1/2} \right\} \\
\leq \ & \frac{1}{n^{3/2}},
\end{aligned}
$$

for all large n. Therefore, $\sup_{t\in[0,1]} \|\xi_d^{x_n}(n+t) - \xi_d^{x_n}(n)\| \leq (3\log n)^{1/2}$ almost surely. This implies that almost surely for all large T, there is at least a particle which lies in the cube centered at the origin with side length $2\lfloor (\lambda T \log T)^{1/d} \rfloor + (3\log T)^{1/2}$. As a consequence,

$$
\limsup_{T\to\infty} \frac{R_d(T)}{(T\log T)^{1/d}} \leq d^{1/2}\lambda^{1/d}, \qquad \text{a.s.,}
$$

yielding the upper bound in (1.3). □

Proof of (1.2): upper bound. The proof of the upper bound in (1.2) is along similar lines as in the case $d \geq 3$, so let us outline the argument, and emphasize on the modifications. First, the size of the cube $C_d(T)$ is changed: we should replace $C_d(T)$ by $C_2(T) := \{x \in \mathbb{Z}^2 : |x| \leq (\lambda T)^{1/2}(\log T)^{-1/2}(\log\log T)^{1/2}\}$. The reason for this change in the choice of $C_d(T)$ is that the estimate $p_d(T) \sim 1/(\gamma_d T)$ for $d \geq 3$ is now replaced by $p_2(T) \sim (\log T)/(2\pi T)$, as stated in (1.7). For the new choice of $C_2(T)$, we apply Proposition 2.1 to $k := \lfloor \log\log T \rfloor$ (instead of $\lfloor \log T \rfloor$ in dimension $d \geq 3$) to see that, if the constant $\lambda > 0$ is chosen sufficiently large, then for all large T,

$$
\mathbb{P}\left\{ C_2(T) \cap \xi_2(T) = \emptyset \right\} \leq \frac{c_{14}\, \lambda \log\log T}{(\log T)^4},
$$

where $c_{14} \in (0,\infty)$ is a constant. Taking the subsequence $T_n := \exp(n^{1/3})$ and applying the Borel–Cantelli lemma, this yields that almost surely for all large n, $C_2(T_n) \cap \xi_2(T_n) \neq \emptyset$. On the other hand, a Borel–Cantelli argument says that for large n, the increment size of a given particle during $[T_n, T_{n+1}]$ cannot exceed $(T_{n+1} - T_n)^{1/2}(3\log n)^{1/2}$ (thus cannot exceed $c_{15}\, T_n^{1/2}(\log T_n)^{-1}(\log\log T_n)^{1/2}$ a fortiori, for some constant c_{15}). Consequently, almost surely for all large T, there is at least a particle lying in the cube centered at the origin with side length $2\lfloor(\lambda T)^{1/2}(\log T)^{-1/2}(\log\log T)^{1/2}\rfloor + c_{15}\, T^{1/2}(\log T)^{-1}(\log\log T)^{1/2}$. This yields the upper bound in (1.2). $\square$

Remark. The argument in this section of course applies also in dimension $d = 1$, and gives that

$$\limsup_{T\to\infty} \frac{R_1(T)}{T^{1/2}\log\log T} \le c_{16}, \qquad \text{a.s.},$$

for some constant $c_{16} > 0$. However, this is a poor estimate, since by considering only the particle starting from the origin and using the usual iterated logarithm law, we know that $\limsup_{T\to\infty} \frac{R_1(T)}{T^{1/2}(\log\log T)^{1/2}} \le 2^{1/2}$ almost surely. $\square$

4 Proof of (1.2) and (1.3): lower bounds

To prove the lower bounds in (1.2) and (1.3), we first study a single particle $(S_d^x(t),\ t \ge 0)$ which is a continuous-time rate one random walk starting from $x \in \mathbb{Z}^d$. We write $q_t(x, y)$ for the probability density of the random walk: $q_t(x, y) := \mathbb{P}\{S_d^x(t) = y\}$.

For any non-empty subset A of $\mathbb{Z}^d$, let $\mathrm{diam}(A) := \sup\{\|x - y\| : x \in A,\ y \in A\}$ denote the diameter of A, and let $q_t(x, A) := \mathbb{P}\{S_d^x(t) \in A\}$. We start with two preliminary estimates.

Lemma 4.1 *Let $d \ge 1$, and let A be a non-empty subset of $\mathbb{Z}^d$. Let $0 < s < T$ be such that $(T - s)^{1/2} \ge \mathrm{diam}(A)$. Then*

$$\mathbb{P}\left\{\xi_d(T) \cap A = \emptyset \,|\, \mathcal{F}_s\right\} \ge \exp\left\{-c_{17} \sum_{x\in\xi_d(s)} q_{T-s}(x, A)\right\}, \qquad \text{a.s.,} \tag{4.1}$$

where $c_{17} = c_{17}(d) \in (0, \infty)$ is a constant depending only on d, and $\mathcal{F}_s := \sigma\{\xi_d^x(t)\ t \in [0, s],\ x \in \mathbb{Z}^d\}$.

