A joint functional law for the Wiener process and principal value

by

Endre Csáki[1,2], Antónia Földes[3] and Zhan Shi[2]

Alfréd Rényi Institute of Mathematics, City University of New York & Université Paris VI

Summary. We present a joint functional iterated logarithm law for the Wiener process and the principal value of its local times.

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1 Introduction

Let $\{W(t);\, t \geqslant 0\}$ be a one-dimensional standard Wiener process with $W(0) = 0$, and let $\{L(t,x);\, t \geqslant 0,\, x \in \mathbb{R}\}$ denote its local time process, jointly continuous in t and x. For any Borel function $f \geqslant 0$,

$$\int_0^t f(W(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\,\mathrm{d}x, \qquad t \geqslant 0.$$

Put $L(t,0) = L(t)$ and

$$\begin{aligned} U_t(x) &:= \frac{W(xt)}{\sqrt{2t\log\log t}}, \\ V_t(x) &:= \frac{L(xt)}{\sqrt{2t\log\log t}}, \qquad x \in [0,1]. \end{aligned}$$

We consider $x \mapsto U_t(x)$ and $x \mapsto V_t(x)$ as elements of the space $\mathcal{C} = \mathcal{C}[0,1]$ of continuous functions with metric

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Recall the celebrated functional law of the iterated logarithm (FLIL) for W due to Strassen [15]:

Theorem A *With probability one, the set $\{U_t\}_{t\geqslant 1}$ is relatively compact in $\mathcal{C}$, with limit set equal to*

$$\mathcal{S} := \left\{ f \in \mathcal{C} :\ f(0) = 0,\ f \text{ is absolutely continuous, with } \int_0^1 (f'(x))^2\,\mathrm{d}x \leqslant 1 \right\}.$$

Using that $\{L(t),\, t \geqslant 0\}$ has the same distribution as $\{\sup_{s\in[0,t]} W(s),\, t \geqslant 0\}$, one can easily obtain (cf. Csáki and Révész [7], Mueller [13], Chen [3]),

Theorem B *With probability one, the set $\{V_t\}_{t\geqslant 1}$ is relatively compact in $\mathcal{C}$, with limit set equal to*

$$\mathcal{S}_M := \{g \in \mathcal{S} :\ g \text{ is non-decreasing}\}.$$

In Csáki and Révész [7] a joint FLIL was given for the vector $\{(U_t(x), V_t(x)),\, x \in [0,1]\}_{t\geqslant 1}$ on the space $\mathcal{C}^{(2)} := \mathcal{C} \times \mathcal{C}$ with metric

$$d((f_1,g_1),(f_2,g_2)) = \sup_{x\in[0,1]} \sqrt{(f_1(x) - f_2(x))^2 + (g_1(x) - g_2(x))^2}.$$

Theorem C *With probability one, the set* $\{(U_t, V_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\mathcal{S}_J^{(2)} := \left\{ (f,g) :\ f \in \mathcal{S}, g \in \mathcal{S}_M, \int_0^1 (f'(x))^2 + (g'(x))^2)\, \mathrm{d}x \leqslant 1,\ f(x)g'(x) = 0 \text{ a.e.} \right\}.$$

We are interested in studying similar joint FLIL for the Wiener process and the process

$$Y(t) = \int_0^t \frac{\mathrm{d}s}{W(s)}, \qquad t \geqslant 0.$$

Rigorously speaking, the integral $\int_0^t \mathrm{d}s/W(s)$ should be considered in the sense of Cauchy's principal value, i.e., $Y(t)$ is defined by

$$Y(t) = \lim_{\varepsilon \to 0^+} \int_0^t \frac{\mathrm{d}s}{W(s)} \mathbb{1}_{\{|W(s)|\geqslant\varepsilon\}} = \int_0^\infty \frac{L(t,x) - L(t,-x)}{x}\, \mathrm{d}x. \tag{1.1}$$

Since $x \mapsto L(t,x)$ is Hölder continuous of order ν, for any $\nu < 1/2$, the integral on the right hand side of (1.1) is well-defined.

The study of Cauchy's principal value of Brownian local time goes back at least to Itô and McKean [12], and has become very active since the late 70s, due to applications in various branches of stochastic analysis. For a detailed account of various motivations, historical facts and general properties of principal values of local times, we refer to the recent collection of research papers in Yor [17], to Chapter 10 of the lecture notes by Yor [18], and to the survey paper by Yamada [16].

The process $Y(\cdot)$ defined in (1.1) is almost surely continuous, having zero quadratic variation. It is easily seen that $Y(\cdot)$ inherits a scaling property from Brownian motion, namely, for any fixed $a > 0$, $t \mapsto a^{-1/2}Y(at)$ has the same law as $t \mapsto Y(t)$. Although the aforementioned zero quadratic variation property distinguishes $Y(\cdot)$ from Brownian motion (in particular, $Y(\cdot)$ is not a semimartingale), it is a kind of folklore that Y behaves somewhat like a Brownian motion. Hu and Shi [11] proved a law of the iterated logarithm for $Y(\cdot)$:

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{8t \log\log t}} = 1 \qquad \text{a.s.}$$

FLIL for Y was not known before. Here we show that similarly to Theorem C, a joint FLIL for W and Y holds. Introduce

$$Z_t(x) = \frac{Y(xt)}{\sqrt{8t\log\log t}}, \qquad 0 \leqslant x \leqslant 1.$$

Our main result is

Theorem 1.1 *With probability one the set* $\{(U_t, Z_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$, *with limit set equal to*

$$\widetilde{\mathcal{S}}_J^{(2)} = \left\{(f,g) : f \in \mathcal{S}, g \in \mathcal{S}, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \leqslant 1,\ f(x)g'(x) = 0 \text{ a.e.}\right\}.$$

Some consequences are as follows.

Corollary 1.2 *With probability one, the set* $\{Z_t\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}$, *with limit set equal to* $\mathcal{S}$ *given in Theorem A.*

Corollary 1.3 *With probability one, the set* $\{(U_t(1), Z_t(1))\}_{t\geqslant 1}$ *is relatively compact in* $\mathbb{R}^2$ *with limit set equal to*

$$\{(x,y) \in \mathbb{R}^2 : |x| + |y| \leqslant 1\}.$$

The organization of the paper is as follows: In Section 2 we present some preliminary results for the distribution of the Wiener process and principal value, as well as certain estimates for the increments of the processes concerned. In Section 3 we prove Theorem 1.1. In Section 4 we prove the Corollaries. Some further remarks and consequences are given in Section 5.

Throughout the paper, for any $x \in \mathbb{R}$, we denote by $\mathbb{P}^x$ the probability under which the Wiener process W starts from $W(0) = x$ (thus $\mathbb{P} = \mathbb{P}^0$); unimportant constants (which are finite and positive) are denoted by the letter c with subscript.

2 Preliminaries

2.1 Distribution results for Wiener process and principal value

First recall some results for principal value. Biane and Yor [1] proved the following result: Let $\{B(s),\, 0 \leqslant s \leqslant 1\}$ be a Brownian bridge, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{B(s)} < x\right) &= \frac{|x|}{2}\sum_{n=1}^{\infty} \exp\left(-\frac{(2n-1)^2x^2}{8}\right) \\ &\geqslant \frac{|x|}{2}\exp\left(-\frac{x^2}{8}\right). \end{aligned} \tag{2.1}$$

It follows that for $0 < \alpha < \beta$

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{B(s)} \in (\alpha, \beta)\right) \geqslant 2\left(\exp\left(-\frac{\alpha^2}{8}\right) - \exp\left(-\frac{\beta^2}{8}\right)\right). \tag{2.2}$$

It was proved in [5] (cf. (2.11), (2.14) and (2.16) there) that for any $\delta > 0$ there exists $c_1(\delta) > 0$ such that for all $s > 0$ and $x > 0$,

$$\sup_{z\in\mathbb{R}} \mathbb{P}^z(|Y(s)| > x) \leqslant c_1(\delta) \exp\left(-\frac{x^2}{(8+\delta)s}\right). \tag{2.3}$$

Lemma 2.1 *Let $s > 0$, $\lambda > 0$, $\delta > 0$ and $0 < \varepsilon < 1$. For $(a, \alpha, z) \in \mathbb{R}^3$, define*

$$I = I(a, \alpha, z) := \mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ \ \alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda\right). \tag{2.4}$$

Then

$$I \leqslant \frac{\lambda}{\sqrt{s}} \exp\left(-\frac{(|a-z| - 2\varepsilon\lambda)^2 - 4\varepsilon^2\lambda^2}{2s}\right). \tag{2.5}$$

Moreover, if $|\alpha| \geqslant 4\varepsilon\lambda$, then

$$I \leqslant c_1(\delta) \exp\left(-\frac{(|\alpha| - 4\varepsilon\lambda)^2}{(8+\delta)s}\right), \tag{2.6}$$

where $c_1(\delta)$ is the constant in (2.3).

