Strong limit theorems for a simple random walk on the 2-dimensional comb

Endre Csáki[1]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O.B. 127, H-1364, Hungary. E-mail address: csaki@renyi.hu

Miklós Csörgő [2]
School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario, Canada K1S 5B6. E-mail address: mcsorgo@math.carleton.ca

Antónia Földes[3]
Department of Mathematics, College of Staten Island, CUNY, 2800 Victory Blvd., Staten Island, New York 10314, U.S.A. E-mail address: foldes@mail.csi.cuny.edu

Pál Révész[1]
Institut für Statistik und Wahrscheinlichkeitstheorie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/107 A-1040 Vienna, Austria. E-mail address: reveszp@renyi.hu

Abstract

We study the path behaviour of a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ that is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the x-axis. In particular, we prove a strong approximation result for such a random walk which, in turn, enables us to establish strong limit theorems, like the joint Strassen type law of the iterated logarithm of its two components, as well as their marginal Hirsch type behaviour.

Key words: Random walk; 2-dimensional comb; Strong approximation; 2-dimensional Wiener process; Iterated Brownian motion; Laws of the iterated logarithm

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1 Introduction and main results

Consider a simple symmetric random walk on the integer lattice $\mathbb{Z}^2$, i.e., if a moving particle is in $\mathbf{x}$ at time n, then at time $n+1$ it moves to any one of its 4 neighbouring locations with equal probabilities, independently of how the location $\mathbf{x}$ was achieved. Let $\mathbf{S}_n = \mathbf{S}(n)$ be the location of the particle after n steps, i.e., at time n, and assume that $\mathbf{S}_0 = \mathbf{0}$. One of the most classical strong theorems on random walks on $\mathbb{Z}^2$ is the famous recurrence theorem of Pólya [25] that states

$$\mathbf{P}(\mathbf{S}_n = \mathbf{0} \text{ i.o.}) = 1.$$

By a simple generalization of this recurrence theorem, one can also conclude that the respective paths of two independent random walks on the integer lattice $\mathbb{Z}^2$ meet infinitely often with probability 1.

Recently Krishnapur and Peres [21] presented a fascinating class of graphs where simple random walks continue to be recurrent, but the respective paths of two independent random walks meet only finitely many times with probability 1. In particular, the 2-dimensional comb lattice $\mathbb{C}^2$, that is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the x-axis, has this property. In a forthcoming paper we will return to studying some related properties of simple random walks on combs. As far as we know, the first paper that discusses the properties of a random walk on a particular tree that has the form of a comb is Weiss and Havlin [32].

A formal way of describing a simple random walk $\mathbf{C}(n)$ on the above 2-dimensional comb lattice $\mathbb{C}^2$ can be formulated via its transition probabilities as follows:

$$\mathbf{P}(\mathbf{C}(n+1) = (x, y \pm 1) \mid \mathbf{C}(n) = (x,y)) = \frac{1}{2}, \quad \text{if } y \neq 0, \tag{1.1}$$

$$\mathbf{P}(\mathbf{C}(n+1) = (x \pm 1, 0) \mid \mathbf{C}(n) = (x,0)) = \mathbf{P}(\mathbf{C}(n+1) = (x, \pm 1) \mid \mathbf{C}(n) = (x,0)) = \frac{1}{4}. \tag{1.2}$$

Unless otherwise stated, we assume that $\mathbf{C}(0) = \mathbf{0}$. The coordinates of the just defined vector valued simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ will be denoted by $C_1(n), C_2(n)$, i.e., $\mathbf{C}(n) := (C_1(n), C_2(n))$.

A compact way of describing the just introduced transition probabilities for this simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ is via defining

$$p(\mathbf{u}, \mathbf{v}) := \mathbf{P}(\mathbf{C}(n+1) = \mathbf{v} \mid \mathbf{C}(n) = \mathbf{u}) = \frac{1}{\deg(\mathbf{u})}, \tag{1.3}$$

for locations $\mathbf{u}$ and $\mathbf{v}$ that are neighbours on $\mathbb{C}^2$, where $\deg(\mathbf{u})$ is the number of neighbours of $\mathbf{u}$, otherwise $p(\mathbf{u}, \mathbf{v}) := 0$. Consequently, the non-zero transition probabilities are equal to $1/4$ if $\mathbf{u}$ is on the horizontal axis and they are equal to $1/2$ otherwise.

Weiss and Havlin [32] derived the asymptotic form for the probability that $\mathbf{C}(n) = (x, y)$ by appealing to a central limit argument. For further references along these lines we refer to Bertacchi [1]. Here we call attention to Bertacchi and Zucca [2], who obtained space-time asymptotic estimates for the n-step transition probabilities $p^{(n)}(\mathbf{u}, \mathbf{v}) := \mathbf{P}(\mathbf{C}(n) = \mathbf{v} \mid \mathbf{C}(0) = \mathbf{u})$, $n \geq 0$, from $\mathbf{u} \in \mathbb{C}^2$

to $\mathbf{v} \in \mathbb{C}^2$, when $\mathbf{u} = (2k, 0)$ or $(0, 2k)$ and $\mathbf{v} = (0, 0)$. Using their estimates, they concluded that, if k/n goes to zero with a certain speed, then $p^{(2n)}((2k, 0), (0, 0))/p^{(2n)}((0, 2k), (0, 0)) \to 0$, as $n \to \infty$, an indication that suggests that the particle in this random walk spends most of its time on some tooth of the comb. The latter suggestion in turn provides a heuristic insight into the above mentioned conclusion of Krishnapur and Peres [21] that the respective paths of two independent random walks on $\mathbb{C}^2$ can not meet infinetely many times with probability 1. A further insight along these lines was provided by Bertacchi [1], where she analyzed the asymptotic behaviour of the horizontal and vertical components $C_1(n)$, $C_2(n)$ of $\mathbf{C}(n)$ on $\mathbb{C}^2$, and concluded that the expected values of various distances reached in n steps are of order $n^{1/4}$ for $C_1(n)$ and of order $n^{1/2}$ for $C_2(n)$. Moreover, this conclusion, in turn, also led her to study the asymptotic law of the random walk $\mathbf{C}(n) = (C_1(n), C_2(n))$ on $\mathbb{C}^2$ via scaling the components $C_1(n), C_2(n)$ by $n^{1/4}$ and $n^{1/2}$, respectively. Namely, defining now the continuous time process $\mathbf{C}(nt) = (C_1(nt), C_2(nt))$ by linear interpolation, Bertacchi [1] established the following remarkable weak convergence result.

Theorem A

$$\left(\frac{C_1(nt)}{n^{1/4}}, \frac{C_2(nt)}{n^{1/2}}; t \geq 0\right) \xrightarrow{\text{Law}} (W_1(\eta_2(0,t)), W_2(t); t \geq 0), \quad n \to \infty, \tag{1.4}$$

where W_1, W_2 are two independent Wiener processes (Brownian motions) and $\eta_2(0,t)$ is the local time process of W_2 at zero, and $\xrightarrow{\text{Law}}$ denotes weak convergence on $C([0,\infty), \mathbb{R}^2)$ endowed with the topology of uniform convergence on compact intervals.