Proof of Lemma 4.1. Consider the following modified model of particle system: until time s, it is our coalescing random walk $(\xi_d^x(t);\ t \in [0, s])$, and for $t \in [s, T]$, the particles keep moving independently *without coalescence.* We denote by $\overline{\xi}_d(T)$ (which of course depends on s) the set of all the sites which are occupied at time T by the new system of particles.

Without loss of generality, we make a coupling for the two models into a same probability space, so that $\overline{\xi}_d(T) \supset \xi_d(T)$. Thus

$$\mathbb{P}\left\{\xi_d(T) \cap A = \emptyset \,|\, \mathcal{F}_s\right\} \geq \mathbb{P}\left\{\,\overline{\xi}_d(T) \cap A = \emptyset \,|\, \mathcal{F}_s\right\}. \tag{4.2}$$

Given $\mathcal{F}_s$, $\overline{\xi}_d(T)$ is by definition the set of all the sites in $\mathbb{Z}^d$ occupied by independent random walks (without coalescence) at time $(T-s)$ starting from every site of $\xi_d(s)$. Accordingly,

$$\mathbb{P}\left\{\,\overline{\xi}_d(T) \cap A = \emptyset \,|\, \mathcal{F}_s\right\} = \prod_{x \in \xi_d(s)} [\, 1 - q_{T-s}(x, A)\,]. \tag{4.3}$$

Since $(T-s)^{1/2} \geq \operatorname{diam}(A)$, we have $\sup_{x \in \mathbb{Z}^d} q_{T-s}(x, A) \leq c_{18} < 1$ for some constant $c_{18} = c_{18}(d)$. As a consequence, there exists $c_{19} = c_{19}(d) \in (0, \infty)$ such that $1 - q_{T-s}(x, A) \geq \exp\{-c_{19}\, q_{T-s}(x, A)\}$ for all $x \in \mathbb{Z}^d$. Plugging this into (4.3) and (4.2) yields the lemma. □

Lemma 4.2 *Let $d \geq 1$, and let A be a subset of $\mathbb{Z}^d$ containing at least two points. Let $u > 0$ and $v > 0$, and let $\Xi := \sum_{x \in \xi_d(v)} q_u(x, A)$. Then*

$$\mathbb{E}(\Xi) = p_d(v)\, \#(A). \tag{4.4}$$

Furthermore, for any integer $k \geq 1$,

$$\begin{aligned} \mathbb{E}\left\{\, [\, \Xi - \mathbb{E}(\Xi)\,]^{2k}\,\right\} \;&\leq\; k^{2k} p_d(v)\, \#(A) \left(c_{20}\, \frac{\operatorname{diam}(A)}{u^{1/2}} \right)^{2k-1} \\ &\quad + \left(c_{20}\, k\, p_d(v)\, \#(A)\, \frac{\operatorname{diam}(A)}{u^{1/2}} \right)^{k}, \end{aligned} \tag{4.5}$$

where $c_{20} = c_{20}(d) \in (0, \infty)$ is a constant depending only on d.

Proof of Lemma 4.2. Since $\Xi = \sum_{x \in \mathbb{Z}^d} q_u(x, A)\, \mathbf{1}_{\{x \in \xi_d(v)\}}$, it follows from Fubini's theorem that

$$\mathbb{E}(\Xi) = \sum_{x \in \mathbb{Z}^d} q_u(x, A)\, \mathbb{P}\{x \in \xi_d(v)\} = \sum_{x \in \mathbb{Z}^d} q_u(x, A)\, p_d(v). \tag{4.6}$$

By symmetry, $q_u(x, y) = q_u(y, x)$, so that

$$\sum_{x \in \mathbb{Z}^d} q_u(x, A) = \sum_{x \in \mathbb{Z}^d} \sum_{y \in A} q_u(x, y) = \sum_{y \in A} \sum_{x \in \mathbb{Z}^d} q_u(y, x) = \sum_{y \in A} 1 = \#(A), \tag{4.7}$$

which, in view of (4.6), implies (4.4).

To check (4.5), we apply Proposition 2.1 to $T := v$ and $a(x) := q_u(x, A)$ to see that, for any integer $k \geq 1$,

$$\mathbb{E}\Big\{ [\, \Xi_n - \mathbb{E}(\Xi_n)\,]^{2k} \Big\} \leq (2k)^{2k} \sum_{x\in A} q_u^{2k}(x, A)\, p_d(v) + \left(c_5\, k \sum_{x\in A} q_u^2(x, A) p_d(v) \right)^k .$$

For any $b > 1$, in view of (4.7), we have

$$\sum_{x\in A} q_u^b(x, A) \leq \left(\sup_{x\in\mathbb{Z}^d} q_u^{b-1}(x, A) \right) \sum_{x\in\mathbb{Z}^d} q_u(x, A) \leq \#(A) \sup_{x\in\mathbb{Z}^d} q_u^{b-1}(x, A).$$

Since $\sup_{x\in\mathbb{Z}^d} q_u(x, A) \leq c_{21}\, u^{-1/2}\mathrm{diam}(A)$ for some $c_{21} = c_{21}(d) \in (0, \infty)$, this yields (4.5), and completes the proof of Lemma 4.2. $\square$