Proof: Observe that

$$I \leqslant \mathbb{P}^z(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda) = \mathbb{P}\left(\frac{a-z}{\sqrt{s}} \leqslant N \leqslant \frac{a-z+2\varepsilon\lambda}{\sqrt{s}}\right),$$

where N is a standard normal variable. Hence (2.5) follows from a straightforward Gaussian estimate.

Now for $|\alpha| \geqslant 4\varepsilon\lambda$, we have

$$I \leqslant \mathbb{P}^z(\alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda) \leqslant \mathbb{P}^z(|Y(s)| \geqslant |\alpha| - 4\varepsilon\lambda),$$

which implies (2.6) by means of (2.3). □

For the lower estimates we prove several lemmas.

Lemma 2.2 *For $\alpha > 0$, $\beta - \alpha > 4$, $0 < \delta < 1$ we have*

$$\mathbb{P}(|W(1)| \leqslant 1,\, \alpha \leqslant Y(1) \leqslant \beta) \geqslant c_2(\delta) \exp\left(-\frac{(\alpha+1)^2}{8(1-\delta)}\right), \tag{2.7}$$

where $c_2(\delta)$ is a constant depending only on δ.

Proof: Let

$$\begin{aligned} G &:= \sup\{t :\, t \leqslant 1,\, W(t) = 0\}, \\ B(s) &:= \frac{W(sG)}{\sqrt{G}}, \qquad s \in [0,1]. \end{aligned}$$

It is known that $(B(s),\ s \in [0,1])$, G and $(\frac{W(G+s(1-G))}{\sqrt{1-G}},\ s \in [0,1])$ are independent, and that $(B(s),\ s \in [0,1])$ is a (standard) Brownian bridge.

We have

$$\begin{aligned} & \mathbb{P}(|W(1)| \leqslant 1,\, \alpha \leqslant Y(1) \leqslant \beta) \\ \geqslant\ & \mathbb{P}(|W(1)| \leqslant 1,\, \alpha + 1 \leqslant Y(G) \leqslant \beta - 1,\, |Y(1) - Y(G)| \leqslant 1,\, G \geqslant 1 - \delta) \\ =\ & \int_{1-\delta}^{1} \mathbb{P}(|W(1)| \leqslant 1,\, \alpha + 1 \leqslant Y(\kappa) \leqslant \beta - 1,\, |Y(1) - Y(\kappa)| \leqslant 1 \,|\, G = \kappa)\, \mathbb{P}(G \in \mathrm{d}\kappa) \\ =\ & \int_{1-\delta}^{1} \mathbb{P}(\alpha + 1 \leqslant Y(\kappa) \leqslant \beta - 1 \,|\, G = \kappa) \times \\ & \qquad \times \mathbb{P}(|W(1)| \leqslant 1,\, |Y(1) - Y(\kappa)| \leqslant 1 \,|\, G = \kappa)\, \mathbb{P}(G \in \mathrm{d}\kappa). \end{aligned}$$

Since under the condition $G = \kappa$, $Y(\kappa)/\sqrt{\kappa}$ has the same distribution as $\int_0^1 \mathrm{d}s/B(s)$, where B is a Brownian bridge, we get from (2.2)

$$\begin{aligned} \mathbb{P}(\alpha + 1 \leqslant Y(\kappa) \leqslant \beta - 1 \,|\, G = \kappa) &\geqslant 2\left(\exp\left(-\frac{(\alpha+1)^2}{8\kappa}\right) - \exp\left(-\frac{(\beta-1)^2}{8\kappa}\right)\right) \\ &\geqslant 2(1 - \mathrm{e}^{-1}) \exp\left(-\frac{(\alpha+1)^2}{8\kappa}\right) \\ &\geqslant 2(1 - \mathrm{e}^{-1}) \exp\left(-\frac{(\alpha+1)^2}{8(1-\delta)}\right). \end{aligned}$$

This gives (2.7), with

$$c_2(\delta) := 2(1 - \mathrm{e}^{-1})\mathbb{P}(|W(1)| \leqslant 1,\, |Y(1) - Y(G)| \leqslant 1,\, G \geqslant 1 - \delta).$$

The lemma is proved. □

Now we introduce the notation

$$T_b := \inf\{t:\ t \geqslant 0,\, W(t) = b\}. \tag{2.8}$$

By the reflection principle, we have for all $u > 0$ and $(a, z) \in \mathbb{R}^2$,

$$\mathbb{P}^z(T_a \leqslant u) = 2\overline{\Phi}\left(\frac{|z-a|}{\sqrt{u}}\right), \tag{2.9}$$

where $\overline{\Phi}(x) := \mathbb{P}(N > x)$ is the standard Gaussian tail distribution function.

In the sequel we shall use the inequalities:

$$\overline{\Phi}(x) \geqslant \frac{1}{(2\pi)^{1/2}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right), \qquad x \geqslant 1, \tag{2.10}$$

$$\overline{\Phi}(x) \leqslant \frac{1}{2}\exp\left(-\frac{x^2}{2}\right), \qquad x > 0. \tag{2.11}$$

(For (2.11), see Proposition II.1.8 of Revuz and Yor [14].)

Lemma 2.3 *For $s > 0$, $0 < \delta < 1$, $z \in \mathbb{R}$ we have*

$$\mathbb{P}^z(T_0 \leqslant \delta s,\, |Y(T_0)| \leqslant 2\sqrt{s}) \geqslant c_3(\delta)\, \overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}}\right). \tag{2.12}$$

Proof: By symmetry, it suffices to prove (2.12) for $z > 0$ (there is nothing to prove if $z = 0$). Assuming first $z > \sqrt{s}$, we have

$$\begin{aligned}
&\mathbb{P}^z(T_0 \leqslant \delta s,\, |Y(T_0)| \leqslant 2\sqrt{s}\,)\\
\geqslant\ &\mathbb{P}^z(T_0 - T_{\sqrt{s}} \leqslant \delta(1-\delta)s,\, T_{\sqrt{s}} \leqslant \delta^2 s,\, Y(T_0) - Y(T_{\sqrt{s}}) \leqslant \sqrt{s})\\
=\ &\mathbb{P}^{\sqrt{s}}(T_0 \leqslant \delta(1-\delta)s, Y(T_0) \leqslant \sqrt{s}\,)\, \mathbb{P}^z(T_{\sqrt{s}} \leqslant \delta^2 s),
\end{aligned}$$

where we used the fact that $T_{\sqrt{s}} \leqslant \delta^2 s$ implies $Y(T_{\sqrt{s}}) \leqslant T_{\sqrt{s}}/\sqrt{s} \leqslant \delta^2\sqrt{s} < \sqrt{s}$.

By scaling, $\mathbb{P}^{\sqrt{s}}(T_0 \leqslant \delta(1-\delta)s, Y(T_0) \leqslant \sqrt{s}\,)$ is a positive constant depending only on δ. In view of (2.9), we have proved (2.12) in case $z > \sqrt{s}$.