Here, and throughout as well, $C(I, \mathbb{R}^d)$, respectively $D(I, \mathbb{R}^d)$, stands for the space of $\mathbb{R}^d$-valued, $d = 1, 2$, continuous, respectively càdlàg, functions defined on an interval $I \subseteq [0, \infty)$. $\mathbb{R}^1$ will throughout be denoted by $\mathbb{R}$.

Recall that if $\{W(t),\, t \geq 0\}$ is a standard Wiener process (Brownian motion), then its two-parameter local time process $\{\eta(x,t),\, x \in \mathbb{R},\, t \geq 0\}$ can be defined via

$$\int_A \eta(x,t)\, dx = \lambda\{s : 0 \leq s \leq t,\, W(s) \in A\} \tag{1.5}$$

for any $t \geq 0$ and Borel set $A \subset \mathbb{R}$, where $\lambda(\cdot)$ is the Lebesgue measure, and $\eta(\cdot,\cdot)$ is frequently referred to as Wiener or Brownian local time.

The iterated stochastic process $\{W_1(\eta_2(0,t));\, t \geq 0\}$ provides an analogue of the equality in distribution $t^{-1/2}W(t) \stackrel{\text{Law}}{=} X$ for each fixed $t > 0$, where W is a standard Wiener process and X is a standard normal random variable. Namely, we have (cf., e.g., (1.7) and (1.8) in [10])

$$\frac{W_1(\eta_2(0,t))}{t^{1/4}} \stackrel{\text{Law}}{=} X|Y|^{1/2}, \quad t > 0 \text{ fixed}, \tag{1.6}$$

where X and Y are independent standard normal random variables.

It is of interest to note that the iterated stochastic process $\{W_1(\eta_2(0,t));\, t \geq 0\}$ has first appeared in the context of studying the so-called second order limit law for additive functionals of a standard Wiener process W. Namely, let $g(x)$ be an integrable function on the real line and consider

$$G(t) = \int_0^t g(W(s))\,ds = \int_{-\infty}^{\infty} g(x)\eta(x,t)\,dx, \quad t \geq 0,$$

where $\eta(x,t)$ is the two-time parameter local time process of W. We recall that Papanicolaou et al. [24], Ikeda and Watanabe [19], Kasahara [20] and Borodin [5] established a weak convergence result on $C([0,\infty),\mathbb{R})$ endowed with the topology of uniform convergence on compact intervals, which reads as follows:

$$\theta^{-1/4}(G(\theta t) - \bar{g}\eta(0,\theta t)) \xrightarrow{\text{Law}} \sigma W_1(\eta_2(0,t)), \quad \theta \to \infty. \tag{1.7}$$

where $\bar{g} = \int_{-\infty}^{\infty} g(x)\,dx$, $W_1(\cdot)$ is a Wiener process, $\eta_2(0,\cdot)$ is a Wiener local time at zero, such that W_1 and η_2 are independent processes, and σ is an explicitly given constant in terms of g.

For a related review of first and second order limit laws we refer to Csáki et al. [11], where the authors also established a strong approximation version of (1.7), and for its simple symmetric random walk case as well, on the real line. In both cases the method developed in Csáki et al. [10] for approximating a centered Wiener local time process by a Wiener sheet whose time clock is an independent Wiener local time at zero, proved to be an appropriate tool for achieving the latter goal. From strong approximation results like those in the just mentioned two papers, one can establish various strong limit laws for the processes in hand. In this regard we note, e.g., that for the process $W_1(\eta_2(0,t))$ as in (1.7), Csáki et al. [10] concluded the following strong asymptotic law:

$$\limsup_{t\to\infty} \frac{W_1(\eta_2(0,t))}{t^{1/4}(\log\log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.} \tag{1.8}$$

For further studies and related results along similar lines we refer to Csáki et al. [12] and references therein.

The investigations that are presented in this paper for the random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ were inspired by the above quoted weak limit law of Bertacchi [1] as in Theorem A and the strong approximation methods and conclusions of Csáki et al. [10], [11], [12].

Bertacchi's method of proof for establishing the joint weak convergence statement of Theorem A is based on showing that, on an appropriate probability space, each of the components converges in probability uniformly on compact intervals to the corresponding components of the conclusion of (1.4) (cf. Proposition 6.4 of Bertacchi [1]). This approach was also the key idea in Cherny et al. [7] for establishing their multivariate extensions of the Donsker-Prokhorov invariance principle (cf. Theorems 2.1 and 2.2 in [7]) that is based on the Skorokhod embedding [30] scheme.

In this paper we extend this approach so that we provide joint strong invariance principles as in Corollaries 1.1 and 1.4. In particular, Corollary 1.4 in turn leads to the joint functional law of the iterated logarithm for the random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ as in Theorem 1.4 via that of Theorem 1.3 for the limiting processes. Also, (1.23), (1.24) and Corollaries 1.5, and 1.6 fully describe the respective marginal limsup and functional laws of the iterated logarithm behaviour of the first and second components of (1.22). Theorem 1.5 describes the joint set of limit points of the two components of $\mathbf{C}(n)$.

As to the liminf behaviour of the max functionals of these two components, following Nane [23] and Hirsch [18] (cf. Theorem H and Theorem I below), in Corollary 1.8 we conclude Hirsch type behaviour of the respective components of the random walk process $\mathbf{C}(n)$ on the 2-dimensional comb lattice $\mathbb{C}^2$. For $|C_2(\cdot)|$ we have Chung's other law of the iterated logarithm as in (1.33), but we could not conclude a similar law for the max functional of $|C_1(\cdot)|$. In Theorem 1.6 and Corollary 1.10 however, we give a Hirsch type (cf. [18]) liminf result for the max functionals of $|W_1(\eta_2(0,\cdot))|$ and $|C_1(\cdot)|$.

In this section we will now present our results and their corollaries, and will also relate them to earlier ones which, just like Theorem A, will be labeled by letters. The results that we believe to be new, will be designated by numbers, and their proofs will be detailed in Sections 3-6. Preceding these sections, in Section 2 we present, without proofs, preliminary results that will be used in the just mentioned sections in our proofs. We note in passing that the preliminary result of Proposition 2.1 may be known, but for the sake of completeness, we also present our proof of it.

Our first result is a strong approximation for the random walk $\mathbf{C}(n) = (C_1(n), C_2(n))$ on $\mathbb{C}^2$.