Proof of (1.3): lower bound. Assume $d \geq 3$. Consider the subsequence $T_n := n^\alpha$, where $\alpha > (d+2)/(d-2)$. Let $C_d(T) := \{\, x \in \mathbb{Z}^d : \ |x| \leq (\lambda T \log T)^{1/d} \,\}$. Here,

$$\lambda = \lambda(d, \alpha) := \frac{\gamma_d}{2^{d+1} c_{17}\, \alpha}, \tag{4.8}$$

where c_{17} is the constant in (4.1). Let

$$A_n := \{\, \xi_d(T_n) \cap C_d(T_n) = \emptyset \,\} .$$

For each n, A_n is measurable with respect to $\mathcal{F}_{T_n}$. If we could show that

$$\sum_n \mathbb{P}\{\, A_{n+1} \,|\, \mathcal{F}_{T_n} \,\} = \infty, \qquad \text{a.s.}, \tag{4.9}$$

then according to Lévy's version of the Borel–Cantelli lemma (see for example Shiryaev [21], p. 486), we would have $\mathbb{P}\{A_n \text{ i.o.}\} = 1$, which, in turn, would imply that for $d \geq 3$,

$$\limsup_{T\to\infty} \frac{R_d(T)}{(T \log T)^{1/d}} \geq \lambda^{1/d}, \qquad \text{a.s.}$$

This would yield the lower bound in (1.3).

It remains to verify (4.9). Applying Lemma 4.1 to $s := T_n$, $T := T_{n+1}$ and $A := C_d(T_{n+1})$ implies that, when n is sufficiently large (so that the condition $(T - s)^{1/2} \geq \mathrm{diam}(A)$ is fulfilled; recalling that $\alpha > (d+2)/(d-2)$)

$$\begin{aligned} \mathbb{P}\{\, A_{n+1} \,|\, \mathcal{F}_{T_n} \,\} \quad &\geq \quad \exp\left\{ -c_{17} \sum_{x\in\xi_d(T_n)} q_{T_{n+1}-T_n}(x, C_d(T_{n+1})) \right\} \\ &:= \quad \exp\{-c_{17}\, \Xi_n\}, \end{aligned}$$

with obvious notation. Applying Lemma 4.2 to $u := T_{n+1} - T_n$, $v := T_n$, $A := C_d(T_{n+1})$ and $k = 1$, and in light of Chebyshev's inequality, we obtain:

$$\mathbb{P}\{ |\Xi_n - \mathbb{E}(\Xi_n)| > \mathbb{E}(\Xi_n) \} \le \frac{\mathrm{Var}(\Xi_n)}{[\,\mathbb{E}(\Xi_n)\,]^2} \le \frac{2c_{20}\,\mathrm{diam}(C_d(T_{n+1}))}{(T_{n+1}-T_n)^{1/2} p_d(T_n)\,\#(C_d(T_{n+1}))}.$$

In view of (1.8), this yields

$$\mathbb{P}\{ |\Xi_n - \mathbb{E}(\Xi_n)| > \mathbb{E}(\Xi_n) \} \le \frac{c_{22}}{n^{(\alpha-1)/2-(\alpha/d)}(\log n)^{(d-1)/d}}.$$

Since $(\alpha-1)/2-(\alpha/d) > 1$ (recalling that $\alpha > (d+2)/(d-2)$), the expression on the right hand side is summable in n. By the Borel–Cantelli lemma (and (4.4) for the expression of $\mathbb{E}(\Xi_n)$), almost surely for all large n,

$$\Xi_n \le 2\mathbb{E}(\Xi_n) = 2p_d(T_n)\,\#(C_d(T_{n+1})) \sim \frac{2^{d+1}\alpha\lambda}{\gamma_d}\,\log n.$$

Therefore, almost surely for all large n, $\Xi_n \le (2^{d+2}\alpha\lambda/\gamma_d)\log n$. Since $\mathbb{P}\{ A_{n+1} \,|\, \mathcal{F}_{T_n} \} \ge \exp\{-c_{17}\,\Xi_n\}$, and in view of (4.8), we obtain:

$$\sum_n \mathbb{P}\{ A_{n+1} \,|\, \mathcal{F}_{T_n} \} = \infty, \qquad \text{a.s.}$$

This yields (4.9), and completes the proof of the lower bound in (1.3). $\square$