If $0 < z \leqslant \sqrt{s}$, we have, by scaling,

$$\begin{aligned}
\mathbb{P}^z(T_0 \leqslant \delta s,\, |Y(T_0)| \leqslant 2\sqrt{s}\,) &= \mathbb{P}^1\left(T_0 \leqslant \frac{\delta s}{z^2},\, |Y(T_0)| \leqslant \frac{2\sqrt{s}}{z}\right)\\
&\geqslant \mathbb{P}^1\left(T_0 \leqslant \delta,\, |Y(T_0)| \leqslant 2\right)\\
&=: c_4(\delta),
\end{aligned}$$

from which (2.12) follows. $\square$

Lemma 2.4 *Let $s > 0$, $\varepsilon > 0$, $\lambda > 0$, $0 < \delta < 1$, $(\alpha, z) \in \mathbb{R}^2$ be such that $\varepsilon\lambda > 8\sqrt{s}$. Then we have*

$$\begin{aligned} &\mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda) \\ \geqslant\ & c_5(\delta) \exp\left(-\frac{(|\alpha| + 2\varepsilon\lambda)^2}{8s(1-\delta)^2}\right) \overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}}\right). \end{aligned} \tag{2.13}$$

Proof: Define, for $n \geqslant 1$,

$$I_{\lambda,z}(\alpha, n) := \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda).$$

Note that $I_{\lambda,z}(\alpha, n)$ is non-decreasing in n. Moreover,

$$\begin{aligned} I_{\lambda,z}(\alpha, n) &\geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda,\, T_0 \leqslant \delta s) \\ &= \int_0^{\delta s} \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) \\ &\geqslant \int_0^{\delta s} \mathbb{P}^z(A_\tau \cap B_\tau(n) \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau), \end{aligned}$$

where

$$\begin{aligned} A_\tau &:= \{|Y(\tau)| \leqslant 2\sqrt{s}\}, \\ B_\tau(n) &:= \{|W(s)| \leqslant \varepsilon\lambda,\, \alpha + 2\sqrt{s} \leqslant Y(s) - Y(\tau) \leqslant \alpha + n\varepsilon\lambda - 2\sqrt{s}\}. \end{aligned}$$

Under the condition $\{W(0) = z,\, T_0 = \tau\}$, A_τ and $B_\tau(n)$ are independent, so that

$$I_{\lambda,z}(\alpha, n) \geqslant \int_0^{\delta s} \mathbb{P}^z(A_\tau \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) \times \inf_{\tau \in (0,\, \delta s)} \mathbb{P}^z(B_\tau(n) \,|\, T_0 = \tau).$$

By Lemma 2.3,

$$\begin{aligned} \int_0^{\delta s} \mathbb{P}^z(A_\tau \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) &= \mathbb{P}^z(|Y(T_0)| \leqslant 2\sqrt{s},\, T_0 \leqslant \delta s) \\ &\geqslant c_3(\delta)\, \overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}}\right), \end{aligned}$$

whereas according to Lemma 2.2, and by scaling,

$$\begin{aligned} &\mathbb{P}^z(B_\tau(1) \,|\, T_0 = \tau) \\ =\ & \mathbb{P}(|W(s-\tau)| \leqslant \varepsilon\lambda,\, \alpha + 2\sqrt{s} \leqslant Y(s-\tau) \leqslant \alpha + \varepsilon\lambda - 2\sqrt{s}) \\ \geqslant\ & \mathbb{P}\left(|W(1)| \leqslant 1,\, \frac{\alpha + 2\sqrt{s}}{\sqrt{s-\tau}} \leqslant Y(1) \leqslant \frac{\alpha + \varepsilon\lambda - 2\sqrt{s}}{\sqrt{s-\tau}}\right). \end{aligned}$$

Assume $\alpha \geqslant 0$ for the moment. By Lemma 2.2,

$$\mathbb{P}^z(B_\tau(1) \,|\, T_0 = \tau) \geqslant c_2(\delta) \exp\left(-\frac{(\alpha+\varepsilon\lambda)^2}{8s(1-\delta)^2}\right),$$

which yields

$$I_{\lambda,z}(\alpha,1) \geqslant c_6(\delta) \exp\left(-\frac{(\alpha+\varepsilon\lambda)^2}{8s(1-\delta)^2}\right) \overline{\Phi}\left(\frac{||z|-\sqrt{s}|}{\delta\sqrt{s}}\right), \qquad \alpha \geqslant 0, \tag{2.14}$$

with $c_6(\delta) := c_3(\delta)c_2(\delta)$. Since $I_{\lambda,z}(\alpha,4) \geqslant I_{\lambda,z}(\alpha,1)$, this yields (2.13) in case $\alpha \geqslant 0$.

To treat the case $\alpha \leqslant -\varepsilon\lambda$, we observe that

$$\begin{aligned} I_{\lambda,z}(\alpha,4) &\geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha+\varepsilon\lambda) \\ &= \mathbb{P}^{-z}(|W(s)| \leqslant \varepsilon\lambda,\, -\alpha-\varepsilon\lambda \leqslant Y(s) \leqslant -\alpha), \end{aligned}$$

the last identity following via replacing W by $-W$. This gives $I_{\lambda,z}(\alpha,4) \geqslant I_{\lambda,-z}(-\alpha-\varepsilon\lambda,1)$. Since $-\alpha-\varepsilon\lambda \geqslant 0$, we are entitled to apply (2.14) to deduce (2.13).

It remains to study the situation $\alpha \in (-\varepsilon\lambda, 0)$. In this case,

$$I_{\lambda,z}(\alpha,4) \geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha+\varepsilon\lambda \leqslant Y(s) \leqslant \alpha+2\varepsilon\lambda) = I_{\lambda,z}(\alpha+\varepsilon\lambda,1),$$

which yields (2.13) in view of (2.14).

Lemma 2.4 is proved. □

Lemma 2.5 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $\varepsilon^2\lambda^2 \geqslant 2s$, $az>0$, and*

$$|z| > \frac{s}{2\varepsilon\lambda} + 3\varepsilon\lambda, \quad |a| > \frac{s}{2\varepsilon\lambda} + 3\varepsilon\lambda$$

we have

$$\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda,\, |Y(s)| \leqslant 2\varepsilon\lambda) \geqslant \frac{1}{2}\exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right). \tag{2.15}$$

Proof: It suffices to prove the lemma for $z > \frac{s}{2\varepsilon\lambda}+\varepsilon\lambda$ and $a > \frac{s}{2\varepsilon\lambda}+\varepsilon\lambda$ (then by symmetry, it will also cover the case $a<0$ and $z<0$). We have,

$$\begin{aligned} &\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda,\, |Y(s)| \leqslant 2\varepsilon\lambda) \\ \geqslant\ &\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u) > \frac{s}{2\varepsilon\lambda},\, a \leqslant W(s) \leqslant a+2\varepsilon\lambda\right) \\ =\ &\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda) \end{aligned}$$

$$-\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u)\leqslant \frac{s}{2\varepsilon\lambda},\ a\leqslant W(s)\leqslant a+2\varepsilon\lambda\right).$$

By the reflection principle,

$$\begin{aligned}
&\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u)\leqslant \frac{s}{2\varepsilon\lambda},\ a\leqslant W(s)\leqslant a+2\varepsilon\lambda\right)\\
=\ &\mathbb{P}^z\left(\frac{s}{\varepsilon\lambda}-a-2\varepsilon\lambda\leqslant W(s)\leqslant\frac{s}{\varepsilon\lambda}-a\right)\\
\leqslant\ &\mathbb{P}\left(\frac{W(s)}{\sqrt{s}}\leqslant-\frac{a+z-\frac{s}{\varepsilon\lambda}}{\sqrt{s}}\right)\\
\leqslant\ &\frac{1}{2}\exp\left(-\frac{\left(a+z-\frac{s}{\varepsilon\lambda}\right)^2}{2s}\right),
\end{aligned}$$

the last inequality following from (2.11). On the other hand,

$$\begin{aligned}
\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda) &= \mathbb{P}\left(\frac{a-z}{\sqrt{s}}\leqslant\frac{W(s)}{\sqrt{s}}\leqslant\frac{a-z+2\varepsilon\lambda}{\sqrt{s}}\right)\\
&\geqslant \frac{2\varepsilon\lambda}{\sqrt{2\pi s}}\exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right)\\
&\geqslant \exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right).
\end{aligned}$$

Since $a+z-s/(\varepsilon\lambda)\geqslant|a-z|+2\varepsilon\lambda$, we obtain (2.15). $\square$

Lemma 2.6 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $az<0$, $|a|>2\varepsilon\lambda+\sqrt{s}$ and $\min(\varepsilon\lambda/2,\,|z|)>\sqrt{s}$, we have*

$$\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leqslant 2\varepsilon\lambda)\geqslant c_7(\delta)\,\overline{\Phi}\left(\frac{|a-z|+2\varepsilon\lambda}{(1-\delta)\sqrt{s}}\right). \tag{2.16}$$

Proof: First we show for $a>\sqrt{u}$, $\varepsilon\lambda>2\sqrt{u}$,

$$P(u):=\mathbb{P}(a\leqslant W(u)\leqslant a+2\varepsilon\lambda,\ |Y(u)|\leqslant 2\sqrt{u})\geqslant c_8(\delta)\,\exp\left(-\frac{a^2}{2(1-\delta)u}\right). \tag{2.17}$$

