On the behavior of random walk around heavy points

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Abstract: Consider a symmetric aperiodic random walk in Z^d, $d \geq 3$. There are points (called heavy points) where the number of visits by the random walk is close to its maximum. We investigate the local times around these heavy points and show that they converge to a deterministic limit as the number of steps tends to infinity.

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Running title: Random walk around heavy points

1. Introduction and main results

Consider a random walk $\{S_n\}_{n=1}^{\infty}$ starting at the origin on the d-dimensional integer lattice Z^d, i.e. $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$, $n = 1, 2, \ldots$, where X_k, $k = 1, 2, \ldots$ are i.i.d. random variables with distribution

$$\mathbf{P}(X_1 = x) = p(x), \quad x \in Z^d. \tag{1.1}$$

The random walk is called simple symmetric if $p(e_i) = 1/(2d)$, $i = 1, \ldots, 2d$, where $e_1, \ldots, e_d$ is a system of orthogonal unit vectors in Z^d and $e_i = -e_{i-d}$, $i = d+1, \ldots, 2d$.

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Denote by Q the covariance matrix of X_1, and let $|Q|$ be its determinant and let Q^{-1} its inverse. Let

$$\|x\|^2 := xQ^{-1}x. \tag{1.2}$$

For simple symmetric random walk $\|x\|^2 = |x|^2 := x_1^2 + \cdots + x_d^2$, where $x = (x_1, \ldots, x_d)$.

Recall the following definitions and basic properties from Spitzer [9].

A random walk is aperiodic if for

$$R^+ = \{x \in Z^d : \mathbf{P}(S_n = x) > 0 \,\text{for some}\, n \geq 0\}$$

we have

$$\{x : x = y - z, \text{ for some}\, y \in R^+, \, z \in R^+\} = Z^d.$$

A random walk is strongly aperiodic if for each $x \in Z^d$

$$\{y : y = x + z, \text{ where}\, p(z) > 0\} = Z^d.$$

We assume throughout the paper that the random walk is aperiodic (but not necessarily strongly aperiodic) and symmetric, i.e. $p(x) = p(-x)$, $x \in Z^d$.

For $d \geq 3$ the random walk is transient, i.e.

$$\gamma := \mathbf{P}(S_i \neq 0, \, i = 1, 2, \ldots) > 0. \tag{1.3}$$

Define

$$\gamma_x := \mathbf{P}(S_i \neq x, \, i = 1, 2, \ldots), \quad x \in Z^d. \tag{1.4}$$

We shall impose the following moment conditions:

$$\sum_{x \in Z^d} |x|^2 p(x) < \infty, \qquad d = 3, \tag{1.5}$$

$$\sum_{x \in Z^d} |x|^2 \log(|x| + 1) p(x) < \infty, \qquad d = 4, \tag{1.6}$$

$$\sum_{x \in Z^d} |x|^{d-2} p(x) < \infty, \qquad d \geq 5, \tag{1.7}$$

where $|x|$ is the Euclidean distance.

The Green function is defined by

$$G(x) := \sum_{n=0}^{\infty} \mathbf{P}(S_n = x), \quad x \in Z^d. \tag{1.8}$$

We have the identities

$$\gamma = \frac{1}{G(0)}, \qquad 1 - \gamma_x = \frac{G(x)}{G(0)},\ x \neq 0.$$

We need the following asymptotic property for the Green function in the case of aperiodic random walk with mean 0, satisfying the moment conditions (1.5), (1.6), (1.7) for $d \geq 3$.

$$G(x) \sim c_d |Q|^{-1/2} \|x\|^{2-d}, \quad |x| \to \infty \tag{1.9}$$

with some constant c_d. See Spitzer [9], p. 308 for $d = 3$, p. 339, Problem 5 for $d > 3$, or Uchiyama [10] for strongly aperiodic case and use Spitzer's trick ([9], p. 310) to reduce the aperiodic case to strongly aperiodic case. For simple random walk see Révész [8].

In this paper we are interested in studying local times of the random walk defined by the number of visits as follows.

$$\xi(x, n) := \sum_{k=1}^{n} I\{S_k = x\}, \quad n = 1, 2, \dots,\ x \in Z^d, \tag{1.10}$$

where $I\{A\}$ denotes the indicator of A.

Since the random walk is transient for $d \geq 3$, typically there is only a finite number of visits to a fixed site, even for infinite time. More precisely we have the distribution

$$\mathbf{P}(\xi(0, \infty) = k) = \gamma(1-\gamma)^k, \quad k = 0, 1, 2, \dots \tag{1.11}$$

Cf. Erdős and Taylor [4] for simple random walk. The general case is similar.

There are however (random) points where the random walk accumulates a higher number of visits. Consider the maximal local time

$$\xi(n) := \max_{x \in Z^d} \xi(x, n), \quad n = 1, 2, \dots \tag{1.12}$$

and also

$$\eta(n) := \max_{0 \leq j \leq n} \xi(S_j, \infty), \quad n = 1, 2, \dots \tag{1.13}$$

Erdős and Taylor [4] proved for simple random walk and $d \geq 3$

$$\lim_{n \to \infty} \frac{\xi(n)}{\log n} = \lambda := -\frac{1}{\log(1-\gamma)} \qquad \text{a.s.} \tag{1.14}$$

Following the proof of Erdős and Taylor, without any new idea, one can prove that (1.14) holds for general aperiodic random walk and also

$$\lim_{n\to\infty} \frac{\eta(n)}{\log n} = \lambda \qquad \text{a.s.} \tag{1.15}$$

For general treatment of similar strong theorems for local and occupation times see [3].

(1.14) means that there are sites where the local time up to time n is around $\lambda \log n$. These will be called heavy points. We are interested in the problem what happens around these heavy points. We may ask whether it is possible that in a close neighborhood of a heavy point there is another heavy point? Or an empty point (not visited at all up to time n)? We shall see that the answers for both questions happen to be negative.