Theorem 1.1 *On an appropriate probability space for the random walk $\{\mathbf{C}(n) = (C_1(n), C_2(n));$ $n = 0, 1, 2, \ldots\}$ on $\mathbb{C}^2$,one can construct two independent standard Wiener processes $\{W_1(t);\, t \geq 0\}$, $\{W_2(t);\, t \geq 0\}$ so that, as $n \to \infty$, we have with any $\varepsilon > 0$*

$$n^{-1/4}|C_1(n) - W_1(\eta_2(0,n))| + n^{-1/2}|C_2(n) - W_2(n)| = O(n^{-1/8+\varepsilon}) \quad a.s.,$$

where $\eta_2(0,\cdot)$ is the local time process at zero of $W_2(\cdot)$.

Throughout this paper the notation $\|\cdot\|$ will stand for the $\|\cdot\|_p$ norm in $\mathbb{R}^d$, with $p \geq 1$. Our choice will usually be $p = 1$ or 2.

Consider now the net of random walk processes $\{\mathbf{C}([nt]) := (C_1([nt]), C_2([nt]));\, 0 \leq t\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$, where $[x]$ stands for the integer part of x. Thus, for each fixed $n \geq 1$, the net of random vectors $\{\mathbf{C}([nt]);\, 0 \leq t\}$ are functions: $[0,\infty) \longrightarrow \mathbb{R}^2$ that are random elements of the space $D([0,\infty), \mathbb{R}^2)$, and each of the components $\{C_1([nt]);\, 0 \leq t\}$ and $\{C_2([nt]);\, 0 \leq t\}$ of $\{\mathbf{C}([nt]);\, 0 \leq t\}$ are random elements of the space $D([0,\infty), \mathbb{R})$. As an immediate consequence of Theorem 1.1, we conclude the following strong invariance principle.

Corollary 1.1 *On the probability space of* Theorem 1.1, *we have almost surely, as $n \to \infty$,*

$$\sup_{t \in A} \left\| \left(\frac{C_1([nt]) - W_1(\eta_2(0,nt))}{n^{1/4}}, \frac{C_2([nt]) - W_2(nt)}{n^{1/2}} \right) \right\| \to 0, \tag{1.9}$$

for all compact intervals $A \subset [0,\infty)$.

We note in passing that Corollary 1.1 also holds true for the continuous time processes as in Theorem A. Consequently, when viewed this way, Corollary 1.1 amounts to an almost sure version of Proposition 6.4 of Bertacchi [1], and yields Theorem A that is Theorem 6.1 of Bertacchi [1].

In its present form, Corollary 1.1 also yields a weak convergence on the space $D([0,\infty),\mathbb{R}^2)$ endowed with a uniform topology that is defined as follows.

For functions $(f_1(t), f_2(t))$, $(g_1(t), g_2(t))$ in the function space $D([0,\infty),\mathbb{R}^2)$, and for compact subsets A of $[0,\infty)$, we define

$$\Delta = \Delta(A, (f_1, f_2), (g_1, g_2)) := \sup_{t\in A} \|(f_1(t) - g_1(t), f_2(t) - g_2(t))\|,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^2$.

We also define the measurable space $(D([0,\infty),\mathbb{R}^2), \mathcal{D})$, where $\mathcal{D}$ is the σ-field generated by the collection of all Δ-open balls of $D([0,\infty),\mathbb{R}^2)$, where a ball is a subset of $D([0,\infty),\mathbb{R}^2)$ of the form

$$\{(f_1, f_2) :\ \Delta(A, (f_1, f_2), (g_1, g_2)) < r\}$$

for some $(g_1, g_2) \in D([0,\infty),\mathbb{R}^2)$, some $r > 0$, and some compact interval A of $[0,\infty)$.

In view of these two definitions, Corollary 1.1 yields a weak convergence result that is determined by the following functional convergence in distribution statement.

Corollary 1.2 *As $n \to \infty$*

$$h\left(\frac{C_1([nt])}{n^{1/4}}, \frac{C_2([nt])}{n^{1/2}}\right) \to_d h(W_1(\eta_2(0,t)), W_2(t)) \tag{1.10}$$

for all $h : D([0,\infty),\mathbb{R}^2) \longrightarrow \mathbb{R}^2$ that are $(D([0,\infty),\mathbb{R}^2), \mathcal{D})$ measurable and Δ-continuous, or Δ-continuous except at points forming a set of measure zero on $(D([0,\infty),\mathbb{R}^2), \mathcal{D})$ with respect to the measure generated by $(W_1(\eta_2(0,t)), W_2(t))$, where W_1, W_2 are two independent Wiener processes and $\eta_2(0,t)$ is the local time process of W_2 at zero, and $\to_d$ denotes convergence in distribution.

As an example, on taking $t = 1$ in Theorem A or, equivalently, in Corollary 1.2, we obtain the following convergence in distribution result.

Corollary 1.3 *As $n \to \infty$*

$$\left(\frac{C_1(n)}{n^{1/4}}, \frac{C_2(n)}{n^{1/2}}\right) \to_d (W_1(\eta_2(0,1)), W_2(1)). \tag{1.11}$$

Concerning the joint distribution of the limiting vector valued random variable, we have

$$(W_1(\eta_2(0,1)), W_2(1)) =_d (X|Y|^{1/2}, Z),$$

where $(|Y|, Z)$ has the joint distribution of the vector $(\eta_2(0,1), W_2(1))$, X is equal in distribution to the random variable $W_1(1)$, and is independent of $(|Y|, Z)$.

As to the joint density of $(|Y|, Z)$, we have (cf. 1.3.8 on p. 127 in Borodin and Salminen [6])

$$\mathbf{P}(|Y| \in dy, Z \in dz) = \frac{1}{\sqrt{2\pi}}(y + |z|)e^{-\frac{(y+|z|)^2}{2}}\, dy\, dz, \quad y \geq 0,\ z \in \mathbb{R}.$$

Now, on account of the independence of X and $(|Y|, Z)$, the joint density function of the random variables $(X, |Y|, Z)$ reads as follows.

$$\mathbf{P}(X \in dx, |Y| \in dy, Z \in dz) = \frac{1}{2\pi}(y + |z|)e^{-\frac{x^2+(y+|z|)^2}{2}} dx\, dy\, dz, \quad y \geq 0,\ x, z \in \mathbb{R}.$$

By changing variables, via calculating the joint density function of the random variables $U = X|Y|^{1/2}, Y, Z$, and then integrating it out with respect to $y \in [0, \infty)$, we arrive at the joint density function of the random variables $(U = X|Y|^{1/2}, Z)$, which reads as follows.

$$\mathbf{P}(X|Y|^{1/2} \in du, Z \in dz) = \frac{1}{2\pi} \int_0^\infty \frac{y + |z|}{y^{1/2}} e^{-\frac{u^2}{2y} - \frac{(y+|z|)^2}{2}} dy\, du\, dz \quad u, z \in \mathbb{R}. \tag{1.12}$$

Clearly, Z is a standard normal random variable. The marginal distribution of $U = X|Y|^{1/2}$ is of special interest in that this random variable first appeared in the conclusion of Dobrushin's classical Theorem 2 of his fundamental paper [16], that was first to deal with the so-called second order limit law for additive functionals of a simple symmetric random walk on the real line. In view of the above joint density function in (1.12), on integrating it out with respect to z over the real line, we are now to also conclude Dobrushin's formula for the density function of this random variable, which we have also introduced already in the context of (1.6).