Define $G_{\sqrt{u}}:=\sup\{t\leqslant u:\ W(t)=\sqrt{u}\}$. Then

$$\begin{aligned}
P(u) &\geqslant \int_0^{\delta u}\mathbb{P}(a\leqslant W(u)\leqslant a+2\varepsilon\lambda,\ |Y(u)|\leqslant 2\sqrt{u}\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(G_{\sqrt{u}}\in \mathrm{d}v)\\
&\geqslant \int_0^{\delta u}\mathbb{P}(|Y(v)|\leqslant\sqrt{u}\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(a\leqslant W(u)\leqslant a+2\varepsilon\lambda\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(G_{\sqrt{u}}\in \mathrm{d}v).
\end{aligned}$$

Under the condition $\{G_{\sqrt{u}} = v\}$, $\{M(r) := \frac{W(v+r(u-v))-\sqrt{u}}{\sqrt{u-v}},\ r \in [0,1]\}$ is a standard Brownian meander, and from the well-known identity (Biane and Yor [1]) $\mathbb{P}(M(1) \leqslant x) = 1 - \exp(-x^2/2)$, we get that, for $v \in [0, \delta u]$, $a > \sqrt{u}$ and $\varepsilon\lambda > 2\sqrt{u}$,

$$
\begin{aligned}
\mathbb{P}(a \leqslant W(u) \leqslant a + 2\varepsilon\lambda \,|\, G_{\sqrt{u}} = v) &= \mathbb{P}\left(\frac{a-\sqrt{u}}{\sqrt{u-v}} \leqslant M(1) \leqslant \frac{a-\sqrt{u}+2\varepsilon\lambda}{\sqrt{u-v}}\right) \\
&= \exp\left(-\frac{(a-\sqrt{u})^2}{2(u-v)}\right) - \exp\left(-\frac{(a-\sqrt{u}+2\varepsilon\lambda)^2}{2(u-v)}\right) \\
&\geqslant c_9 \exp\left(-\frac{(a-\sqrt{u})^2}{2(u-v)}\right) \\
&\geqslant c_9 \exp\left(-\frac{a^2}{2(1-\delta)u}\right),
\end{aligned}
$$

where $c_9 > 0$ is an absolute constant. Hence

$$
P(u) \geqslant c_{10}(\delta) \exp\left(-\frac{a^2}{2(1-\delta)u}\right),
$$

with

$$
\begin{aligned}
c_{10}(\delta) &:= c_9 \int_0^{\delta u} \mathbb{P}(|Y(v)| \leqslant \sqrt{u} \,|\, G_{\sqrt{u}} = v)\, \mathbb{P}(G_{\sqrt{u}} \in \mathrm{d}v) \\
&= c_9\, \mathbb{P}(G_{\sqrt{u}} \leqslant \delta u,\ |Y(G_{\sqrt{u}})| \leqslant \sqrt{u}\,),
\end{aligned}
$$

which, by scaling, does not depend on u. This yields (2.17).

We now start proving (2.16). Let $\varepsilon\lambda > 2\sqrt{s}$. Let T_0 and $T_{-\sqrt{s}}$ be as in (2.8). It suffices to prove (2.16) for $z < -\sqrt{s}$ and $a > \sqrt{s}$ (then by symmetry, it will also cover the case $z > \sqrt{s}$, $a < -2\varepsilon\lambda - \sqrt{s}\,$). Since $|Y(T_{-\sqrt{s}})| \leqslant \sqrt{s}$ under $\mathbb{P}^z$ (recalling that $z < -\sqrt{s}$), we have

$$
\begin{aligned}
&\mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda\right) \\
\geqslant\ &\mathbb{P}^z\Big(T_0 - T_{-\sqrt{s}} \leqslant \delta s,\ |Y(T_0) - Y(T_{-\sqrt{s}})| \leqslant \sqrt{s}, \\
&\qquad a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s) - Y(T_{-\sqrt{s}})| \leqslant 2\sqrt{s}\Big).
\end{aligned}
$$

By the strong Markov property at times $T_{-\sqrt{s}}$ and T_0, we get:

$$
\begin{aligned}
&\mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda\right) \\
\geqslant\ &\int_0^{\delta s} \left(\int_0^{s-y} P(s-h-y)\, \mathbb{P}^z(T_{-\sqrt{s}} \in \mathrm{d}h)\right) \mathbb{P}^{-\sqrt{s}}\left(T_0 \in \mathrm{d}y,\ |Y(T_0)| \leqslant \sqrt{s}\right).
\end{aligned}
\tag{2.18}
$$

By (2.17),

$$
P(s-h-y) \geqslant c_8(\delta) \exp\left(-\frac{a^2}{2(1-\delta)(s-h-y)}\right)
$$

$$\begin{aligned}
&\geqslant\ 2c_8(\delta)\,\overline{\Phi}\left(\frac{a}{\sqrt{(1-\delta)(s-h-y)}}\right)\\
&=\ 2c_8(\delta)\,\mathbb{P}^{-z-\sqrt{s}}\left(W(s-h-y)>\frac{a}{\sqrt{1-\delta}}-z-\sqrt{s}\right),
\end{aligned}$$

the second inequality being a consequence of (2.11). Therefore, for $y\in[0,\delta s]$,

$$\begin{aligned}
&\int_0^{s-y} P(s-h-y)\,\mathbb{P}^z(T_{-\sqrt{s}}\in \mathrm{d}h)\\
\geqslant\ & 2c_8(\delta)\int_0^{s-y}\mathbb{P}^{-z-\sqrt{s}}\left(W(s-h-y)>\frac{a}{\sqrt{1-\delta}}-z-\sqrt{s}\right)\mathbb{P}^z\left(T_{-\sqrt{s}}\in \mathrm{d}h\right)\\
=\ & 2c_8(\delta)\,\mathbb{P}\left(W(s-y)>\frac{a}{\sqrt{1-\delta}}-z-\sqrt{s}\right)\\
=\ & 2c_8(\delta)\,\overline{\Phi}\left(\frac{\frac{a}{\sqrt{1-\delta}}-z-\sqrt{s}}{\sqrt{s-y}}\right)\\
\geqslant\ & 2c_8(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right),
\end{aligned}$$

(recalling that $z<0$). Plugging this into (2.18), we get

$$\begin{aligned}
&\mathbb{P}^z\left(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leqslant 2\varepsilon\lambda\right)\\
\geqslant\ & 2c_8(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right)\mathbb{P}^{-\sqrt{s}}\left(T_0\leqslant \delta s,\ |Y(T_0)|\leqslant\sqrt{s}\right)\\
=\ & c_{11}(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right),
\end{aligned}$$

where $c_{11}(\delta):=2c_8(\delta)\,\mathbb{P}^{-1}(T_0\leqslant\delta,\ |Y(T_0)|\leqslant 1)$. This yields (2.16). $\square$

Lemma 2.7 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $\varepsilon\lambda>2\sqrt{s}$ and $|a|>2\varepsilon\lambda+\sqrt{s}$, we have*

$$\begin{aligned}
&\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leq 2\varepsilon\lambda)\\
\geqslant\ & c_{12}(\delta)\,\exp\left(-\frac{a^2}{2(1-\delta)^2 s}\right)\overline{\Phi}\left(\frac{||z|-\sqrt{s}|}{\delta\sqrt{s}}\right),
\end{aligned}\tag{2.19}$$

with a constant $c_{12}(\delta)>0$.

Proof: Again, it suffices to treat the case $a>\sqrt{s}$. In this case, we have

$$\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leq 2\varepsilon\lambda)$$

$$\geqslant \int_0^{\delta s} \mathbb{P}^z(|Y(T_0)| \le 2\sqrt{s},\, T_0 \in \, \mathrm{d}h)\, \mathbb{P}(a \le W(s-h) \le a + 2\varepsilon\lambda,\, |Y(s-h)| \le 2\sqrt{s-h}\,)$$
$$\geqslant \; c_{13}(\delta) \exp\left(-\frac{a^2}{2(1-\delta)^2 s}\right) \mathbb{P}^z(|Y(T_0)| \le 2\sqrt{s},\, T_0 \le \delta s),$$

hence (2.19) follows from Lemma 2.3. □

2.2 Increments

Recall the results for the increments of Wiener process (cf. [9]) and principal value (cf. [4]).