In [2] we investigated the joint asymptotic behavior of local times of two neighboring sites for simple random walk and found that the vector

$$\left(\frac{\xi(x,n)}{\log n}, \frac{\xi(x+e_1,n)}{\log n} \right)$$

is essentially in the domain

$$\{y \geq 0, z \geq 0 : -(y+z)\log(y+z) + y\log y + z\log z - (y+z)\log\alpha \leq 1\},$$

where

$$\alpha := \frac{1-\gamma}{2-\gamma}.$$

One can see that the only point in this domain with $y = \lambda$ is $z = \lambda(1-\gamma)$, which tells us that if a point is heavy, i.e. its local time is around $\lambda \log n$, then the local time of any of its neighbors should be around $\lambda(1-\gamma)\log n$, i.e. cannot fluctuate too much, at least asymptotically. We say that the local time around a heavy point is asymptotically deterministic. Our concern is to investigate this phenomenon further and determine the asymptotic value of local times of sites x with $\|x\| \leq r_n$, where r_n may tend to infinity at a certain rate.

Define

$$m_x = \begin{cases} 1 & \text{if} \quad x = 0, \\ \frac{(1-\gamma_x)^2}{1-\gamma} & \text{if} \quad x \neq 0. \end{cases} \tag{1.16}$$

m_x is, in fact, the expectation of the local time at x between two consecutive returns to zero (see Remark 2.1).

We shall consider the "balls" (which are, in fact, ellipsoids in Euclidean space)

$$B(r) = \{x : \|x\| \leq r\}, \tag{1.17}$$

where $\|x\|$ is defined by (1.2).

Theorem 1.1. *Let $d \geq 5$ and $k_n = (1-\delta_n)\lambda \log n$. Let $r_n > 0$ and $\delta_n > 0$ be selected such that δ_n is non-increasing, r_n is non-decreasing, and for any $c > 0$ $r_{[cn]}/r_n < C$ with some $C > 0$ and for*

$$\beta_n := r_n^{2d-4} \frac{\log \log n}{\log n} \tag{1.18}$$

$$\lim_{n\to\infty} \beta_n = 0, \qquad \lim_{n\to\infty} \delta_n r_n^{2d-4} = 0. \tag{1.19}$$

Define the random set of points

$$\mathcal{A}_n = \{z \in Z^d : \xi(z,n) \geq k_n\}. \tag{1.20}$$

Then we have for symmetric aperiodic random walk

$$\lim_{n\to\infty} \sup_{z\in\mathcal{A}_n} \sup_{x\in B(r_n)} \left| \frac{\xi(z+x,n)}{m_x \lambda \log n} - 1 \right| = 0 \quad \text{a.s.} \tag{1.21}$$

Theorem 1.2. *Let $d \geq 3$ and $k_n = (1-\delta_n)\lambda \log n$. Let $r_n > 0$ and $\delta_n > 0$ be selected such that δ_n is non-increasing, r_n is non-decreasing, and for any $c > 0$ $r_{[cn]}/r_n < C$ for some $C > 0$ and for*

$$\beta_n := r_n^{2d-4} \frac{\log \log n}{\log n} \tag{1.22}$$

$$\lim_{n\to\infty} \beta_n = 0, \qquad \lim_{n\to\infty} \delta_n r_n^{2d-4} = 0. \tag{1.23}$$

Define the random set of indices

$$\mathcal{B}_n = \{j \leq n : \xi(S_j, \infty) \geq k_n\}. \tag{1.24}$$

Then we have for symmetric aperiodic random walk

$$\lim_{n\to\infty} \sup_{j\in\mathcal{B}_n} \sup_{x\in B(r_n)} \left| \frac{\xi(S_j+x,\infty)}{m_x \lambda \log n} - 1 \right| = 0 \quad \text{a.s.} \tag{1.25}$$

Remark 1.1 *For a given ω, $\mathcal{A}_n$ or $\mathcal{B}_n$ can be empty. In this case $\sup_{z\in\mathcal{A}_n}$ or $\sup_{j\in\mathcal{B}_n}$ is automatically considered to be 0.*

Corollary 1.1 *Let $A \subset Z^d$ be a fixed set.*

(i) *If $d \geq 5$ and $z_n \in \mathcal{A}_n$, then*

$$\lim_{n\to\infty} \frac{\sum_{x\in A} \xi(x+z_n,n)}{\log n} = \lambda \sum_{x\in A} m_x \quad \text{a.s.}$$

(ii) *If* $d\geq 3$ *and* $j_n\in\mathcal{B}_n$, *then*

$$\lim_{n\to\infty}\frac{\sum_{x\in A}\xi(x+S_{j_n},\infty)}{\log n}=\lambda\sum_{x\in A}m_x \quad \text{a.s.}$$

From our Theorems it is obvious that the critical case is around $r_n\sim(\log n)^{1/(2d-4)}$. It follows that for smaller r_n the ball $S_j+B(r_n)$ is completely covered for $j\in\mathcal{B}_n$ with probability 1. We have the following Corollary.

Corollary 1.2 *For* $j\in\mathcal{B}_n$ *let* $R(n,j)$ *denote the largest number such that* $S_j+B(R(n,j))$ *is completely covered by the random walk* $S_0,S_1,S_2,\ldots$, *i.e.* $\xi(S_j+x,\infty)>0$, $x\in B(R(n,j))$. *Then for any* $\varepsilon>0$ *we have* $R(n,j)\geq(\log n)^{(1-\varepsilon)/(2d-4)}$ *almost surely.*

We conjecture that for $j\in\mathcal{B}_n$ we have $R(n,j)\leq(\log n)^{(1+\varepsilon)/(2d-4)}$. Our next result is one step in this direction, showing that in Theorems 1.2 the power $1/(2d-4)$ of $\log n$ cannot be improved in general.