$$\begin{aligned} \mathbf{P}(U \in du) &= \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{y + z}{\sqrt{y}} e^{-\frac{u^2}{2y} - \frac{(y+z)^2}{2}} dy\, dz\, du \\ &= \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{y}} e^{-\frac{u^2}{2y} - \frac{y^2}{2}} dy\, du = \frac{2}{\pi} \int_0^\infty e^{-\frac{u^2}{2v^2} - \frac{v^4}{2}} dv\, du. \end{aligned}$$

Continuing with the use of Theorem 1.1, or that of Corollary 1.1, we now conclude another strong invariance principle that will enable us to establish functional laws of the iterated logarithm for the continuous version of the random walk process $\{\mathbf{C}(ns) = (C_1(ns), C_2(ns));\ 0 \leq s \leq 1\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$, that is defined by linear interpolation as in Theorem A.

Corollary 1.4 *On the probability space of* Theorem 1.1, *on* $C([0,1], \mathbb{R}^2)$ *we have almost surely, as* $n \to \infty$,

$$\sup_{0 \leq s \leq 1} \left\| \left(\frac{C_1(ns) - W_1(\eta_2(0, ns))}{n^{1/4}(\log\log n)^{3/4}}, \frac{C_2(ns) - W_2(ns)}{(n \log\log n)^{1/2}} \right) \right\| \to 0. \tag{1.13}$$

Our just stated strong invariance principle clearly parallels the first such result in history, that was established by Strassen [31] via using the Skorokhod [30] embedding theorem. It reads as follows.

Theorem B *Given i.i.d. random variables* $X_1, X_2, \ldots$ *with mean* 0 *and variance* 1, *and their successive partial sums* $S(n)$, $n = 0, 1, 2, \ldots, S(0) = 0$, *there is a probability space with* $\widehat{S}(n)$, $n = 0, 1, 2, \ldots, \widehat{S}(0) = 0$, *and a standard Wiener process* $\{W(t);\ t \geq 0\}$ *on it so that*

$$\{\widehat{S}(nt); 0 \leq t \leq 1,\ n = 0, 1, 2, \ldots\} =_d \{S(nt); 0 \leq t \leq 1,\ n = 0, 1, 2, \ldots\},$$

where $S(nt)$ are random elements in the space $C([0,1],\mathbb{R})$ of continuous real valued functions, obtained by linear interpolation, and as $n \to \infty$,

$$\sup_{0\le t\le 1} \frac{|\widehat{S}(nt) - W(nt)|}{(n\log\log n)^{1/2}} \to 0 \quad a.s.$$

In the same paper, Strassen also established his famous functional law of the iterated logarithm for a standard Wiener process (cf. Theorem C below), and then concluded it also for partial sums of i.i.d. random variables as well (cf. Theorem D), via his just stated strong invariance principle as in Theorem B.

In this regard, let $\mathcal{S}$ be the Strassen class of functions, i.e., $\mathcal{S} \subset C([0,1],\mathbb{R})$ is the class of absolutely continuous functions (with respect to the Lebesgue measure) on $[0,1]$ for which

$$f(0) = 0 \qquad \text{and} \qquad \int_0^1 \dot{f}^2(x)dx \le 1. \tag{1.14}$$

Theorem C *The net*

$$\left\{\frac{W(xt)}{(2t\log\log t)^{1/2}};\ 0 \le x \le 1\right\}_{t\ge 3},$$

as $t \to \infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R})$ and the set of its limit points is the class of functions $\mathcal{S}$.

Theorem D *The sequence of random functions*

$$\left\{\frac{S(xn)}{(2n\log\log n)^{1/2}};\ 0 \le x \le 1\right\}_{n\ge 3},$$

as $n \to \infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R})$ and the set of its limit points is the class of functions $\mathcal{S}$.

In view of Theorem C and our strong invariance principle as stated in Corollary 1.4, we are now to study the set of limit points of the net of random vectors

$$\left(\frac{W_1(\eta_2(0,xt))}{2^{3/4}t^{1/4}(\log\log t)^{3/4}}, \frac{W_2(xt)}{(2t\log\log t)^{1/2}};\ 0 \le x \le 1\right)_{t\ge 3}, \tag{1.15}$$

as $t \to \infty$. This will be accomplished in Theorem 1.3. In order to achieve this, we define the *Strassen class* $\mathcal{S}^2$ as the set of $\mathbb{R}^2$ valued, absolutely continuous functions

$$\{(f(x), g(x));\ 0 \le x \le 1\} \tag{1.16}$$

for which $f(0) = g(0) = 0$ and

$$\int_0^1 (\dot{f}^2(x) + \dot{g}^2(x))dx \le 1. \tag{1.17}$$

For the sake of presenting now our intermediate result of Theorem 1.2 to that of Theorem 1.3, we need also $\mathcal{S}_M \subset \mathcal{S}$, the class of non-decreasing functions in the Strassen class of functions $\mathcal{S}$.

Theorem 1.2 *Let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent standard Wiener processes starting from 0, and let $\eta_2(0,\cdot)$ be the local time process of $W_2(\cdot)$ at zero. Then the net of random vectors*

$$\left(\frac{W_1(xt)}{(2t\log\log t)^{1/2}}, \frac{W_2(xt)}{(2t\log\log t)^{1/2}}, \frac{\eta_2(0,xt)}{(2t\log\log t)^{1/2}}; 0 \le x \le 1\right)_{t\ge 3}, \tag{1.18}$$

as $t \to \infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R}^3)$ and its limit points is the set of functions

$$\mathcal{S}^{(3)} := \Big\{(f(x), g(x), h(x)) : (f,g) \in \mathcal{S}^2, h \in \mathcal{S}_M,$$

$$\int_0^1 (\dot f^2(x) + \dot g^2(x) + \dot h^2(x))\,dx \le 1, \text{ and } g(x)\dot h(x) = 0 \text{ a.e.}\Big\} \tag{1.19}$$

Theorem 1.3 *Let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent standard Wiener processes starting from 0, and let $\eta_2(0,\cdot)$ be the local time process at zero of $W_2(\cdot)$. Then the net of random vectors*

$$\left(\frac{W_1(\eta_2(0,xt))}{2^{3/4}t^{1/4}(\log\log t)^{3/4}}, \frac{W_2(xt)}{(2t\log\log t)^{1/2}}; 0 \le x \le 1\right)_{t\ge 3}, \tag{1.20}$$

as $t \to \infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R}^2)$ and its limit points is the set of functions

$$\begin{aligned}\mathcal{S}^{(2)} := &\Big\{(f(h(x)), g(x)) : (f,g) \in \mathcal{S}^2, h \in \mathcal{S}_M,\\ &\int_0^1 (\dot f^2(x) + \dot g^2(x) + \dot h^2(x))\,dx \le 1, g(x)\dot h(x) = 0 \text{ a.e.}\Big\}\\ = &\Big\{(k(x), g(x)) : k(0) = g(0) = 0,\ k, g \in \dot C([0,1],\mathbb{R})\\ &\int_0^1 (|3^{3/4}2^{-1/2}\dot k(x)|^{4/3} + \dot g^2(x))\,dx \le 1,\ \dot k(x) g(x) = 0 \text{ a.e.}\Big\},\end{aligned}$$

where $\dot C([0,1],\mathbb{R})$ stands for the space of absolutely continuous functions in $C([0,1],\mathbb{R})$.