As $T \to \infty$, we have almost surely

$$\sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} |X(t+s) - X(t)| = \mathcal{O}\left(\sqrt{a_T(\log(T/a_T) + \log\log T)}\right), \tag{2.20}$$

and for fixed T, as $\delta \to 0$ we have almost surely

$$\sup_{0\leqslant t\leqslant T} \sup_{0\leqslant s\leqslant \delta} |X(t+s) - X(t)| = \mathcal{O}(\sqrt{\delta \log(1/\delta)}). \tag{2.21}$$

Here in (2.20) and (2.21) X can be either W or Y.

3 Proof of Theorem 1.1

According to (2.20) for Y,

$$\lim_{\delta\to 0} \sup_{t\geqslant 1} \sup_{0\leqslant x,\, x'\leqslant 1,\, |x-x'|\leqslant\delta} |Z_t(x) - Z_t(x')| \to 0, \qquad \text{a.s.}$$

Now the relative compactness of $\{Z_t\}_{t\geqslant 1}$ in $\mathcal{C}$ follows from the Arzelà–Ascoli theorem. This fact and Theorem A imply that $\{(U_t, Z_t)\}$ is relatively compact in $\mathcal{C}^{(2)}$. Our further proof will consist of two steps:

(1) With probability one any $(f,g) \notin \widetilde{\mathcal{S}}_J^{(2)}$ is not a limit point.

(2) With probability one every $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ is a limit point.

Proof of (1): Obviously, if either $f \notin \mathcal{S}$, or $g(0) \neq 0$, then (f,g) cannot be a limit point almost surely. So from now on we assume that $f \in \mathcal{S}$ and $g(0) = 0$. Let $x_0 \in (0,1]$ be a point, where $f(x_0) \neq 0$. Since f is continuous, there exists an interval $(x_1, x_2) \subset [0,1]$ such that $x_0 \in (x_1, x_2]$ and $f(x) \neq 0$ for all $x \in (x_1, x_2)$. We show that if (f, g) is a limit point, then g is constant in (x_1, x_2). Since (f, g) is a limit point, there exists a sequence $\{t_n\}_{n \geqslant 1}$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$|W(xt_n)| \geqslant c_{14} \sqrt{2t_n \log\log t_n}, \quad x \in (x_1, x_2)$$

for some $c_{14} > 0$ and for every $x \in (x_1, x_2)$

$$\begin{aligned} \left| \frac{Y(xt_n) - Y(x_0 t_n)}{\sqrt{8t_n \log\log t_n}} \right| &= \frac{1}{\sqrt{8t_n \log\log t_n}} \left| \int_{x_0 t_n}^{xt_n} \frac{\mathrm{d}s}{W(s)} \right| \\ &\leqslant \frac{|xt_n - x_0 t_n|}{4c_{14}\, t_n \log\log t_n} \to 0 = g(x) - g(x_0), \end{aligned}$$

as $n \to \infty$. So $g(x) = g(x_0)$ for every $x \in (x_1, x_2)$. So if (f, g) is a limit point and g is absolutely continuous (which is not guaranteed so far), then we must have $f(x)g'(x) = 0$ a.e.

To this end, we need a lemma.

Lemma 3.1 *Let (f, g) be such that $f \in \mathcal{S}$, $g(0) = 0$ and either g is not absolutely continuous or $f(x)g'(x) = 0$ a.e., and*

$$\int_0^1 \left((f'(x))^2 + (g'(x))^2 \right) \mathrm{d}x > 1, \tag{3.1}$$

holds. Then there exists a partition $x_0 = 0 < x_1 < \ldots < x_{k-1} < x_k = 1$ of $[0, 1]$ such that for any $\delta > 0$ small enough, we have

$$\Lambda_k := \sum_{i=1}^{k} \left(\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} \mathbb{1}_{\{g_i - g_{i-1} = 0\}} + \frac{8}{8 + \delta} \frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}} \right) > 1, \tag{3.2}$$

where $f_i := f(x_i)$ and $g_i := g(x_i)$.

Proof: If g is not absolutely continuous, then we can clearly find a partition $x_0 = 0 < x_1 < \ldots < x_{j-1} < x_j = 1$ of $[0, 1]$ such that

$$\sum_{i=1}^{k} \frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}} > 1 + \frac{\delta}{8},$$

so we have also (3.2). If g is absolutely continuous and (3.1) holds, then we can find a partition $x_0 = 0 < x_1 < \ldots < x_{j-1} < x_j = 1$ of $[0, 1]$ such that

$$\sum_{i=1}^{j} \left(\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} + \frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}} \right) > 1$$

holds. Moreover, for any small enough $\delta > 0$, we have also

$$\sum_{i=1}^{j}\left(\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} + \frac{8}{8+\delta}\frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}}\right) > 1.$$

For the ith interval of the above partition consider the following three cases: (i) $f_{i-1} = f_i$, (ii) $f_{i-1} \neq f_i$, and $f_{i-1}f_i \geqslant 0$, (iii) $f_{i-1} \neq f_i$, and $f_{i-1}f_i < 0$. In case (i) we can simply write

$$\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} = \frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}}\mathbb{1}_{\{g_i - g_{i-1} = 0\}}.$$

In case (ii) let $x_i' = \max\{x \leqslant x_i : f(x) = f(x_{i-1})\}$ and $x_i'' = \min\{x \geqslant x_i' : f(x) = f(x_i)\}$. (It is possible that $x_i' = x_{i-1}$ or $x_i'' = x_i$.) Consider the refinement of the partition by replacing (x_{i-1}, x_i) with (x_{i-1}, x_i'), (x_i', x_i''), (x_i'', x_i). In the interval (x_i', x_i'') $f(x)$ must strictly be between f_{i-1} and f_i, so $f(x) \neq 0$, hence $g'(x) = 0$ for all $x \in (x_i', x_i'')$, thus $g(x_i') = g(x_i'')$. Using the elementary inequality

$$\frac{(a+b)^2}{c+d} \leqslant \frac{a^2}{c} + \frac{b^2}{d},$$

we may write

$$\begin{aligned}
\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} &\leqslant \frac{(f(x_i') - f_{i-1})^2}{x_i' - x_{i-1}} + \frac{(f(x_i'') - f(x_i'))^2}{x_i'' - x_i'} + \frac{(f_i - f(x_i''))^2}{x_i - x_i''} \\
&= \frac{(f(x_i') - f_{i-1})^2}{x_i' - x_{i-1}}\mathbb{1}_{\{g(x_i') - g_{i-1} = 0\}} + \frac{(f(x_i'') - f(x_i'))^2}{x_i'' - x_i'}\mathbb{1}_{\{g(x_i'') - g(x_i') = 0\}} \\
&+ \frac{(f_i - f(x_i''))^2}{x_i - x_i''}\mathbb{1}_{\{g_i - g(x_i'') = 0\}}. \qquad (3.3)
\end{aligned}$$

In case (iii) let $x_i' = \min\{x \geqslant x_{i-1} : f(x) = 0\}$ and $x_i'' = \max\{x \leqslant x_i : f(x) = 0\}$. Consider again the refinement of the partition by replacing (x_{i-1}, x_i) with (x_{i-1}, x_i'), (x_i', x_i''), (x_i'', x_i). In the first and the last of these three intervals $f(x) \neq 0$, hence $g'(x) = 0$, thus $g(x_i') = g_{i-1}$ and $g(x_i'') = g_i$. On the other hand, $f(x_i') = f(x_i'') = 0$. So we again have (3.3).

By repeating this argument, we get finally a partition for which (3.2) holds. This completes the proof of the Lemma. □

Returning to the main course of the proof, choose $\varepsilon > 0$ such that

$$\begin{aligned}
&\Lambda_k - 20\varepsilon\sum_{i=1}^{k}\frac{1}{x_i - x_{i-1}} > 1, \\
&(f_{i-1} - \varepsilon, f_{i-1} + \varepsilon) \quad \text{and} \quad (f_i - \varepsilon, f_i + \varepsilon) \quad \text{are disjoint if } f_i \neq f_{i-1},
\end{aligned}$$

$$|g_i - g_{i-1}| > 6\varepsilon \quad \text{if} \quad g_i \neq g_{i-1}.$$

Here $f_i = f(x_i)$ and $g_i = g(x_i)$, $i = 1, \ldots, k$. We may also assume that $|f_i - f_{i-1}| \leqslant 1$ and $|g_i - g_{i-1}| \leqslant 1$, $i = 1, \ldots, k$, otherwise (f, g) cannot be a limit point by the usual law of the iterated logarithm.