Theorem 1.3. *For simple symmetric random walk let* $\{x_n\}$ *be a sequence such that* $|x_n|\sim c(\log n)^{1/(2d-4)}$ *for some* $c>0$. *Then with probability one there exist infinitely many* n *such that*

$$\xi(S_n,\infty)\geq\lambda\left(\log n+\left(\frac{d-4}{d-2}-\varepsilon\right)\log\log n\right),\quad \xi(S_n+x_n,\infty)=0.$$

Consequently, $n\in\mathcal{B}_n$ *and* $R(n,n)\leq c(\log n)^{1/(2d-4)}$ *infinitely often with probability one.*

2. Preliminary facts and results

First we present some more notations. For $x\in Z^d$ let T_x be the first hitting time of the point x, i.e. $T_x=\min\{i\geq 1: S_i=x\}$ with the convention that $T_x=\infty$ if there is no i with $S_i=x$. Denote $T_0=T$.

Introduce further

$$q_x:=\mathbf{P}(T<T_x), \tag{2.1}$$
$$s_x:=\mathbf{P}(T_x<T). \tag{2.2}$$

In words, q_x is the probability that the random walk, starting from 0, returns to 0, before hitting x (including $T<T_x=\infty$), and s_x is the probability that the random walk, starting from 0, hits x, before returning to 0 (including $T_x<T=\infty$).

Now we give the joint distribution of $\xi(0,\infty)$ and $\xi(x,\infty)$ in the following form.

Lemma 2.1. *For $x \neq 0$, $v < \log(1/(1-\gamma))$, $k = 0, 1, 2, \dots$*

$$\mathbf{E}(e^{v\xi(x,\infty)}; \xi(0,\infty) = k) = \left(q_x + \frac{s_x^2 e^v}{1 - q_x e^v}\right)^k (1 - q_x - s_x)\left(1 + \frac{s_x e^v}{1 - q_x e^v}\right) \tag{2.3}$$

$$= \gamma(1-\gamma)^k (\varphi(v))^k \psi(v), \tag{2.4}$$

where

$$\varphi(v) := \frac{1 - \frac{(1-\gamma)^2 - (1-\gamma_x)^2}{\gamma(1-\gamma)}(e^v - 1)}{1 - \frac{1-\gamma-(1-\gamma_x)^2}{\gamma}(e^v - 1)}, \tag{2.5}$$

$$\psi(v) := \frac{1 - \frac{\gamma_x - \gamma}{\gamma}(e^v - 1)}{1 - \frac{1-\gamma-(1-\gamma_x)^2}{\gamma}(e^v - 1)}, \tag{2.6}$$

Proof. Observe that

$$\mathbf{P}\left(\sum_{n=1}^{T} I\{S_n = x\} = j, T < \infty\right) = \begin{cases} q_x & \text{if } j = 0, \\ s_x^2 q_x^{j-1} & \text{if } j = 1, 2, \dots \end{cases} \tag{2.7}$$

and

$$\mathbf{P}\left(\sum_{n=1}^{T} I\{S_n = x\} = j,\, T = \infty\right) = \begin{cases} 1 - q_x - s_x & \text{if } j = 0, \\ s_x(1 - q_x - s_x) q_x^{j-1} & \text{if } j = 1, 2, \dots \end{cases} \tag{2.8}$$

Obviously

$$\xi(x,\infty) = Z_1 + \ldots + Z_{\xi(0,\infty)} + \hat{Z},$$

where $Z_1, \dots, Z_{\xi(0,\infty)}$ are the local times of x between consecutive returns to 0 and $\hat{Z}$ is the local time of x after the last return to zero. Hence (2.3) follows from (2.7) and (2.8). (2.4) can be obtained by using

$$q_x = 1 - \frac{\gamma}{1 - (1-\gamma_x)^2}, \tag{2.9}$$

$$s_x = (1-\gamma_x)(1-q_x). \tag{2.10}$$

(Cf. [1] or [8] for simple random walk, the general case being similar).

Remark 2.1 *It is easy to see that our condition* $v < \log(1/(1-\gamma))$ *implies* $q_x e^v < 1$, *needed to obtain* (2.3). *Furthermore*

$$\varphi(v) = \mathbf{E}\left(e^{v\sum_{n=1}^{T} I\{S_n=x\}} \mid T < \infty\right),$$

$$\psi(v) = \mathbf{E}\left(e^{v\sum_{n=1}^{T} I\{S_n=x\}} \mid T = \infty\right)$$

and

$$m_x = \mathbf{E}\left(\sum_{n=1}^{T} I\{S_n = x\} \mid T < \infty\right).$$

Further properties of q_x and s_x for simple symmetric random walk are given in the next Lemma.

Lemma 2.2. *For simple symmetric random walk and* $x \in Z^d$

$$\gamma_x \geq \gamma, \tag{2.11}$$

$$\frac{1-\gamma}{2-\gamma} \leq q_x \leq 1-\gamma, \tag{2.12}$$

$$1 - q_x - s_x \geq \frac{\gamma}{2-\gamma}, \tag{2.13}$$

$$q_x(n) := \mathbf{P}(T < \min(n, T_x)) = q_x + \frac{O(1)}{n^{d/2-1}}. \tag{2.14}$$

Proof. For (2.11) see [1], Lemma 2.4 and for (2.14) see [1], Lemma 2.5. (2.12) and (2.13) can be easily obtained from (2.9), (2.10) and (2.11).

The next result gives an estimation of φ and ψ, where the error term is uniform in x.