Proof of (1.2): lower bound. The proof of the lower bound in (1.2) follows similar lines as in the case of $d \ge 3$. We feel free to write only an outline of the argument. Take $T_n := \mathrm{e}^n$, and let $C_2(T) := \{ x \in \mathbb{Z}^d : \; |x| \le (\lambda T)^{1/2}(\log T)^{-1/2}(\log\log T)^{1/2} \}$. Consider $A_n := \{ \xi_d(T_n) \cap C_d(T_n) = \emptyset \}$. Again, as in the case of $d \ge 3$, we get via Lemma 4.1 that $\mathbb{P}\{ A_{n+1} \,|\, \mathcal{F}_{T_n} \} \ge \exp(-c_{17}\,\Xi_n)$, where $\Xi_n := \sum_{x\in\xi_d(T_n)} q_{T_{n+1}-T_n}(x, C_d(T_{n+1}))$. This time, we apply Lemma 4.2 to $k = 3$ to see that

$$\mathbb{P}\{ |\Xi_n - \mathbb{E}(\Xi_n)| > \mathbb{E}(\Xi_n) \} \le \frac{\mathbb{E}\{ [\,\Xi_n - \mathbb{E}(\Xi_n)\,]^6\}}{[\,\mathbb{E}(\Xi_n)\,]^6} \le \frac{c_{23}(\log n)^{9/2}}{n^{3/2}},$$

which is summable for n. The Borel–Cantelli lemma yields that almost surely for all large n, $\Xi_n \le 2\mathbb{E}(\Xi_n) = 2p_d(T_n)\#(C_d(T_{n+1})) \sim (4\lambda\mathrm{e}/\pi)\log n$. It is therefore possible to choose the constant $\lambda > 0$ to be so small that $\sum_n \exp(-c_{17}\,\Xi_n) = \infty$ almost surely. This yields $\sum_n \mathbb{P}\{ A_{n+1} \,|\, \mathcal{F}_{T_n} \} = \infty$ a.s., and implies the lower bound in (1.2). $\square$

Remark. Applying Lemmas 4.1 and 4.2 in dimension $d = 1$ gives that

$$\limsup_{T\to\infty} \frac{R_1(T)}{T^{1/2}} \geq c_{24}, \qquad \text{a.s.},$$

for some constant $c_{24} > 0$. Again, this does not yield the optimal rate function for $R_1(T)$, which should be $(T \log\log T)^{1/2}$. Therefore, the argument using Proposition 2.1 leads to the correct rate function for $R_d(T)$ for all dimensions except for $d = 1$. Fortunately, in the next Section, we will use some special features in dimension $d = 1$ to obtain not only the correct rate function for $R_1(T)$, but also the correct constant. $\square$

5 Proof of Theorem 1.1: the one-dimensional case

In this section, we assume $d = 1$, and prove the one-dimensional part (i.e., identity (1.1)) in Theorem 1.1. In dimension $d = 1$, if ξ_1^x and ξ_1^y (particles starting from x and y, respectively) coalesce together before time T, then any particle whose starting position is between x and y also coalesces with them before time T. This special property will considerably simplify the study, and will allow us to obtain even the correct constant in the iterated logarithm law for $R_1(T)$.

The proof of (1.1) is divided into two parts.

Proof of (1.1): upper bound. Fix $\delta \in (0, 1/2)$ and let

$$\begin{aligned} \varphi(t) = \varphi_\delta(t) &:= \sqrt{(1+4\delta)t \log\log t}\,, \\ a_k = a_k(T) &:= k\lfloor \delta\varphi(T)\rfloor, \qquad k = 0, 1, 2, \cdots \end{aligned}$$

We first estimate $\mathbb{P}\{E(T)\}$, where

$$E(T) := \bigcap_{k=0}^{N} \left\{|\xi_1^{-a_k}(T)| > \varphi(T)\right\} \cap \bigcap_{k=0}^{N} \left\{|\xi_1^{a_k}(T)| > \varphi(T)\right\}$$

with $N := \left\lfloor \frac{1}{\delta} \right\rfloor$. Clearly, $\{R_1(T) > \varphi(T)\} \subset E(T)$.

By symmetry,

$$\begin{aligned} \mathbb{P}\{E(T)\} &= 2\mathbb{P}\left\{\xi_1^0(T) > \varphi(T),\ E(T)\right\} \\ &= 2\mathbb{P}\left\{\xi_1^0(T) > \varphi(T),\ |\xi_1^{-a_k}(T)| > \varphi(T),\ \forall 1 \leq k \leq N\right\}. \end{aligned}$$

It turns out to be more convenient to estimate $\mathbb{P}\{E(T)\}$ in terms of independent random walks (without coalescence), instead of the original coalescing random walks $(\xi_1(t),\ t \geq 0)$.

Let $(S_1^x(t)+x,\ t\geq 0)_{x\in\mathbb{Z}}$ be a family of independent (continuous-time) simple random walks with $S_1^x(0)=0$. We will be working on the independent random walks $(S_1^x+x)_{x\in\mathbb{Z}^d}$ (without coalescence) instead of the original coalescing random walks $(\xi_1(t),\ t\geq 0)$.