To illustrate somewhat the intrinsic stochastic nature of Theorem 1.3, we call attention to the result of Csáki et al. [10] that we quoted in (1.8). The latter amounts to saying that, marginally, the iterated process that is the first component of the net of random vectors in (1.20) satisfies a law of the iterated logarithm. Moreover, it was shown in Csáki et al. [14] (cf. their Theorem 2.2) that the following functional version of this law of the iterated logarithm holds true as well for the first component of the net of random vectors in (1.20).

Theorem E *The net*

$$\left\{\frac{W_1(\eta_2(0,xt))}{2^{5/4}3^{-3/4}t^{1/4}(\log\log t)^{3/4}};\ 0\le x\le 1\right\}_{t\ge 3},$$

as $t\to\infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R})$ and the set of its limit points $\mathcal{S}(4/3)\subset C([0,1],\mathbb{R})$ is the class of absolutely continuous functions (with respect to the Lebesgue measure) on $[0,1]$ for which

$$f(0)=0 \qquad \textit{and} \qquad \int_0^1 |\dot f(x)|^{4/3}\,dx\le 1. \tag{1.21}$$

As to the second component of the net of random vectors in (1.20), Strassen's functional law of the iterated logarithm obtains (cf. Theorem C).

Now, Theorem 1.3 establishes a functional law of the iterated logarithm jointly for the two components in the net of (1.20) so that their set of limit points is the set of functions $\mathcal{S}^{(2)}$, which however is not equal to the cross product of the just mentioned function classes $\mathcal{S}(4/3)$ and $\mathcal{S}$ of Theorem E and Theorem C, respectively.

Theorem 1.3 is of importance not only on its own, for combining it with Corollary 1.4, it leads to a similarly important conclusion for the net of our random walk processes on the 2-dimensional comb lattice $\mathbb{C}^2$ that reads as follows.

Theorem 1.4 *For the random walk $\{\mathbf{C}(n)=(C_1(n),C_2(n));\ n=1,2,\ldots\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$ we have that the sequence of random vector-valued functions*

$$\left(\frac{C_1(xn)}{2^{3/4}n^{1/4}(\log\log n)^{3/4}},\frac{C_2(xn)}{(2n\log\log n)^{1/2}};\ 0\le x\le 1\right)_{n\ge 3} \tag{1.22}$$

is almost surely relatively compact in the space $C([0,1],\mathbb{R}^2)$ and its limit points is the set of functions $\mathcal{S}^{(2)}$ as in Theorem 1.3.

As a consequence, this theorem in combination with (1.8) and Corollary 1.4 implies

$$\limsup_{n\to\infty}\frac{C_1(n)}{n^{1/4}(\log\log n)^{3/4}}=\frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.} \tag{1.23}$$

Moreover, via Corollary 1.4, Theorem E in this context implies a functional version of the latter law of the iterated logarithm for the first component of the sequence of random vectors in (1.22), which reads as follows.

Corollary 1.5 *The sequence*

$$\left\{\frac{C_1(xn)}{2^{5/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}};\ 0\le x\le 1\right\}_{n\ge 3},$$

as $n\to\infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R})$, and the set of its limit points is $\mathcal{S}(4/3)$, as in Theorem E.

As to the second component in (1.22), the classical law of the iterated logarithm for the Wiener process in combination with Corollary 1.4 implies

$$\limsup_{n\to\infty} \frac{C_2(n)}{(2n\log\log n)^{1/2}} = 1 \quad \text{a.s.} \tag{1.24}$$

Moreover, Theorem C in combination with Corollary 1.4 implies the next functional law of the iterated logarithm.

Corollary 1.6 *The sequence*

$$\left(\frac{C_2(xn)}{(2n\log\log n)^{1/2}},\ 0 \le x \le 1\right)_{n\ge 3},$$

as $n \to \infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R})$, and the set of its limit points is the class of functions $\mathcal{S}$.

Now, *à la* Theorem 1.3, Theorem 1.4 establishes a joint functional law of the iterated logarithm for the two components of the random vectors in (1.22), but again so that their set of limit points is the set of functions $\mathcal{S}^{(2)}$, i.e., not the cross product of the function classes $\mathcal{S}(4/3)$ and $\mathcal{S}$.

In order to illustrate the case of a joint functional law of the iterated logarithm for the two components of the random vectors in (1.22), we give the following example. Let

$$k(x, B, K_1) = k(x) = \begin{cases} \dfrac{Bx}{K_1} & \text{if} \quad 0 \le x \le K_1, \\ B & \text{if} \quad K_1 < x \le 1, \end{cases} \tag{1.25}$$

$$g(x, A, K_2) = g(x) = \begin{cases} 0 & \text{if} \quad 0 \le x \le K_2, \\ \dfrac{(x-K_2)A}{1-K_2} & \text{if} \quad K_2 < x \le 1, \end{cases} \tag{1.26}$$

where $0 \le K_1 \le K_2 \le 1$, and we see that $\dot{k}(x)g(x) = 0$. Hence, provided that for A, B, K_1 and K_2

$$\frac{3\,B^{4/3}}{2^{2/3}K_1^{1/3}} + \frac{A^2}{(1-K_2)} \le 1,$$

we have $(k,g) \in \mathcal{S}^{(2)}$. Consequently, in the two extreme cases,

(i) when $K_1 = K_2 = 1$, then $|B| \le 2^{1/2}3^{-3/4}$ and on choosing $k(x) = \pm 2^{1/2}3^{-3/4}x$, $0 \le x \le 1$, then $g(x) = 0$, $0 \le x \le 1$, and

(ii) when $K_1 = K_2 = 0$, then $|A| \le 1$ and on choosing $g(x) = \pm x$, $0 \le x \le 1$, then $k(x) = 0$, $0 \le x \le 1$.

Concerning now the joint limit points of $C_1(n)$ and $C_2(n)$ a consequence of Theorem 1.4 reads as follows.