Lemma 2.3. *For* $\log(1-\gamma(1-\gamma)) < v < \log(1+\gamma(1-\gamma))$ *we have*

$$\varphi(v) = \exp(m_x(v + O(v^2))), \quad v \to 0, \tag{2.15}$$

where O *is uniform in* x,

$$\psi(v) \leq \frac{1 + |e^v - 1|}{1 - |e^v - 1|/\gamma}. \tag{2.16}$$

Proof. Write

$$\varphi(v) = \frac{1-u}{1-y}$$

with

$$u = \frac{(1-\gamma)^2 - (1-\gamma_x)^2}{\gamma(1-\gamma)}(e^v - 1), \quad y = \frac{1-\gamma-(1-\gamma_x)^2}{\gamma}(e^v - 1).$$

Then it is easy to see that

$$y - u = m_x(e^v - 1),$$

and

$$|u| \le \frac{|e^v - 1|}{\gamma(1-\gamma)}, \quad |y| \le \frac{|e^v - 1|}{\gamma(1-\gamma)}.$$

By Taylor series

$$\log\frac{1-u}{1-y} = \log(1-u) - \log(1-y) = y - u + \frac{y^2-u^2}{2} + \frac{y^3-u^3}{3} + \dots$$

$$= (y-u)\left(1 + \frac{y+u}{2} + \frac{y^2+uy+u^2}{3} + \dots\right).$$

Since $e^v - 1 = v + O(v^2)$, we have

$$\left|\log\frac{1-u}{1-y} - m_x(e^v-1)\right| \le m_x|e^v-1|\left(\frac{|e^v-1|}{\gamma(1-\gamma)} + \left(\frac{|e^v-1|}{\gamma(1-\gamma)}\right)^2 + \dots\right) = m_x O(v^2),$$

where O is independent of x. Hence (2.15) follows. (2.16) is obvious.

3. Proof of Theorem 1.2

Observe that $k_n \sim \lambda \log n$. Let $n_\ell = [e^\ell]$, $\xi(z) = \xi(z,\infty)$ and define the events

$$A_j = \left\{\xi(S_j) \ge k_{n_\ell},\ \sup_{x\in B(r_{n_{\ell+1}})}\left(\frac{\xi(S_j+x)}{m_x k_{n_\ell}} - 1\right) \ge \varepsilon\right\}$$

$$\mathbf{P}\left(\bigcup_{j=0}^{n_{\ell+1}} A_j\right) \le \sum_{j=0}^{n_{\ell+1}} \mathbf{P}(A_j) \le \sum_{j=0}^{n_{\ell+1}} \sum_{x\in B(r_{n_{\ell+1}})} \mathbf{P}(A_j^{(x)}),$$

where

$$A_j^{(x)} = \{\xi(S_j) \geq k_{n_\ell},\, \xi(S_j + x) \geq (1+\varepsilon) m_x k_{n_\ell}\}\,.$$

Consider the random walk obtained by reversing the original walk at S_j, i.e. let $S_i' := S_{j-i} - S_j$, $i = 0, 1, \ldots, j$ and extend it to infinite time, and also the forward random walk $S_i'' := S_{j+i} - S_j$, $i = 0, 1, 2, \ldots$ Then $\{S_0', S_1', \ldots\}$ and $\{S_0'', S_1'', \ldots\}$ are independent random walks and so are their respective local times ξ' and ξ". Moreover,

$$\xi(S_j) = \xi"(0) + \xi(S_j, j) \leq \xi"(0) + \xi'(0) + 1,$$

$$\xi(S_j + x) = \xi"(x) + \xi(S_j + x, j) \leq \xi"(x) + \xi'(x).$$

Here ξ' and ξ" are independent and have the same distribution as ξ.

Hence

$$\begin{aligned} \mathbf{P}(A_j^{(x)}) &= \mathbf{P}(\xi"(0) + \xi'(0) \geq k_{n_\ell} - 1,\, \xi"(x) + \xi'(x) \geq (1+\varepsilon) m_x k_{n_\ell}) \\ &= \sum \mathbf{P}(\xi"(0) = k_1, \xi'(0) = k_2, \xi"(x) + \xi'(x) \geq (1+\varepsilon) m_x k_{n_\ell}), \end{aligned}$$

where the summation goes for $k_1 + k_2 \geq k_{n_\ell} - 1$. Using exponential Markov inequality, Lemma 2.1, independence of ξ" and ξ' and elementary calculus, we get

$$\begin{aligned} \mathbf{P}(A_j^{(x)}) &\leq \sum \mathbf{E}\left(e^{v(\xi"(x) + \xi'(x))}, \xi"(0) = k_1, \xi'(0) = k_2\right) e^{-v(1+\varepsilon) m_x k_{n_\ell}} \\ &= \sum (\varphi(v))^{k_1 + k_2} \gamma^2 (1-\gamma)^{k_1 + k_2} \psi^2(v) e^{-v(1+\varepsilon) m_x k_{n_\ell}} \\ &= \gamma^2 \psi^2(v) e^{-v(1+\varepsilon) m_x k_{n_\ell}} \sum (\varphi(v)(1-\gamma))^{k_1 + k_2} \\ &= \gamma^2 \psi^2(v) e^{-v(1+\varepsilon) m_x k_{n_\ell}} (\varphi(v)(1-\gamma))^{k_{n_\ell}} \\ &\times \left(\frac{k_{n_\ell}}{\varphi(v)(1-\gamma)(1-\varphi(v)(1-\gamma))} + \frac{1}{(1-\varphi(v)(1-\gamma))^2}\right). \end{aligned}$$