Let $I=I(T):=\max\{i\geq 1:\ S_1^{-a_i}(t)-a_i\geq S_1^0(t)$ for some $t\in[0,T]\}+1$. In words, $S_1^{-a_I}-a_I$ is the random walk starting from the largest point (among $S_1^{-a_i}-a_i$, $i\geq 1$) which does not meet S_1^0 during time interval $[0,T]$. We have

$$\mathbb{P}\{E(T)\}\leq 2\sum_{k=1}^N r_k(T)+2\mathbb{P}\left\{S_1^0(T)>\varphi(T),\ I>N\right\},$$

where

$$r_k(T):=\mathbb{P}\left\{S_1^0(T)>\varphi(T),\ I=k,\ |S_1^{-a_k}(T)-a_k|>\varphi(T)\right\}.$$

It is easy to estimate $\mathbb{P}\{S_1^0(T)>\varphi(T),\ I>N\}$. Indeed, if $I>N$, then the random walks $S_1^{-a_N}-a_N$ and S_1^0 meet during $[0,T]$, so that

$$\begin{aligned}\mathbb{P}\left\{S_1^0(T)>\varphi(T),\ I>N\right\} &\leq \mathbb{P}\left\{S_1^{-a_N}(T)-a_N>\varphi(T)\right\}\\ &= \mathbb{P}\left\{S_1^0(T)>a_N+\varphi(T)\right\}\\ &\leq \exp\left(-(1+o(1))\frac{(a_N+\varphi(T))^2}{2T}\right).\end{aligned}$$

Therefore,

$$\mathbb{P}\{E(T)\}\leq 2\sum_{k=1}^N r_k(T)+2\exp\left(-(1+o(1))\frac{(a_N+\varphi(T))^2}{2T}\right).$$

We now estimate $r_k(T)$ for $1\leq k\leq N$. The term $r_1(T)$ is special. Indeed, if $I=1$, then

$$\begin{aligned}r_1(T) &\leq \mathbb{P}\left\{S_1^0(T)>\varphi(T),\ |S_1^{-a_1}(T)-a_1|>\varphi(T)\right\}\\ &= \mathbb{P}\left\{S_1^0(T)>\varphi(T)\right\}\mathbb{P}\left\{|S_1^{-a_1}(T)-a_1|>\varphi(T)\right\}\\ &\leq \exp\left(-(1+o(1))\frac{(\varphi(T)-a_1)^2}{T}\right).\end{aligned}$$

Accordingly,

$$\begin{aligned}\mathbb{P}\{E(T)\} &\leq 2\sum_{k=2}^N r_k(T)+2\exp\left(-(1+o(1))\frac{(\varphi(T)-a_1)^2}{T}\right)\\ &\qquad +2\exp\left(-(1+o(1))\frac{(a_N+\varphi(T))^2}{2T}\right).\end{aligned}\tag{5.1}$$

To estimate $r_k(T)$ for $2 \le k \le N$, we note that if $2 \le I < \infty$, then $\sup_{t\in[0,T]}[S_1^{-a_j}(t) - S_1^0(t)] \ge a_j$ for all $j \le I-1$. In particular,

$$r_k(T) \le \mathbb{P}\left\{ S_1^0(T) > \varphi(T),\ \sup_{t\in[0,T]}[S_1^{-a_{k-1}}(t) - S_1^0(t)] \ge a_{k-1},\ |S_1^{-a_k}(T) - a_k| > \varphi(T) \right\}.$$

Recall that S_1^0, $S_1^{-a_{k-1}}$ and $S_1^{-a_k}$ are three independent random walks on $\mathbb{Z}$ all starting from 0. So the probability of $|S_1^{-a_k}(T) - a_k| > \varphi(T)$ can be splitted from the right hand side. It is easily seen that

$$\begin{aligned} \mathbb{P}\left\{|S_1^{-a_k}(T) - a_k| > \varphi(T)\right\} &\le 2\mathbb{P}\left\{S_1^{-a_k}(T) > \varphi(T) - a_k\right\} \\ &\le 2\exp\left(-(1+o(1))\frac{(\varphi(T)-a_k)^2}{2T}\right). \end{aligned}$$

Therefore, if we write X and Y for two independent random walks on $\mathbb{Z}$ both starting from 0, then

$$\begin{aligned} r_k(T) &\le 2\exp\left(-(1+o(1))\frac{(\varphi(T)-a_k)^2}{2T}\right) \times \\ &\qquad \times \mathbb{P}\left\{X(T) > \varphi(T),\ \sup_{t\in[0,T]}[Y(t) - X(t)] \ge a_{k-1}\right\} \\ &:= 2\exp\left(-(1+o(1))\frac{(\varphi(T)-a_k)^2}{2T}\right)\widetilde{r}_k(T), \end{aligned} \tag{5.2}$$

with obvious notation.

Let us estimate $\widetilde{r}_k(T)$. According to a result of Csörgő *et al.* [11] (cf. also Csörgő and Horváth [12]), which is the continuous time version of the well-known Komlós–Major–Tusnády [15] approximation theorem, there exists a coupling for X and a standard Wiener process W such that

$$\mathbb{P}\left\{\sup_{t\in[0,T]}|X(t) - W(t)| \ge A\log T + x\right\} \le B\exp(-Cx),$$

for all $T \ge 1$, $x \ge 0$ and some constants A, B, C. From this we can conclude that

$$\mathbb{P}\left\{\sup_{t\in[0,T]}|X(t) - W(t)| \ge T^{1/2}\right\} \le \frac{c_{25}}{T},$$

for all $T \ge 1$ and some constant c_{25}. Similar result is true for $Y(t)$ with another Wiener process, independent of W. Therefore, if W_1 and W_2 denote a pair of independent standard