Corollary 1.7 *The sequence*

$$\left(\frac{C_1(n)}{n^{1/4}(\log\log n)^{3/4}}, \frac{C_2(n)}{(2n\log\log n)^{1/2}}\right)_{n\geq 3}$$

is almost surely relatively compact in the rectangle

$$R = \left[-\frac{2^{5/4}}{3^{3/4}}, \frac{2^{5/4}}{3^{3/4}}\right] \times [-1, 1]$$

and the set of its limit points is equal to the domain

$$D = \{(u, v) : \ k(1) = u, \ g(1) = v, \ (k(\cdot), g(\cdot)) \in \mathcal{S}^{(2)}\}. \tag{1.27}$$

It is of interest to find a more explicit description of D. In order to formulate the corresponding result for describing also the intrinsic nature of the domain D we introduce the following notations:

$$F(B, A, K) = \frac{3B^{4/3}}{2^{2/3}K^{1/3}} + \frac{A^2}{1-K} \quad (0 \leq B, A, K \leq 1), \tag{1.28}$$

$$D_1(K) = \{(u, v) : \ F(|u|, |v|, K) \leq 1\},$$

$$D_2 = \bigcup_{K\in(0,1)} D_1(K). \tag{1.29}$$

Theorem 1.5 *The two domains D in* (1.27) *and D_2 in* (1.29) *are the same.*

Remark 1. Let

(i) $A = A(B, K)$ be defined by the equation

$$F(B, A(B, K), K) = 1, \tag{1.30}$$

(ii) $K = K(B)$ be defined by the equation

$$A(B, K(B)) = \max_{0\leq K\leq 1} A(B, K). \tag{1.31}$$

Then clearly

$$D_2 = \{(B, A) : \ |A| \leq A(|B|, K(|B|)\}.$$

The explicit form of $A(B, K)$ can be easily obtained, and that of $K(B)$ can be obtained by the solution of a cubic equation. Hence, theoretically, we have the explicit form of D_2. However this explicit form is too complicated. A picture of D_2 can be given by numerical methods (Fig. 1.)

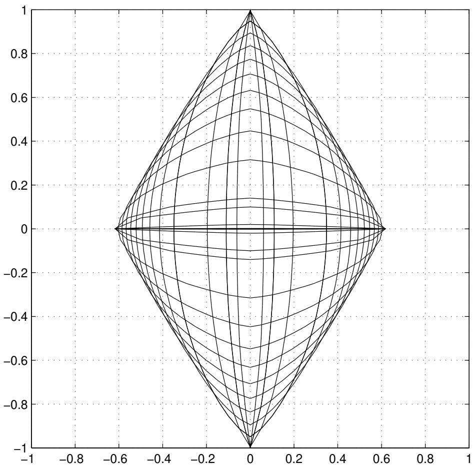

Figure 1: A picture of D_2.

Concerning almost sure properties of a standard Wiener process $W(\cdot)$, we now mention the so-called *other law of the iterated logarithm* that was first established by Chung [8] for partial sums of independent random variables. In terms of a standard Wiener process, it reads as follows.

Theorem F

$$\liminf_{t\to\infty}\left(\frac{8\log\log t}{\pi^2 t}\right)^{1/2}\sup_{0\le s\le t}|W(s)|=1 \qquad a.s. \tag{1.32}$$

On account of Theorem 1.1, the same other law of the iterated logarithm obtains for $C_2(n)$ as well.

Corollary 1.8

$$\liminf_{n\to\infty}\left(\frac{8\log\log n}{\pi^2 n}\right)^{1/2}\max_{0\le k\le n}|C_2(k)|=1 \qquad a.s. \tag{1.33}$$

In view of (1.8) and (1.23), one wonders about possibly having other laws of the iterated logarithm for the respective first components $W_1(\eta_2(0,t))$ and $C_1(n)$ as well. Concerning the iterated process $\{W_1(\eta_2(0,t));\ t\ge 0\}$, from the more general Theorem 2.1 of Nane [23], in our context the following result obtains.

Theorem G *As $u\downarrow 0$,*

$$\mathbf{P}(\sup_{0\le t\le 1}|W_1(\eta_2(0,t))|\le u)\sim cu^2 \tag{1.34}$$

with some positive constant c. Consequently, for small u we have

$$c_1u^2\le \mathbf{P}(\sup_{0\le t\le 1}|W_1(\eta_2(0,t))|\le u)\le c_2u^2 \tag{1.35}$$

with some positive constants c_1 *and* c_2.

It is worthwile to note that from the well-known formula (cf. Erdős and Kac [17] and footnote 3 in their paper)

$$\mathbf{P}(\sup_{0\le s\le t}|W_1(s)|\le ut^{1/2})=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\exp\left(-\frac{(2k-1)^2\pi^2}{8u^2}\right)$$

one arrives at

$$\frac{2}{\pi}\exp\left(-\frac{\pi^2}{8u^2}\right)\le \mathbf{P}(\sup_{0\le s\le t}|W_1(s)|\le ut^{1/2})\le \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8u^2}\right),$$

for all $u>0$ and $t>0$.

Now, the above mentioned other law of the iterated logarithm of Chung [8] for Wiener process can be based on the latter inequality. Hence, comparing it with the small ball inequality (1.35), Nane [23] concludes that one can not expect to have a Chung type LIL for the iterated process $W_1(\eta_2(0,t))$. Instead, we give a Hirsch type (cf. [18]) liminf result in Theorem 1.6 below. Nane [23] obtains a Hirsch type integral test for one-sided maximum of a class of iterated process which in our context reads as follows.

Theorem H *Let* $\beta(t)>0; t\ge 0$ *be a non-increasing function. Then we have almost surely that*

$$\liminf_{t\to\infty}\frac{\sup_{0\le s\le t}W_1(\eta_2(0,s))}{t^{1/4}\beta(t)}=0\quad or\quad \infty$$

according as the integral $\int_1^\infty \beta(t)/t\,dt$ *diverges or converges.*

For the sake of comparison we note that, when it is applied to Wiener process, then Hirsch's integral test [18] obtains as follows.

Theorem I *With* $\beta(\cdot)$ *as in* Theorem H, *we have almost surely*

$$\liminf_{t\to\infty}\frac{\sup_{0\le s\le t}W_2(s)}{t^{1/2}\beta(t)}=0\quad or\quad \infty$$

according as the integral $\int_1^\infty \beta(t)/t\,dt$ *diverges or converges.*

In view of Theorems H and I, with the help of our Theorem 1.1, for the random walk process $\{\mathbf{C}(n)=(C_1(n),C_2(n));\ n=0,1,2,\ldots\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$, we now conclude the following results.

Corollary 1.9 *Let* $\beta(n)$, $n=1,2,\ldots$, *be a non-increasing sequence of positive numbers. Then we have almost surely that*

$$\liminf_{n\to\infty}\frac{\max_{0\le k\le n}C_1(k)}{n^{1/4}\beta(n)}=0\quad or\quad \infty$$

and

$$\liminf_{n\to\infty} \frac{\max_{0\le k\le n} C_2(k)}{n^{1/2}\beta(n)} = 0 \quad or \quad \infty$$

according as the series $\sum_1^\infty \beta(n)/n$ *diverges or converges.*

Based on Theorem G, we can obtain the following result.