By (2.15) we obtain for all j

$$\begin{aligned} \mathbf{P}(A_j^{(x)}) &\leq \gamma^2 \psi^2(v) \left(\frac{k_{n_\ell}}{\varphi(v)(1-\gamma)(1-\varphi(v)(1-\gamma))} + \frac{1}{(1-\varphi(v)(1-\gamma))^2}\right) \\ &\times e^{-m_x v k_{n_\ell}(\varepsilon + O(v))} (1-\gamma)^{k_{n_\ell}}. \end{aligned}$$

Choose $v_0 > 0$ small enough such that

$$\varepsilon + O(v_0) > 0, \quad e^{v_0} < 1 + \gamma(1-\gamma), \quad \varphi(v_0) < \frac{1}{1-\gamma}.$$

Using $x \in B(r_{n_{\ell+1}})$ and (1.9) we get

$$m_x k_{n_\ell} = \frac{(1-\gamma_x)^2}{1-\gamma}(\lambda \log n_\ell (1-\delta_{n_\ell})) \geq \frac{C_1(1-\delta_{n_\ell})\log n_\ell}{\|x\|^{2d-4}} \geq \frac{C_1(1-\delta_{n_\ell})\log n_\ell}{r_{n_{\ell+1}}^{2d-4}},$$

where here and in the sequel $C_1, C_2, \dots$ will denote positive constants whose values are unimportant in our proofs.

By the above assumptions

$$\begin{aligned}
\mathbf{P}(A_j^{(x)}) &\leq C_2 k_{n_\ell} e^{-m_x v_0 k_{n_\ell}(\varepsilon + O(v_0))}(1-\gamma)^{k_{n_\ell}} \\
&\leq C_2 k_{n_\ell} \exp\left(-(1-\delta_{n_\ell})\log n_\ell \left(\frac{C_3}{r_{n_{\ell+1}}^{2d-4}} + 1\right)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=0}^{n_{\ell+1}} \sum_{x \in B(r_{n_{\ell+1}})} \mathbf{P}(A_j^{(x)}) &\leq C_4 n_{\ell+1} r_{n_{\ell+1}}^d k_{n_\ell} \exp\left(-(1-\delta_{n_\ell})\log n_\ell \left(\frac{C_3}{r_{n_{\ell+1}}^{2d-4}} + 1\right)\right) \\
&\leq C_4 \frac{n_{\ell+1}}{n_\ell} k_{n_\ell} r_{n_{\ell+1}}^d \exp\left(-\frac{C_3 \log n_\ell}{r_{n_{\ell+1}}^{2d-4}} + \delta_{n_\ell}\log n_\ell\right) \\
&= C_4 \frac{n_{\ell+1}}{n_\ell} k_{n_\ell} r_{n_{\ell+1}}^d \exp\left(-\frac{\log n_\ell}{r_{n_\ell}^{2d-4}}\left(C_3\left(\frac{r_{n_\ell}}{r_{n_{\ell+1}}}\right)^{2d-4} - \delta_{n_\ell} r_{n_\ell}^{2d-4}\right)\right) \\
&\leq C_4 \frac{n_{\ell+1}}{n_\ell} k_{n_\ell} r_{n_{\ell+1}}^d \exp\left(-C_5 \frac{\log n_\ell}{r_{n_\ell}^{2d-4}}\right) \leq C_6 (\log n_\ell)^{3-\frac{C_7}{\beta_{n_\ell}}},
\end{aligned}$$

where in the last two lines we used the conditions of the Theorem for r_n and δ_n. Consequently

$$\mathbf{P}\Big(\bigcup_{j=0}^{n_{\ell+1}} A_j\Big) \leq \sum_{j=0}^{n_{\ell+1}} \sum_{x \in B(r_{n_{\ell+1}})} \mathbf{P}(A_j^{(x)}) \leq C_6 \ell^{3-\frac{C_7}{\beta_{n_\ell}}} \leq \frac{C_6}{\ell^2}$$

for large enough ℓ which is summable in ℓ. By Borel-Cantelli lemma for large ℓ if $\xi(S_j) \geq k_{n_\ell}$, then $\xi(S_j + x) \leq (1+\varepsilon) m_x k_{n_\ell}$ for all $x \in B(r_{n_{\ell+1}})$.

Let now $n_\ell \leq n < n_{\ell+1}$ and $x \in B(r_{n_{\ell+1}})$. $\xi(S_j) \geq k_n, j \leq n$ implies $\xi(S_j) \geq k_{n_\ell}$, i.e.

$$\xi(S_j + x) \leq (1+\varepsilon) m_x k_{n_\ell} \leq (1+\varepsilon) m_x k_n. \tag{3.1}$$

The lower bound is similar, with slight modifications. We call S_j new if $S_i \neq S_j$, $i = 1, 2, \ldots, j-1$. Define the events

$$D_j = \left\{ \xi(S_j) \geq k_{n_\ell}, \sup_{x \in B(r_{n_{\ell+1}})} \left(1 - \frac{\xi(S_j + x)}{m_x k_{n_{\ell+1}}}\right) \geq \varepsilon \right\},$$

$$D_j^{(x)} = \{S_j \,\text{new}, \, \xi(S_j) \geq k_{n_\ell}, \xi(S_j + x) \leq (1-\varepsilon) m_x k_{n_{\ell+1}}\}.$$

Observe that

$$\bigcup_{\{j: 0 \leq j \leq n_{\ell+1}\}} D_j = \bigcup_{\{j: 0 \leq j \leq n_{\ell+1}, \, S_j \,\text{new}\}} D_j.$$

Considering again the forward random walk, we have

$$\xi(S_j) = \xi\text{''}(0) + 1, \; \xi(S_j + x) \geq \xi\text{''}(x).$$

Hence by Markov's inequality

$$\begin{aligned}
\mathbf{P}(D_j^{(x)}) &\leq \sum_{k = k_{n_\ell} - 1}^{\infty} \mathbf{P}(\xi\text{''}(0) = k, \xi\text{''}(x) \leq (1-\varepsilon) m_x k_{n_{\ell+1}}) \\
&\leq \sum_{k = k_{n_\ell} - 1}^{\infty} (\varphi(-v)(1-\gamma))^k \psi(-v) \exp(v(1-\varepsilon) m_x k_{n_{\ell+1}}) \\
&\leq \frac{\psi(-v)}{(1-\gamma)\varphi(-v)(1 - (1-\gamma)\varphi(-v))} ((1-\gamma)\varphi(-v))^{k_{n_\ell}} e^{v(1-\varepsilon) m_x k_{n_{\ell+1}}}.
\end{aligned}$$

Proceeding as above we finally conclude after somewhat simpler calculations than the previous one, that for large enough n, $\xi(S_j) \geq k_n$ implies $\xi(S_j + x) \geq (1-\varepsilon) m_x k_n$.