Wiener processes, then

$$
\begin{aligned}
\widetilde{r}_k(T) &\leq \mathbb{P}\left\{W_1(T) > (1+o(1))\varphi(T),\ \sup_{t\in[0,T]}[W_2(t)-W_1(t)] \geq (1+o(1))a_{k-1}\right\} + \frac{c_{25}}{T} \\
&:= \mathbb{P}\left\{W_1(T) > \widetilde{\varphi}(T),\ \sup_{t\in[0,T]}[W_2(t)-W_1(t)] \geq \widetilde{a}_{k-1}\right\} + \frac{c_{25}}{T},
\end{aligned}
$$

where we have written $\widetilde{\varphi}(T) := (1+o(1))\varphi(T)$ and $\widetilde{a}_{k-1} := (1+o(1))a_{k-1}$ for brevity. Let $B_1 := (W_2 - W_1)/\sqrt{2}$ and $B_2 := (W_2 + W_1)/\sqrt{2}$, so that B_1 and B_2 are also independent Wiener processes. Accordingly,

$$
\begin{aligned}
\widetilde{r}_k(T) &\leq \mathbb{P}\left\{B_2(T) - B_1(T) > \sqrt{2}\,\widetilde{\varphi}(T),\ \sup_{t\in[0,T]} B_1(t) \geq \frac{\widetilde{a}_{k-1}}{\sqrt{2}}\right\} + \frac{c_{25}}{T} \\
&\leq \mathbb{P}\left\{B_2(T) + \left(2\sup_{t\in[0,T]} B_1(t) - B_1(T)\right) \geq \sqrt{2}\,\widetilde{\varphi}(T) + \sqrt{2}\,\widetilde{a}_{k-1}\right\} + \frac{c_{25}}{T} \\
&= \mathbb{P}\left\{B_2(1) + \left(2\sup_{t\in[0,1]} B_1(t) - B_1(1)\right) \geq \frac{\sqrt{2}\,(\widetilde{\varphi}(T) + \widetilde{a}_{k-1})}{\sqrt{T}}\right\} + \frac{c_{25}}{T}.
\end{aligned}
$$

The joint density of $B_1(1)$ and $\sup_{t\in[0,1]} B_1(t)$ is known (cf., e.g., Borodin and Salminen [5], p. 147): $\mathbb{P}\{B_1(1) \in \mathrm{d}x,\ \sup_{t\in[0,1]} B_1(t) \in \mathrm{d}y\} = (\frac{2}{\pi})^{1/2}(2y-x)\exp(-\frac{(2y-x)^2}{2})\,\mathbf{1}_{\{y>0,\,y>x\}}$, from which we deduce that for $\lambda \to \infty$,

$$
\mathbb{P}\left\{2\sup_{t\in[0,1]} B_1(t) - B_1(1) > \lambda\right\} \leq \exp\left(-(1+o(1))\frac{\lambda^2}{2}\right), \qquad \lambda \to \infty.
$$

(Alternatively, this can be proved by means of the fact that $s \mapsto 2\sup_{t\in[0,1]} B_1(s) - B_1(s)$, for $s \geq 0$, is a three-dimensional Bessel process; that is, the Euclidean modulus of an $\mathbb{R}^3$-valued Wiener process.) On the other hand, we trivially have

$$
\mathbb{P}\{B_2(1) > \lambda\} \leq \exp\left(-\frac{\lambda^2}{2}\right), \qquad \forall \lambda \geq 0.
$$

Since $B_2(1)$ is independent of $2\sup_{t\in[0,1]} B_1(t) - B_1(1)$, it follows that

$$
\begin{aligned}
\widetilde{r}_k(T) &\leq \exp\left(-(1+o(1))\frac{(\widetilde{\varphi}(T) + \widetilde{a}_{k-1})^2}{2T}\right) + \frac{c_{25}}{T} \\
&\leq \exp\left(-(1+o(1))\frac{(\varphi(T) + a_{k-1})^2}{2T}\right).
\end{aligned}
$$

In view of (5.2), we get that

$$
\begin{aligned}
r_k(T) &\leq 2\exp\left(-(1+o(1))\frac{(\varphi(T)-a_k)^2}{2T} - (1+o(1))\frac{(\varphi(T)+a_{k-1})^2}{2T}\right) \\
&\leq 2\exp\left(-(1-\delta+o(1))(1+4\delta)\log\log T\right).
\end{aligned}
$$

Plugging this into (5.1), and we obtain that for all large T,

$$
\mathbb{P}\{E(T)\} \leq \frac{1}{(\log T)^{1+2\delta}}. \tag{5.3}
$$

This is the main probability estimate we need in the proof of the upper bound for (1.1).