Define the events

$$\begin{aligned} A_t^{(i)} &= \{f_i - \varepsilon \leqslant U_t(x_i) \leqslant f_i + \varepsilon,\ g_i - g_{i-1} - 2\varepsilon \leqslant Z_t(x_i) - Z_t(x_{i-1}) \leqslant g_i - g_{i-1} + 2\varepsilon\} \\ &= \{a_i \leqslant W(s_i) \leqslant b_i,\ \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i\} \end{aligned}$$

with $s_i = x_i t$ and

$$a_i = (f_i - \varepsilon)(2t \log\log t)^{1/2}, \quad b_i = (f_i + \varepsilon)(2t \log\log t)^{1/2},$$

$$\alpha_i = (g_i - g_{i-1} - 2\varepsilon) 2 (2t \log\log t)^{1/2}, \quad \beta_i = (g_i - g_{i-1} + 2\varepsilon) 2 (2t \log\log t)^{1/2}.$$

It follows from Lemma 2.1 putting $\lambda = (2t \log\log t)^{1/2}$ there

$$\mathbb{P}(A_t^{(i)} \,|\, W(s_{i-1}) = z_{i-1}) \leqslant \sqrt{\frac{2 \log\log t}{x_i - x_{i-1}}} \exp\left(-\frac{(f_i - f_{i-1})^2 - 8\varepsilon}{x_i - x_{i-1}} \log\log t \right)$$

and if $g_i \neq g_{i-1}$, then

$$\mathbb{P}(A_t^{(i)} \,|\, W(s_{i-1}) = z_{i-1}) \leqslant c_{15} \exp\left(-\frac{(g_i - g_{i-1})^2 - 20\varepsilon}{(8 + \delta)(x_i - x_{i-1})} 8 \log\log t \right)$$

with some $c_{15} > 0$. So for large enough t we have

$$\begin{aligned} \mathbb{P}(A_t^{(i)} \,|\, W(s_{i-1}) = z_{i-1}) \leqslant\ & c_{16} \sqrt{\frac{\log\log t}{x_i - x_{i-1}}} \Bigg[\exp\left(-\frac{(f_i - f_{i-1})^2 - 8\varepsilon}{x_i - x_{i-1}} \log\log t \right) \mathbb{1}_{\{g_i = g_{i-1}\}} \\ & + \exp\left(-\frac{(g_i - g_{i-1})^2 - 20\varepsilon}{(8 + \delta)(x_i - x_{i-1})} 8 \log\log t \right) \mathbb{1}_{\{g_i \neq g_{i-1}\}} \Bigg]. \end{aligned}$$

It follows that for all large t and some constants $c_{17} > 0$ and $\widetilde{\delta} > 0$,

$$\begin{aligned} \mathbb{P}(\cap_{i=1}^k A_t^{(i)}) &\leqslant c_{17} (\log\log t)^{3k/2} \exp\left(-\left(\Lambda_k - 20\varepsilon \sum_{i=1}^k \frac{1}{x_i - x_{i-1}} \right) \log\log t \right) \\ &\leqslant \exp(-(1 + \widetilde{\delta}) \log\log t). \end{aligned}$$

Let $t = t_n = \exp(n/(\log n))$. Then $\sum_n \mathbb{P}(A_{t_n}) < \infty$. By the Borel–Cantelli lemma,

$$\liminf_{n\to\infty} d\left((U_{t_n}, Z_{t_n}),\ (f, g) \right) \geqslant \varepsilon \qquad \text{a.s.} \tag{3.4}$$

On the other hand, we infer from the increment results in Section 2.3 that

$$
\begin{aligned}
&(3.5) \qquad \lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1}]} \sup_{x\in[0,1]} |U_t(x) - U_{t_n}(x)| &= 0 \qquad \text{a.s.},\\
&(3.6) \qquad \lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1}]} \sup_{x\in[0,1]} |Z_t(x) - Z_{t_n}(x)| &= 0 \qquad \text{a.s.},
\end{aligned}
$$

Combining (3.5)–(3.6) with (3.4) gives that

$$\liminf_{t\to\infty} d\left((U_t, Z_t)\,,\,(f,g)\right) \geqslant \varepsilon \qquad \text{a.s.}$$

for some $\varepsilon > 0$.

Thus we proved that if $(f,g) \notin \widetilde{\mathcal{S}}_J^{(2)}$, then it is not a limit point with probability one, i.e. (f,g) has an open ball neighboorhood of radius ε not containing (U_t, Z_t) for large enough t. However the exceptional ω-set of probability zero may depend on (f,g). Now we prove that the totality of these exceptional ω-sets is still of probability zero. Denote the complement of $\widetilde{\mathcal{S}}_J^{(2)}$ by $\mathcal{D}$ and for each $(f,g) \in \mathcal{D}$ consider the open balls defined above. Their union covers $\mathcal{D}$ and being $\mathcal{C}^{(2)}$ separable, we can select a countable subcover (cf. e.g. [2], p. 217). The union of exceptional ω-sets belonging to this countable subcover is still of probability zero. We call the complement of this last set of probability zero as our universal ω-set. Each $(f,g) \in \mathcal{D}$ has a neighborhood which is completely contained in one of the elements of the countable subcover, hence on the universal set this neighborhood for large enough t does not contain (U_t, Z_t), i.e. (f,g) is not a limit point. This completes the proof of (1). $\square$

Proof of (2): Assume that $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ with strict inequality in the integral criterion, i.e.

$$\int_0^1 \left((f'(x))^2 + (g'(x))^2\right) \mathrm{d}x < 1.$$

For given $\varepsilon_1 > 0$, choose a partition $x_0 = 0 < x_1 \ldots < x_k = 1$ of the interval $[0,1]$ such that

$$
\begin{aligned}
\sup_{1\leqslant i\leqslant k} (x_i - x_{i-1}) &\leqslant \varepsilon_1^2,\\
\sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |f(x) - f_i| &\leqslant \varepsilon_1,\\
\sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |g(x) - g_i| &\leqslant \varepsilon_1,
\end{aligned}
$$

where $f_i = f(x_i)$, $g_i = g(x_i)$. We may assume that if $g_{i-1} \neq g_i$, then $f_{i-1} = f_i = 0$. Otherwise if it happens that $g_{i-1} \neq g_i$ but either $f_{i-1} \neq 0$ or $f_i \neq 0$ (or both), then we can choose $x' = \min\{x : x > x_{i-1},\ f(x) = 0\}$, $x'' = \max\{x : x < x_i,\ f(x) = 0\}$. We must have

$g(x')=g_{i-1}$ and $g(x'')=g_i$ so by refining the original partition by inserting new points x', x'', the new partition satisfies the above assumption. Since

$$\frac{(f(x_i)-f(x_{i-1})^2}{x_{i-1}-x_i}\leqslant\int_{x_{i-1}}^{x_i}(f'(x))^2\,\mathrm{d}x,\qquad \frac{(g(x_i)-g(x_{i-1}))^2}{x_{i-1}-x_i}\leqslant\int_{x_{i-1}}^{x_i}(g'(x))^2\,\mathrm{d}x,$$

(cf. for example [10], p. 52), we have

$$\bar\Lambda_k:=\sum_{i=1}^k\frac{(f_i-f_{i-1})^2+(g_i-g_{i-1})^2}{x_i-x_{i-1}}<1. \tag{3.7}$$

Now choose $0<\delta<1$ such that $\bar\Lambda_k<(1-\delta)^2$ and then choose $\varepsilon>0$ such that

$$\Gamma:=\frac{\bar\Lambda_k}{(1-\delta)^2}+\left(\frac{20\varepsilon}{(1-\delta)^2}+\frac{2\varepsilon}{\delta^2}\right)\sum_{i=1}^k\frac{1}{x_i-x_{i-1}}<1 \tag{3.8}$$

and $5\varepsilon<|f_i|$, $i=1,2,\ldots,k$.