This, combined with (3.1) completes the proof of Theorem 1.2.

4. Proof of Theorem 1.1

Lemma 4.1. *Let $d \geq 5$, $\frac{2}{d-2} < \alpha < 1$, $j \leq n - n^\alpha$, $|x| \leq \log n$. Then with probability 1 there exists an $n_0(\omega)$ such that for $n \geq n_0$ we have*

$$\xi(S_j + x, n) = \xi(S_j + x, \infty).$$

Proof. The proof is essentially the same as that of Theorem 1 (iii) in Erdős and Taylor [5].

Let

$$n_{k+1} = n_k + \left[\frac{1}{2} n_k^\alpha\right].$$

$$A_k = \bigcup_{j \le n_k} \bigcup_{\ell \ge n_k + [\frac{1}{2} n_{k-1}^\alpha]} \bigcup_{x \in B(\log n_{k+1})} \{S_\ell - S_j = x\}.$$

For aperiodic random walk we have (cf. Jain and Pruitt [6])

$$\mathbf{P}(S_n = x) \le C_8 n^{-d/2} \tag{4.1}$$

for all $x \in Z^d$ and $n \ge 1$ with some constant C_8.

Using the fact that $B(\log n_{k+1})$ contains less than $C_9 (\log n_{k+1})^d$ points,

$$\begin{aligned} \mathbf{P}(A_k) &\le C_9 (\log n_{k+1})^d \sum_{j=0}^{n_k} \sum_{\ell = n_k + [\frac{1}{2} n_{k-1}^\alpha]}^{\infty} \frac{C_8}{(\ell - j)^{d/2}} \\ &\le \sum_{j=0}^{n_k} \frac{C_{10} (\log n_{k+1})^d}{(n_k + [\frac{1}{2} n_{k-1}^\alpha] - j)^{d/2-1}} \le \frac{C_{10} (\log n_{k+1})^d}{n_{k-1}^{\alpha(d/2-2)}} \le \frac{C_{10} (\log n_{k-1})^d}{n_{k-1}^{\alpha(d-4)/2}}. \end{aligned} \tag{4.2}$$

We will show now that $\sum_k \mathbf{P}(A_k)$ converges.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(\log n)^d}{n^{\alpha(d-2)/2}} &\ge \sum_k \sum_{n=n_k+1}^{n_{k+1}} \frac{(\log n)^d}{n^{\alpha(d-2)/2}} \ge C_{11} \sum_k \frac{n_{k+1} - n_k}{n_{k+1}^{\alpha(d-2)/2}} (\log n_{k+1})^d \\ &\ge C_{11} \sum_k \frac{\frac{1}{2} n_k^\alpha}{n_{k+1}^{\alpha(d-2)/2}} (\log n_{k+1})^d = C_{12} \sum_k \frac{(\log n_{k+1})^d}{n_{k+1}^{\alpha(d-4)/2}} \left(\frac{n_k}{n_{k+1}}\right)^\alpha. \end{aligned} \tag{4.3}$$

Observe that

$$\left(\frac{n_k}{n_{k+1}}\right)^\alpha = \left(\frac{n_k}{n_k + [\frac{1}{2} n_k^\alpha]}\right)^\alpha \to 1, \quad k \to \infty.$$

Since

$$\sum_{n=1}^{\infty} \frac{(\log n)^d}{n^{\alpha(d-2)/2}}$$

converges, (4.2) and (4.3) imply the convergence of $\sum_k \mathbf{P}(A_k)$. By Borel-Cantelli lemma, if k is big enough, the tube of radius $\log n_{k+1}$ around the path $\{S_j,\ j = 1, 2, \ldots, n_k\}$ is disjoint from the path $\{S_\ell,\ \ell = n_k + [\frac{1}{2} n_{k-1}^\alpha], \ldots\}$.

To finish the proof, let

$$n_{k-1} < n - n^\alpha \le n_k.$$

Then

$$n_{k-1} + 2\left[\frac{n_{k-1}^\alpha}{2}\right] < n_{k-1} + n^\alpha < n,$$

hence

$$n_k + \left[\frac{n_{k-1}^\alpha}{2}\right] < n.$$

Thus with probability 1 for large n the tube of radius $\log n$ around the path $\{S_j,\, j = 1, 2, \ldots, n - [n^\alpha]\}$ is disjoint from the path $\{S_\ell,\, \ell = n, \ldots\}$, i.e. Lemma 4.1 follows.

To prove Theorem 1.1 observe that it suffices to consider points visited before time $n - n^\alpha$, $(2/(d-2) < \alpha < 1)$, since in the time interval $(n - n^\alpha, n)$ the maximal local time is less than $\alpha(1+\varepsilon)\lambda \log n$, hence this point cannot be in $\mathcal{A}_n$. Consequently, Theorem 1.1 follows from Theorem 1.2 and Lemma 4.1.