To complete the proof of the upper bound in question, we consider the subsequence $T_n := \exp(n^{1-\delta})$. According to (5.3), $\sum_n \mathbb{P}\{E(T_n)\} < \infty$, so that by the Borel–Cantelli lemma, almost surely for all large n,

$$
\min_{k:\, 0\leq k\leq N} \left\{|\xi_1^{-a_k}(T_n)| \wedge |\xi_1^{-a_k}(T_n)|\right\} \leq \varphi(T_n), \tag{5.4}
$$

with the usual notation $a \wedge b := \min\{a,b\}$. On the other hand, by the usual estimate for Gaussian tails,

$$
\begin{aligned}
&\mathbb{P}\left\{\max_{k:\, 0\leq k\leq N}\ \sup_{T\in[T_n,T_{n+1}]} \left\{|\xi_1^{-a_k}(T)-\xi_1^{-a_k}(T_n)| \vee |\xi_1^{a_k}(T)-\xi_1^{a_k}(T_n)|\right\} > \delta\varphi(T_n)\right\} \\
\leq\ & 2(N+1)\mathbb{P}\left\{\sup_{s\in[0,T_{n+1}-T_n]} |\xi_1^0(s)| > \delta\varphi(T_n)\right\} \\
\leq\ & 2(N+1)\exp\left(-\frac{\delta^2\varphi^2(T_n)}{2(T_{n+1}-T_n)}\right),
\end{aligned}
$$

which is summable for n, so that by the Borel–Cantelli lemma, almost surely for all large n, all $k \leq N$ and all $T \in [T_n, T_{n+1}]$, we have $|\xi_1^{-a_k}(T)-\xi_1^{-a_k}(T_n)| \leq \delta\varphi(T_n)$ and $|\xi_1^{a_k}(T)-\xi_1^{a_k}(T_n)| \leq \delta\varphi(T_n)$.

In view of (5.4), we deduce that almost surely for all large T,

$$
\min_{k:\, 0\leq k\leq N} \left\{|\xi_1^{-a_k}(T)| \wedge |\xi_1^{-a_k}(T)|\right\} \leq (1+\delta)\,\varphi(T),
$$

so that a fortiori,

$$
\inf_{x\in\mathbb{Z}} |\xi_1^x(T)| \leq (1+\delta)\,\varphi(T).
$$

By definition, this implies

$$
\limsup_{T\to\infty} \frac{R_1(T)}{\varphi(T)} \leq 1+\delta, \qquad \text{a.s.}
$$

Since δ can be arbitrarily close to 0, we obtain the upper bound in (1.1). $\square$

Proof of (1.1): lower bound. Fix $\delta \in (0,1)$ and let $\psi(t) = \psi_\delta(t) := (1-\delta)\sqrt{t \log\log t}$. According to Arratia [1], $T^{-1/2}\xi_1(T)$ converges weakly (as $T \to \infty$) to a (non-Poisson) limit point process. In particular, if we write $R^+(T) := \inf\{x > 0 : \ x \in \xi_1(T)\}$ and $R^-(T) := \sup\{x < 0 : \ x \in \xi_1(T)\}$, then $\mathbb{P}\{R^+(\delta T) \in [\sqrt{\delta T}, 2\sqrt{\delta T}],\, R^-(\delta T) \in [-2\sqrt{\delta T}, -\sqrt{\delta T}]\}$ converges to a (strictly) positive constant, so that

$$c_{26} := \inf_{T \geq 1} \mathbb{P}\left\{ R^+(\delta T) \in [\sqrt{\delta T}, 2\sqrt{\delta T}],\, R^-(\delta T) \in [-2\sqrt{\delta T}, -\sqrt{\delta T}] \right\} > 0.$$

Consider the situation at time δT. Two particles (referred to as $\xi_1^+(\cdot)$ and $\xi_1^-(\cdot)$, respectively) occupy the sites $R^+(\delta T) \in \mathbb{Z}_+$ and $R^-(\delta T) \in \mathbb{Z}_-$ respectively whereas no site in $(R^+(\delta T), R^-(\delta T))$ is occupied. Let us consider the events

$$\begin{aligned} E_+(T) &:= \left\{ \xi_1^+(T) \geq \psi(T),\ \inf_{t \in [\delta T, T]} \xi_1^+(t) > 0 \right\}, \\ E_-(T) &:= \left\{ \xi_1^-(T) \leq -\psi(T),\ \sup_{t \in [\delta T, T]} \xi_1^-(t) < 0 \right\}. \end{aligned}$$

Clearly, $(E_+(T) \cap E_-(T)) \subset \{R_1(T) \geq \psi(T)\}$. Therefore

$$\begin{aligned} \mathbb{P}\{R_1(T) \geq \psi(T)\} &\geq \mathbb{P}\Big\{ E_+(T),\ E_-(T),\ R^+(\delta T) \in [\sqrt{\delta T}, 2\sqrt{\delta T}], \\ &\qquad\qquad R^-(\delta T) \in [-2\sqrt{\delta T}, -\sqrt{\delta T}] \Big\} \\ &\geq c_{26} \inf_{x \in [\sqrt{\delta T}, 2\sqrt{\delta T}]} \left(\mathbb{P}\left\{ \xi_1^x((1-\delta)T) \geq \psi(T),\ \inf_{t \in [0,(1-\delta)T]} \xi_1^x(t) > 0 \right\} \right)^2. \end{aligned} \tag{5.5}$$