Theorem 1.6 *Let* $\beta(t) > 0$, $t \ge 0$, *be a non-increasing function. Then we have almost surely that*

$$\liminf_{t\to\infty} \frac{\sup_{0\le s\le t} |W_1(\eta_2(0,s))|}{t^{1/4}\beta(t)} = 0 \quad or \quad \infty$$

according as the integral $\int_1^\infty \beta^2(t)/t\,dt$ *diverges or converges.*

An immediate consequence, via Theorem 1.1, is the following result.

Corollary 1.10 *Let* $\beta(n)$, $n = 1, 2, \ldots$, *be a non-increasing sequence of positive numbers. Then we have almost surely that*

$$\liminf_{n\to\infty} \frac{\max_{0\le k\le n} |C_1(k)|}{n^{1/4}\beta(n)} = 0 \quad or \quad \infty$$

according as the series $\sum_1^\infty \beta^2(n)/n$ *diverges or converges.*

For some related Hirsch type results for other kind of iterated Brownian motion we may refer to Bertoin [3] and Bertoin and Shi [4].

We note in passing that the above mentioned Hirsch type results for the respective two components of the random walk process $\mathbf{C}(n) = (C_1(n), C_2(n))$, $n = 0, 1, 2, \ldots$ on the 2-dimensional comb lattice $\mathbb{C}^2$ reflect only the marginal behaviour of the 2 components $C_1(n)$ and $C_2(n)$, and say nothing about their joint behaviour in this regard. The latter is an open problem and may even be so for the joint Hirsch type behaviour of a 2-dimensional Wiener process.

2 Preliminaries

Let X_i, $i = 1, 2, \ldots$, be i.i.d. random variables with the distribution $P(X_i = 1) = P(X_i = -1) = 1/2$, and put $S(0) := 0$, $S(n) := X_1 + \ldots + X_n$, $n = 1, 2, \ldots$. Define the local time process of this simple symmetric random walk by

$$\xi(k, n) := \#\{i : 1 \le i \le n,\, S(i) = k\}, \quad k = 0, \pm 1, \pm 2, \ldots, n = 1, 2, \ldots \tag{2.1}$$

We quote the following result by Révész [26], that amounts to the first simultaneous strong approximation of a simple symmetric random walk and that of its local time process on the integer lattice $\mathbb{Z}$.

Theorem J *On an appropriate probability space for a simple symmetric random walk* $\{S(n);\, n = 0,1,2,\ldots\}$ *with local time* $\{\xi(x,n);\, x = 0,\pm1,\pm2,\ldots;\, n = 0,1,2,\ldots\}$ *one can construct a standard Wiener process* $\{W(t);\, t \geq 0\}$ *with local time process* $\{\eta(x,t);\, x \in \mathbb{R};\, t \geq 0\}$ *such that, as* $n \to \infty$, *we have for any* $\varepsilon > 0$

$$S(n) - W(n) = O(n^{1/4+\varepsilon}) \quad a.s.$$

and

$$\sup_{x\in\mathbb{Z}} |\xi(x,n) - \eta(x,n)| = O(n^{1/4+\varepsilon}) \quad a.s.,$$

simultaneously.

Let $\rho(N)$ be the time of the N-th return to zero of the simple symmetric random walk on the integer lattice $\mathbb{Z}$, i.e., $\rho(0) := 0$,

$$\rho(N) := \min\{j > \rho(N-1) :\, S_j = 0\}, \quad N = 1,2,\ldots \tag{2.2}$$

Then, cf. Révész [27], we have the following result of interest for further use in the sequel.

Theorem K *For any* $0 < \varepsilon < 1$ *we have with probability 1 for all large enough* N

$$(1-\varepsilon)\frac{N^2}{2\log\log N} \leq \rho(N) \leq N^2(\log N)^{2+\varepsilon}.$$

We need inequalities for increments of the Wiener process (Csörgő and Révész [15]), Wiener local time (Csáki et al. [9]), and random walk local time (Csáki and Földes [13]).

Theorem L *With any constant* $c_2 < 1/2$ *and some* $c_1 > 0$ *we have*

$$\mathbf{P}\left(\sup_{0\leq s\leq T-h}\sup_{0\leq t\leq h}|W(s+t)-W(s)| \geq x\sqrt{h}\right) \leq \frac{c_1T}{h}e^{-c_2x^2},$$

$$\mathbf{P}\left(\sup_{0\leq s\leq t-h}(\eta(0,h+s)-\eta(0,s)) \geq x\sqrt{h}\right) \leq c_1\left(\frac{t}{h}\right)^{1/2}e^{-c_2x^2},$$

and

$$\mathbf{P}\left(\max_{0\leq j\leq t-a}(\xi(0,a+j)-\xi(0,j)) \geq x\sqrt{a}\right) \leq c_1\left(\frac{t}{a}\right)^{1/2}e^{-c_2x^2}.$$

Note that we may have the same constants c_1, c_2 in the above inequalities. In fact, in our proofs the values of these constants are not important, and it is indifferent whether they are the same or not. We continue using these notations for constants of no interest that may differ from line to line.

Corollary A *Let $0 < a_T \le T$ be a non-decreasing function of T. Then, as $T \to \infty$, we have almost surely*

$$\sup_{0\le t\le T-a_T} \sup_{s\le a_T} |W(t+s) - W(t)| = O(a_T^{1/2}(\log(T/a_T) + \log\log T)),$$

$$\sup_{0\le t\le T-a_T} (\eta(0, t+a_T) - \eta(0,T)) = O(a_T^{1/2}(\log(T/a_T) + \log\log T)),$$

and, as $N \to \infty$, we have almost surely

$$\max_{0\le n\le N-a_N} |\xi(0, n+a_N) - \xi(0,n)| = O(a_N^{1/2}(\log(N/a_N) + \log\log N)).$$

Theorem M *For fixed $x \in \mathbb{Z}$ we have for any $\varepsilon > 0$, as $n \to \infty$ and $N \to \infty$,*

$$|\xi(x,n) - \xi(0,n)| = O(n^{1/4+\varepsilon}),$$

$$\xi(x, \rho(N)) = N + O(N^{1/2+\varepsilon})$$

almost surely.

We need the following Strassen type theorem for random vectors (cf. [27], Theorem 19.3)

Theorem N *Let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent standard Wiener processes. Then, as $t \to \infty$, the net of random vectors*

$$\left(\frac{W_1(xt)}{(2t\log\log t)^{1/2}}, \frac{W_2(xt)}{(2t\log\log t)^{1/2}}; \ 0 \le x \le 1 \right)_{t\ge 3} \tag{2.3}$$

is almost surely relatively compact in the space $C([0,1], \mathbb{R}^2)$, and the set of its limit points is $\mathcal{S}^2$.