5. Proof of Theorem 1.3

First we prove

Lemma 5.1. *Let A_i, B_i be events such that $\sum_i \mathbf{P}(A_i) = \infty$,*

$$\mathbf{P}(A_i A_k) \le c_1 \mathbf{P}(A_i)\mathbf{P}(A_k),$$

and

$$\mathbf{P}(A_i B_i) \ge c_2 \mathbf{P}(A_i)$$

with some constants $c_1, c_2 > 0$. Then

$$\mathbf{P}(A_i B_i \text{ i.o.}) > 0.$$

Proof.

$$\sum_i \mathbf{P}(A_i B_i) \ge c_2 \sum_i \mathbf{P}(A_i) = \infty.$$

On the other hand,

$$\mathbf{P}(A_i B_i A_k B_k) \le \mathbf{P}(A_i A_k) \le \frac{c_1}{c_2^2}\mathbf{P}(A_i B_i)\mathbf{P}(A_k B_k),$$

the Lemma follows by Borel-Cantelli lemma in Spitzer [9], pp. 317.

To prove the Theorem, define the stopping times V_j as in Révész [7]. Let

$$
\begin{aligned}
&\rho_0(t) = t,\\
&\rho_1(t) = \min\{\tau :\ \tau > t,\ S(\tau) = S(t)\},\\
&\rho_2(t) = \min\{\tau :\ \tau > \rho_1(t),\ S(\tau) = S(\rho_1(t)) = S(t)\},\\
&\ \dots,
\end{aligned}
$$

where here and the sequel we denote $S(k) = S_k$.

$$
U(L,t) = \begin{cases} t + L & \text{if } \rho_1(t) - t > L,\\ \rho_1(t) + L & \text{if } \rho_1(t) - t \le L,\ \rho_2(t) - \rho_1(t) > L,\\ \rho_2(t) + L & \text{if } \rho_1(t) - t \le L,\ \rho_2(t) - \rho_1(t) \le L,\ \rho_3(t) - \rho_2(t) > L,\\ \dots, \end{cases}
$$

$$
L_k = (\log(k+2))^\alpha, \quad \left(\alpha > \frac{2}{d-2},\ k = 0,1,2,\dots\right)
$$

$$
V_0 = 0, \quad V_{j+1} = U(L_j, V_j), \quad (j = 0,1,2,\dots)
$$

V_{j+1} is the first time-point after V_j when the random walk has not visited $S(V_j)$ during a time-interval of length L_j.

Let $\{x_n\}$ be a sequence of points in Z^d as in Theorem 1.3 and define the events

$$
A_j = \{\xi(S(V_j), V_{j+1}) - \xi(S(V_j), V_j) = \psi_j,\ \xi(S(V_j) + x_{V_j}, V_{j+1}) - \xi(S(V_j) + x_{V_j}, V_j) = 0\}, \tag{5.1}
$$

$$
B_j = \{\xi(S(V_j) + x_{V_j}, V_j) = \xi(S(V_j) + x_{V_j}, \infty) - \xi(S(V_j) + x_{V_j}, V_{j+1}) = 0\}, \tag{5.2}
$$

where $\psi_j = [\lambda(\log j + \log\log j)]$.

Lemma 5.2. *The events A_j, $j = 1,2,\dots$ are independent and*

$$
\mathbf{P}(A_j) \ge \frac{C_{13}}{j \log j} \tag{5.3}
$$

Proof. Since $\{V_j\}_{j=1}^\infty$ is a sequence of stopping times and A_j depends only on the random walk between V_j and V_{j+1}, independence follows. To show (5.3), let $U_j := U(L_j, 0)$. Consider the random walk starting from V_j as a new origin. Then the original random walk in the interval (V_j, V_{j+1}) has the same distribution as the new random walk in $(0, U_j)$. Hence

$$
\mathbf{P}(A_j \mid V_j = m) = \mathbf{P}(\xi(0, U_j) = \psi_j,\ \xi(x_m, U_j) = 0).
$$

The event $\{\xi(0,U_j)=\psi_j,\ \xi(x_m,U_j)=0\}$ means that there are exactly ψ_j excursions around 0, each of which has length less than L_j, none of them are visiting x_m and in the last section (U_j-L_j,U_j) the random walk starting from 0, does not visit 0 and x_m. Hence applying (2.14) of Lemma 2.2,

$$\begin{aligned}\mathbf{P}(\xi(0,U)=\psi_j,\ \xi(x_m,U)=0)\\ =\left(q_{x_m}+O((\log j)^{-\alpha(d/2-1)})\right)^{\psi_j}\mathbf{P}(\xi(0,L_j)=0,\ \xi(x_m,L_j)=0).\end{aligned}$$

Obviously

$$\mathbf{P}(\xi(0,L_j)=0,\ \xi(x_m,L_j)=0)\geq\mathbf{P}(\xi(0,\infty)=0,\ \xi(x_m,\infty)=0)=1-q_{x_m}-s_{x_m}.$$

>From the inequalities (2.12) and (2.13) of Lemma 2.2 we can get by easy calculation that

$$\mathbf{P}(\xi(0,U_j)=\psi_j,\ \xi(x_m,U_j)=0)\geq C_{14}(q_{x_m})^{\psi_j}\geq C_{15}(1-\gamma)^{\psi_j}\left(1-\frac{(1-\gamma_{x_m})^2}{1-\gamma}\right)^{\psi_j}.$$

Since $L_j\geq 1$, we obviously have $V_j\geq j$, i.e. we can take $m\geq j$. Since

$$(1-\gamma)^{\psi_j}\geq\frac{1}{j\log j}$$

and (cf. (1.9))

$$(1-\gamma_{x_m})^2\sim C_{16}(\log m)^{-1},$$

we have

$$\mathbf{P}(A_j\mid V_j=m)=\mathbf{P}(\xi(0,U_j)=\psi_j,\ \xi(x_m,U_j)=0)\geq\frac{C_{13}}{j\log j},$$

with $C_{13}>0$ independent of m, the lemma follows.