For any $x > 0$, $t > 0$ and $a > 0$, we have, by the reflection principle,

$$\begin{aligned} \mathbb{P}\{\xi_1^x(t) \geq a\} &= \mathbb{P}\left\{ \xi_1^x(t) \geq a,\ \inf_{u \in [0,t]} \xi_1^x(u) > 0 \right\} + \mathbb{P}\left\{ \xi_1^x(t) \geq a,\ \inf_{u \in [0,t]} \xi_1^x(u) \leq 0 \right\} \\ &= \mathbb{P}\left\{ \xi_1^x(t) \geq a,\ \inf_{u \in [0,t]} \xi_1^x(u) > 0 \right\} + \mathbb{P}\left\{ \xi_1^{-x}(t) \geq a \right\}. \end{aligned}$$

Taking $t := (1-\delta)T$ and $a := \psi(T)$ yields that

$$\begin{aligned} &\inf_{x \in [\sqrt{\delta T}, 2\sqrt{\delta T}]} \mathbb{P}\left\{ \xi_1^x((1-\delta)T) \geq \psi(T),\ \inf_{t \in [0,(1-\delta)T]} \xi_1^x(t) > 0 \right\} \\ &\qquad \geq \exp\left(-(1+o(1)) \frac{(\psi(T) - \sqrt{\delta T})^2}{2T} \right). \end{aligned}$$

Plugging this into (5.5), and we get that

$$
\begin{aligned}
\mathbb{P}\left\{R_1(T) \geq \psi(T)\right\} &\geq \exp\left(-(1+o(1))\frac{\psi^2(T)}{T}\right) \\
&\geq \exp\left(-(1-2\delta)\log\log T\right) \\
&= \frac{1}{(\log T)^{1-2\delta}}. \qquad (5.6)
\end{aligned}
$$

Let $T_n := n^n$, and consider a sequence of independent random variables $(R_1^{(n)}(T_n - T_{n-1}),\ n \geq 2)$, such that for any n, $R_1^{(n)}(T_n - T_{n-1})$ is distributed as $R_1(T_n - T_{n-1})$. We can make a coupling for $(R_1^{(n)}(T_n - T_{n-1}),\ n \geq 2)$ and the coalescing random walk $(\xi_1(t),\ t \geq 0)$ such that $R_1(T_n) \geq R_1^{(n)}(T_n - T_{n-1})$ for all $n \geq 2$.

By (5.6) and the Borel–Cantelli lemma, almost surely there exists infinitely many n such that $R_1^{(n)}(T_n - T_{n-1}) \geq \psi(T_n - T_{n-1})$. Therefore,

$$\limsup_{n\to\infty} \frac{R_1(T_n)}{\psi(T_n - T_{n-1})} \geq 1, \qquad \text{a.s.}$$

Since $\psi(T_n - T_{n-1}) \sim \psi(T_n) = (1-\delta)\sqrt{T_n \log\log T_n}$, and since $\delta > 0$ can be as close to 0 as possible, this yields the lower bound in (1.1). $\square$

Remark. Let as before $R^+(T) := \inf\{x > 0 :\ x \in \xi_1(T)\}$ and $R^-(T) := \sup\{x < 0 :\ x \in \xi_1(T)\}$. In words, $R^+(T)$ (resp. $R^-(T)$) is the smallest positive (resp. largest negative) site occupied by the coalescing random walk at time T. By definition, $R_1(T) = R^+(T) \wedge |R^-(T)|$. Our proof of (1.1) also shows that

$$\limsup_{T\to\infty} \frac{X(T)}{\sqrt{T\log\log T}} = 1, \qquad \text{a.s.}$$

where $X(T)$ can be either $R^+(T)$, or $|R^-(T)|$, or $R^+(T) \vee |R^-(T)|$, or $(R^+(T) + |R^-(T)|)/2$.

6 Proof of Theorem 1.3

Let $d \geq 2$. By (1.10)–(1.11), Proposition 2.2 and Chebyshev's inequality, we have, for $\varepsilon > 0$ and $T \geq 3$,

$$\mathbb{P}\left\{|\Lambda_d(T) - \mathbb{E}[\Lambda_d(T)]| \geq \varepsilon\, \mathbb{E}[\Lambda_d(T)]\right\} \leq \frac{c_{27}}{\log T},$$

for some constant c_{27} depending on (d, ε). Taking the subsequence $T = T_k := \exp\{k(\log k)^2\}$, and by means of the Borel–Cantelli lemma, we have, almost surely for all large k,

$$(1-\varepsilon)\mathbb{E}[\Lambda_d(T_k)] \leq \Lambda_d(T_k) \leq (1+\varepsilon)\mathbb{E}[\Lambda_d(T_k)].$$

By the monotonicity of $T \mapsto \Lambda_d(T)$ and again in view of (1.10)–(1.11), this implies that almost surely for all large T,

$$(1-2\varepsilon)\mathbb{E}[\Lambda_d(T)] \le \Lambda_d(T) \le (1+2\varepsilon)\mathbb{E}[\Lambda_d(T)].$$

Since $\varepsilon > 0$ can be arbitrarily close to 0, this yields Theorem 1.3. $\square$

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