Introduce the notations $\lambda=(2t\log\log t)^{1/2}$, $s_i=tx_i$,

$$a_i=(f_i-\varepsilon)\lambda,\qquad b_i=(f_i+\varepsilon)\lambda,$$

$$\alpha_i=2(g_i-g_{i-1}-\varepsilon)\lambda,\qquad \beta_i=2(g_i-g_{i-1}+\varepsilon)\lambda.$$

By using the strong Markov property of the Wiener process, it is readily seen that

$$\begin{aligned}&\mathbb{P}(a_i\leqslant W(s_i)\leqslant b_i,\ \alpha_i\leqslant Y(s_i)-Y(s_{i-1})\leqslant\beta_i,\ i=1,2,\ldots,k)\\ \geqslant\ &\mathbb{P}(a_i\leqslant W(s_i)\leqslant b_i,\ \alpha_i\leqslant Y(s_i)-Y(s_{i-1})\leqslant\beta_i,\ i=1,2,\ldots,k-1)\times\\ &\times\inf_{a_{k-1}\leqslant z_{k-1}\leqslant b_{k-1}}\mathbb{P}(a_k\leqslant W(s_k)\leqslant b_k,\ \alpha_k\leqslant Y(s_k)-Y(s_{k-1})\leqslant\beta_k\,|\,W(s_{k-1})=z_{k-1}).\end{aligned}$$

Iterating this argument we can see that

$$\begin{aligned}&\mathbb{P}(a_i\leqslant W(s_i)\leqslant b_i,\ \alpha_i\leqslant Y(s_i)-Y(s_{i-1})\leqslant\beta_i,\ i=1,2,\ldots,k)\\ \geqslant\ &\prod_{i=1}^k\inf_{a_{i-1}\leqslant z_{i-1}\leqslant b_{i-1}}\mathbb{P}(a_i\leqslant W(s_i)\leqslant b_i,\ \alpha_i\leqslant Y(s_i)-Y(s_{i-1})\leqslant\beta_i\,|\,W(s_{i-1})=z_{i-1}).\end{aligned} \tag{3.9}$$

Next we show that for $i=1,2,\ldots,k$ we have

$$\begin{aligned}&\mathbb{P}(a_i\leqslant W(s_i)\leqslant b_i,\ \alpha_i\leqslant Y(s_i)-Y(s_{i-1})\leqslant\beta_i\,|\,W(s_{i-1})=z_{i-1})\geqslant\frac{c_{18}(\delta)}{(\log\log t)^{1/2}}\times\\ &\times\exp\left(-\left(\frac{(f_i-f_{i-1})^2+(g_i-g_{i-1})^2+20\varepsilon}{(1-\delta)^2(x_i-x_{i-1})}+\frac{2\varepsilon}{\delta^2(x_i-x_{i-1})}\right)\log\log t\right)\end{aligned} \tag{3.10}$$

with some $c_{18}(\delta) > 0$. To see (3.10) we apply Lemmas 2.4–2.7 with $s = s_i - s_{i-1} = t(x_i - x_{i-1})$, $\lambda = (2t \log\log t)^{1/2}$ and t large enough and use the inequalities $|f_i - f_{i-1}| \leqslant 1$, $|g_i - g_{i-1}| \leqslant 1$, $\varepsilon < 1$.

(1) In case $f_i = f_{i-1} = 0$, apply Lemma 2.4 with $\alpha = (g_i - g_{i-1} - \varepsilon)\lambda$, $|z| \leqslant \varepsilon\lambda$ and observe that by (2.10), $\overline{\Phi}$ gives a constant$\times(\log\log t)^{-1/2}$ factor in front of the exponent.

(2) In case $g_i = g_{i-1}$, $f_i f_{i-1} > 0$, apply Lemma 2.5 with $a = (f_i - \varepsilon)\lambda$ and use $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$.

(3) In case $g_i = g_{i-1}$, $f_i f_{i-1} < 0$, apply Lemma 2.6 with $a = (f_i - \varepsilon)\lambda$ and use $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$.

(4) In case $g_i = g_{i-1}$, $f_i = 0$, $f_{i-1} \neq 0$, apply Lemma 2.4 with $\alpha = -2\varepsilon\lambda$, use that $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$ and replace δ by $1 - \delta$.

(5) In case $g_i = g_{i-1}$, $f_i = 0$, $f_{i-1} \neq 0$, apply Lemma 2.7 with $a = (f_i - \varepsilon)\lambda$, $|z| \leqslant \varepsilon\lambda$.

Assembling all these estimations, (3.10) follows. This combined with (3.9) gives

$$\begin{aligned} &\mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\, i = 1, 2, \ldots, k) \\ &\geqslant \frac{(c_{18}(\delta))^k}{(\log\log t)^{k/2}} \exp(-\Gamma \log\log t), \end{aligned} \tag{3.11}$$

where $\Gamma < 1$ is given by (3.8).

Now let $t_i = \exp(7i \log i)$, $i = 1, 2, \ldots$ and define

$$\eta_0 = 0, \quad T_i = \eta_{i-1} + t_i, \quad \eta_i = \inf\{t : t > T_i,\, W(t) = 0\},\ i = 1, 2, \ldots$$

It was shown in [6] that we have almost surely for all large enough n,

$$t_n \leqslant T_n \leqslant t_n \left(1 + \frac{1}{n}\right).$$

Define

$$\begin{aligned} \widehat{W}^{(n)}(t) &= W(t + \eta_{n-1}), \quad t \geqslant 0, \\ \widehat{Y}^{(n)}(t) &= Y(t + \eta_{n-1}) - Y(\eta_{n-1}), \quad t \geqslant 0, \\ \widehat{U}^{(n)}(x) &= \frac{\widehat{W}^{(n)}(xt_n)}{\sqrt{2t_n \log\log t_n}}, \quad x \in [0, 1], \\ \widehat{Z}^{(n)}(x) &= \frac{\widehat{Y}^{(n)}(xt_n)}{\sqrt{2t_n \log\log t_n}}, \quad x \in [0, 1]. \end{aligned}$$

Now let $x_0 = 0 < x_1 < \ldots < x_k = 1$ be a partition as before and consider the events $\widehat{E}_n = \cap_{i=1}^k \widehat{E}_n^{(i)}$ with

$$\widehat{E}_n^{(i)} = \{\widehat{a}_i \leqslant \widehat{W}^{(n)}(\widehat{s}_i) \leqslant \widehat{b}_i,\ \widehat{\alpha}_i \leqslant \widehat{Y}^{(n)}(\widehat{s}_i) - \widehat{Y}^{(n)}(\widehat{s}_{i-1}) \leqslant \widehat{\beta}_i\},$$

$\widehat{s}_i = x_i t_n$,

$$\widehat{a}_i = (f_i - \varepsilon)(2t_n \log\log t_n)^{1/2}, \quad \widehat{b}_i = (f_i + \varepsilon)(2t_n \log\log t_n)^{1/2},$$

$$\widehat{\alpha}_i = (g_i - g_{i-1} - \varepsilon)^+(2t_n \log\log t_n)^{1/2}, \quad \widehat{\beta}_i = (g_i - g_{i-1} + \varepsilon)(2t_n \log\log t_n)^{1/2}.$$

It follows from (3.11) that $\sum_n \mathbb{P}(\widehat{E}_n) = \infty$ and since $\widehat{E}_n$ are independent, we have by the Borel–Cantelli lemma $\mathbb{P}(\widehat{E}_n \text{ i.o.}) = 1$. Since $\varepsilon > 0$ is arbitrary, this implies

$$\begin{aligned}
\liminf_{n\to\infty} \sup_{1\leqslant i\leqslant k} |\widehat{U}^{(n)}(x_i) - f(x_i)| &= 0 \qquad \text{a.s.} \\
\liminf_{n\to\infty} \sup_{1\leqslant i\leqslant k} |\widehat{Z}^{(n)}(x_i) - g(x_i)| &= 0 \qquad \text{a.s.}
\end{aligned}$$

Again, from the increment results in Subsection 2.2 it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i)} |\widehat{U}^{(n)}(x_{i-1}) - \widehat{U}^{(n)}(x)| &\leqslant \varepsilon_1 \qquad \text{a.s.} \\
\limsup_{n\to\infty} \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i)} |\widehat{Z}^{(n)}(x_{i-1}) - \widehat{Z}^{(n)}(x)| &\leqslant \varepsilon_1 \qquad \text{a.s.}
\end{aligned}$$

Since $\varepsilon_1 > 0$ is arbitrary, these yield

$$\liminf_{n\to\infty} d\left((\widehat{U}^{(n)}, \widehat{Z}^{(n)}),\ (f,g)\right) = 0 \qquad \text{a.s.}$$

On the other hand, the increment results in Subsection 2.2 once again yields that, as $n \to \infty$, $d((\widehat{U}^{(n)}, \widehat{Z}^{(n)}),\ (U_{T_n}, Z_{T_n}))$ converges to 0 almost surely. Therefore,

$$\liminf_{n\to\infty} d\left((U_{T_n}, Z_{T_n}),\ (f,g)\right) = 0 \qquad \text{a.s.},$$

Hence, (f,g) is a limit point of (U_t, Z_t) with probability 1.