Lemma 5.3. *Let the events A_j, B_j be defined by* (5.1) *and* (5.2). *Then*

$$\mathbf{P}(A_jB_j)\geq\gamma^2\mathbf{P}(A_j). \tag{5.4}$$

Proof.

$$\begin{aligned}\mathbf{P}(A_jB_j)=\mathbf{E}\mathbf{P}(A_jB_j\mid S(V_j),\ S(V_{j+1}))\\ =\mathbf{E}\left(\mathbf{P}(A_j\mid S(V_j),\ S(V_{j+1}))\mathbf{P}(B_j\mid S(V_j),\ S(V_{j+1}))\right).\end{aligned}$$

We show that

$$\mathbf{P}(B_j\mid S(V_j),\ S(V_{j+1}))\geq\gamma^2,\quad j=1,2,\dots \tag{5.5}$$

Consider the reversed random walk before $S(V_j)$, as in the the proof of Theorem 1.2, i.e. $S'_i = S(V_j - i) - S(V_j)$, and its local time $\xi'(x,n)$ and also the forward random walk starting from $S(V_{j+1})$, i.e. $S''_i = S(V_{j+1}+i) - S(V_{j+1})$, $i = 1, 2, \dots$ and its local time $\xi''(x,n)$. These two random walks are independent and the event B_j means that the first random walk S' does not visit x_{V_j} (up to time V_j) and the second random walk S'' does not visit $S(V_j) + x_{V_j} - S(V_{j+1})$ (for infinite time). Hence

$$\begin{aligned}
&\mathbf{P}(B_j \mid S(V_j),\, S(V_{j+1})) \\
&= \mathbf{P}(\xi'(x_{V_j}, V_j) = 0,\, \xi''(S(V_j) - S(V_{j+1}) + x_{V_j}, \infty) = 0 \mid S(V_j), S(V_{j+1})) \\
&\geq \mathbf{P}(\xi'(x_{V_j}, \infty) = 0)\mathbf{P}(\xi''(S(V_j) - S(V_{j+1}) + x_{V_j}, \infty) = 0 \mid S(V_j), S(V_{j+1})).
\end{aligned}$$

>From (2.11) of Lemma 2.2 it follows that

$$\mathbf{P}(\xi'(x_{V_j}, \infty) = 0) \geq \gamma$$

and similarly

$$\mathbf{P}(\xi''(S(V_j) - S(V_{j+1}) + x_{V_j}, \infty) = 0 \mid S(V_j), S(V_{j+1})) \geq \gamma,$$

hence (5.5) follows, which, in turn, implies (5.4). This proves Lemma 5.3.

Lemma 5.2 and Lemma 5.3 together imply by Lemma 5.1 that

$$\mathbf{P}(A_j B_j \,\text{i.o.}) > 0.$$

Since (cf. Révész [7])

$$V_j = n_j \leq O(1) j (\log j)^{\alpha} \quad \text{a.s.},$$

assuming that $A_j B_j$ occurs, we have

$$\begin{aligned}
\xi(S_{n_j}, \infty) = \xi(S(V_{j+1}), \infty) &\geq \xi(S(V_j), V_{j+1}) - \xi(S(V_j), V_j) \geq \psi_j \geq \\
&\geq \lambda \log n_j - \lambda\alpha \log\log n_j + (1-\varepsilon)\lambda \log\log n_j \geq \\
&\geq \lambda \log n_j + \lambda \left(\frac{d-4}{d-2} - \varepsilon \right) \log\log n_j
\end{aligned}$$

and also $\xi(S_{n_j} + x_{n_j}, \infty) = 0$. Thus we have $\mathbf{P}(D_n \,\text{i.o.}) > 0$, where

$$D_n = \left\{ \xi(S_n, \infty) \geq \lambda \left(\log n + \left(\frac{d-4}{d-2} - \varepsilon \right) \log\log n \right), \quad \xi(S_n + x_n, \infty) = 0 \right\}.$$

Let

$$\widetilde{D}_n = \Big\{\xi(S_n,\infty) \ge \lambda\left(\log n + \left(\frac{d-4}{d-2}-\varepsilon\right)\log\log n\right),$$
$$\xi(S_n+x_n,\infty)-\xi(S_n+x_n,\log n)=0\Big\}.$$

Then we have also $\mathbf{P}(\widetilde{D}_n \text{ i.o.}) > 0$ and since $\widetilde{D}_n$ is a tail event for the random walk, by 0-1 law we have $\mathbf{P}(\widetilde{D}_n \text{ i.o.}) = 1$.

To show that also $\mathbf{P}(D_n \text{ i.o.}) = 1$, we prove the following

Lemma 5.4. *For any $0 < \delta < 1/2$ with probability 1 there exists n_0 such that for $n \ge n_0$ we have*

$$\xi(S_n + x, n^\delta) = 0 \quad \text{for all} \quad |x| \le \log n.$$

Proof. By (4.1) we get

$$\mathbf{P}\left(\bigcup_{|x|\le\log n}\bigcup_{j\le n^\delta}\{S_j = S_n+x\}\right) \le \sum_{|x|\le\log n}\sum_{j\le n^\delta}\mathbf{P}(S_j=S_n+x)$$
$$\le \sum_{|x|\le\log n}\sum_{j\le n^\delta}\frac{C_8}{(n-j)^{d/2}} \le \frac{C_{17}(\log n)^d}{n^{d/2-\delta}},$$

and since this is summable, the lemma follows by Borel-Cantelli lemma. This implies $\mathbf{P}(D_n \text{ i.o.}) = 1$, proving Theorem 1.3.

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