To complete the proof of Theorem 1.1, we have to show that there exists an ω-set of probability one for which every $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ is a limit point.

First we show that there exists a countable dense subset $K \subset \widetilde{\mathcal{S}}_J^{(2)}$. For any $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ and $\varepsilon > 0$, as before, choose a partition $x_0 = 0 < x_1 < \ldots < x_{k-1} < x_k = 1$ such that

$$\sup_{x_{i-1}\leqslant x\leqslant x_i} |f(x) - f(x_i)| \leqslant \varepsilon, \qquad \sup_{x_{i-1}\leqslant x\leqslant x_i} |g(x) - g(x_i)| \leqslant \varepsilon$$

and $g(x_{i-1}) \neq g(x_i)$ implies $f(x_{i-1}) = f(x_i) = 0$. Define $(\widetilde{f}, \widetilde{g}) \in S_J^{(2)}$ such that $\widetilde{f}(x_i) = f(x_i)$, $\widetilde{g}(x_i) = g(x_i)$, $i = 1, 2, \ldots, k$ and let $\widetilde{f}$ and $\widetilde{g}$ be linear in between. Then

$$d((f, g),\, (\widetilde{f}, \widetilde{g})) < 2\sqrt{2}\varepsilon,$$

meaning that the set of pairs (f, g), where both f and g are piecewise linear (with the same cut-off points), is dense. It can be seen that one can choose a countable dense subset $K = \{(f_n, g_n)\}_{n=1}^{\infty}$ (for example by taking all x_i, $f_n(x_i)$, $g_n(x_i)$ rational) such that

$$\int_0^1 (f_n'(x))^2 + (g_n'(x))^2 \,\mathrm{d}x < 1.$$

It follows that there exists an ω-set of probability one such that all $(f_n, g_n) \in K$ are limit points. Next we show that for this ω-set every $(f, g) \in \widetilde{\mathcal{S}}_J^{(2)}$ is a limit point. Since K is dense, for each n we find $(f_n, g_n) \in K$ such that

$$d((f, g),\, (f_n, g_n)) < \frac{1}{n}$$

and since (f_n, g_n) is a limit point, we can find t_n such that $d((f_n, g_n),\, (U_{t_n}, Z_{t_n})) < \frac{1}{n}$. Hence $d((f, g),\, (U_{t_n}, Z_{t_n})) < \frac{2}{n}$. Consequently,

$$\lim_{n\to\infty} (U_{t_n}, Z_{t_n}) = (f, g),$$

i.e., (f, g) is a limit point.

This completes the proof of Theorem 1.1. □

4 Proof of Corollaries

The proof of Corollary 1.2 is obvious. To show Corollary 1.3 we need the following lemma.

Lemma 4.1 *If f and g are absolutely continuous functions and $f(x)g'(x) = 0$ a.e., then*

$$\int_0^1 (f'(x))^2\, \mathbb{1}_{\{g'(x)\neq 0\}} \,\mathrm{d}x = 0. \tag{4.1}$$

Proof: Let

$$\mathcal{A} = \{x \in [0,1] : \ f(x) = 0, \ f'(x) \neq 0\}.$$

For each $x \in \mathcal{A}$, there exists $\delta_x > 0$ such that $f(y) \neq 0$ for all $y \in (x - \delta_x, \ x + \delta_x) \backslash \{x\}$. The intervals $\{(x - \delta_x, \ x + \delta_x)\}_{x \in \mathcal{A}}$ being disjoint and thus containing each a different rational number, they are at most countably many. This means $\mathcal{A}$ is a countable set. Now (4.1) follows immediately.

This proof, more elegant than our original one, was kindly communicated to us by Omer Adelman. □

Now we prove Corollary 1.3. It follows from Lemma 4.1 that if $(f,g) \in \widetilde{S}_J^{(2)}$, then

$$\int_0^1 (f'(x) + g'(x))^2 \, \mathrm{d}x \leqslant 1, \qquad \int_0^1 (f'(x) - g'(x))^2 \, \mathrm{d}x \leqslant 1,$$

from which (cf. [15])

$$|f(1) + g(1)| \leqslant 1, \qquad |f(1) - g(1)| \leqslant 1$$

showing that a limit point cannot be outside the set given in the Corollary.

To show that every point is a limit point, define

$$f(u) = \frac{x(u - 1 + |x|)}{|x|} 1\!\!1_{\{1-|x| \le u \le 1\}}, \qquad g(u) = \frac{yu}{|y|} 1\!\!1_{\{0 \le u \le |y|\}} + y 1\!\!1_{\{|y| \le u \le 1\}}.$$

It is easy to see that $(f,g) \in \widetilde{S}_J^{(2)}$ and $f(1) = x$, $g(1) = y$. So (x,y) is a limit point. □

5 Further consequences: additive functionals

Consider the additive functional

$$A(t) = \int_0^t \psi(W(s)) \, \mathrm{d}s = \int_{\mathbb{R}} \psi(x) L(t,x) \, \mathrm{d}x,$$

where ψ is an integrable function such that $\overline{\psi} := \int_{\mathbb{R}} \psi(x) \, \mathrm{d}x \neq 0$. Then by the ratio ergodic theorem (cf. [12], p. 228)

$$\lim_{t \to \infty} \frac{A(t)}{\overline{\psi} \, L(t)} = 1 \qquad \text{a.s.}$$

Hence, introducing

$$\widetilde{V}_t(x) := \frac{A(xt)}{\overline{\psi} \sqrt{2t \log \log t}},$$

Theorem C implies

Corollary 5.1 *With probability one, the set* $\{(U_t, \widetilde{V}_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\mathcal{S}_J^{(2)} := \left\{(f,g) :\ f \in \mathcal{S}, g \in \mathcal{S}_M, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \leq 1,\ f(x)g'(x) = 0 \text{ a.e.}\right\}.$$

On the other hand, there are additive functionals which can be approximated by the principal value $Y(t)$. Let ψ be a function as above and consider its Hilbert transform:

$$\mathcal{H}(\psi)(x) = \frac{1}{\pi}\,\text{p.v.} \int_{\mathbb{R}} \frac{\psi(y)}{x-y}\,\mathrm{d}y,$$

where p.v. indicates that the integral should be considered as a principal value. It was shown in [8] that if ψ is a Borel function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^\kappa |\psi(x)|\,\mathrm{d}x < \infty,$$

for some $\kappa > 0$, then for all sufficiently small $\varepsilon > 0$, when $t \to \infty$,

$$B(t) := \int_0^t (\mathcal{H}\psi)(W(s))\,\mathrm{d}s = \frac{\overline{\psi}}{\pi}\,Y(t) + o(t^{1/2-\varepsilon}), \qquad \text{a.s.}$$

Introducing the notation

$$\widetilde{Z}_t(x) = \frac{\pi B(xt)}{\overline{\psi}\,\sqrt{8t\log\log t}},$$

we have

Corollary 5.2 *With probability one, the set* $\{(U_t, \widetilde{Z}_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\widetilde{\mathcal{S}}_J^{(2)} = \left\{(f,g) :\ f \in \mathcal{S}, g \in \mathcal{S}, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \leq 1,\ f(x)g'(x) = 0 \text{ a.e.}\right\}.$$

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Endre Csáki
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
P.O.B. 127
H-1364 Budapest
Hungary
`csaki@renyi.hu`

Antónia Földes
Department of Mathematics
City University of New York
2800 Victory Blvd.
Staten Island, New York 10314
U.S.A.
`afoldes@gc.cuny.edu`

Zhan Shi
Laboratoire de Probabilités UMR 7599
Université Paris VI
4 Place Jussieu
F-75252 Paris Cedex 05
France
`zhan@proba.jussieu.fr`