Proposition 2.1 *Let $\{W(t),\, t \ge 0\}$ be a standard Wiener process. Then the following two statements are equivalent.*

(i) *The net*

$$\left\{ \frac{W(xt)}{(2t\log\log t)^{1/2}}; \ 0 \le x \le 1 \right\}_{t\ge 3},$$

as $t \to \infty$, is almost surely relatively compact in the space $C([0,1], \mathbb{R})$ and the set of its limit points is the class of functions $\mathcal{S}$.

(ii) *The net*

$$\left\{ \frac{|W(xt)|}{(2t\log\log t)^{1/2}}; \ 0 \le x \le 1 \right\}_{t\ge 3},$$

as $t \to \infty$, is almost surely relatively compact in the space $C([0,1], \mathbb{R}^+)$ and the set of its limit points is the class of functions $\mathcal{S}^+ := \{|f| : \ f \in \mathcal{S}\}$.

Proof. Clearly, (i), that is the statement of Theorem C, implies (ii).

As to the converse, we first consider the stochastic process $\{V(t,\omega)\, t \geq 0\}$, $\omega \in \Omega_1$, that is living on a probability space $\{\Omega_1, \mathcal{A}_1, P_1\}$ and is equal in distribution to the absolute value of a standard Wiener process. Our aim now is to extend the latter probability space so that it would carry a Wiener process, constructed from the just introduced stochastic process $V(\cdot)$. This construction will be accomplished by assigning random signs to the excursions of this process. In order to realize this construction, we start with introducing an appropriate set of tools.

Let $g(u),\ u \geq 0$ be a nonnegative continuous function with $g(0) = 0$. We introduce the following notations.

$$\begin{aligned}
G_0 &:= G_0(g) = \{u \geq 0 :\ g(u) = 0,\ g(u+v) > 0\ \forall\ 0 < v \leq 1\},\\
G_1 &:= G_1(g) = \{u \geq 0 :\ u \notin G_0,\ g(u) = 0,\ g(u+v) > 0\ \forall\ 0 < v \leq 1/2\},\\
&\ \ \dots\\
G_k &:= G_k(g) = \{u \geq 0 :\ u \notin G_j, j = 0, 1, \dots, k-1,\ g(u) = 0,\ g(u+v) > 0,\ \forall\ 0 < v \leq 1/2^k\},
\end{aligned}$$

$k = 1, 2, \dots$

$$\begin{aligned}
u_{\ell 1} &:= u_{\ell 1}(g) = \min\{u :\ u \in G_\ell\},\\
&\ \ \dots\\
u_{\ell j} &:= u_{\ell j}(g) = \min\{u :\ u > u_{\ell, j-1},\ u \in G_\ell\}, \quad j = 2, 3, \dots\\
v_{\ell j} &:= v_{\ell j}(g) = \min\{u :\ u > u_{\ell j},\ g(u) = 0\}, \qquad j = 1, 2, \dots
\end{aligned}$$

$\ell = 0, 1, 2, \dots$

Let $\{\delta_{\ell j},\ \ell = 0, 1, 2, \dots,\ j = 1, 2, \dots\}$ be a double sequence of i.i.d. random variables with distribution

$$P_2(\delta_{\ell j} = 1) = P_2(\delta_{\ell j} = -1) = \frac{1}{2},$$

that is assumed to be independent of $V(\cdot)$, and lives on the probability space $(\Omega_2, \mathcal{A}_2, P_2)$.

Now, replace the function $g(\cdot)$ by $V(\cdot)$ in the above construction of $u_{\ell j}$ and $v_{\ell j}$ and define the stochastic process

$$W(u) = W(u,\omega) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \delta_{\ell j} V(u) 1_{(u_{\ell j}, v_{\ell j}]}(u), \quad u \geq 0,\ \omega \in \Omega, \tag{2.4}$$

that lives on the probability space

$$(\Omega, \mathcal{A}, \mathbf{P}) := (\Omega_1, \mathcal{A}_1, P_1) \times (\Omega_2, \mathcal{A}_2, P_2).$$

Clearly, $W(\cdot)$ as defined in (2.4) is a standard Wiener process on the latter probability space and $V(u) = |W(u)|$. Consequently, (ii) holds true in terms of the just defined Wiener process $W(\cdot)$ as in (2.4). Hence, in order to show now that (ii) implies (i) in general, it suffices to demonstrate that for any $\{f(x),\, 0 \le x \le 1\} \in \mathcal{S}$, (i) also happens to be true in terms of the same W that we have just defined in (2.4).

In order to accomplish the just announced goal, we first note that it suffices to consider only those $f \in \mathcal{S}$ for which there are finitely many zero-free intervals $(\alpha_i, \beta_i), i = 1, 2, \ldots, m$, in their support $[0, 1]$, since the set of the latter functions is dense in $\mathcal{S}$. Clearly then, such a function $f(\cdot)$ can be written as

$$f(x) = \sum_{i=1}^{m} \varepsilon_i |f(x)| 1_{(\alpha_i, \beta_i]}(x),$$

where $\varepsilon_i, \in \{-1, 1\},\ i = 1, \ldots, m$. On account of having (ii) in terms of $|W(\cdot)|$, for P_1-almost all $\omega \in \Omega_1$ there exists a sequence $\{t_r = t_r(\omega)\}_{r=1}^{\infty}$ with $\lim_{r\to\infty} t_r = \infty$, such that

$$\lim_{r\to\infty} \sup_{0\le x\le 1} \left| \frac{|W(xt_r)|}{(2t_r \log\log t_r)^{1/2}} - |f(x)| \right| = 0, \tag{2.5}$$

with $W(\cdot)$ as in (2.4).

On recalling the construction of the latter $W(\cdot)$ via the excursion intervals $(u_{\ell j}, v_{\ell j}]$, we conclude that, for r large enough, there exists a finite number of excursion intervals $(u(r,i), v(r,i)]$, $i = 1, 2, \ldots, m$, such that

$$\lim_{r\to\infty} \frac{u(r,i)}{t_r} = \alpha_i, \qquad \lim_{r\to\infty} \frac{v(r,i)}{t_r} = \beta_i,$$

for each $\omega \in \Omega_1$ for which (2.5) and the construction of the excursion intervals $(u_{\ell j}, v_{\ell j}]$ hold true.

The finite set of the just defined intervals $(u(r,i), v(r,i)]$ is a subset of the excursion intervals $(u_{\ell j}, v_{\ell j}]$ that are paired with double sequence of i.i.d. random variables $\delta_{\ell j}$ in the construction of $W(\cdot)$ as in (2.4). Let $\delta(r,i)$ denote the $\delta_{\ell j}$ that belongs to $(u(r,i), v(r,i))$. Since these random variables are independent, there exists a subsequence $\delta(r_N, i)$, $N = 1, 2, \ldots$ such that we have

$$\delta(r_N, i) = \varepsilon_i, \qquad i = 1, \ldots, m, \quad N = 1, 2, \ldots. \tag{2.6}$$

P_2-almost surely.

Hence on account of (2.5) and (2.6), we have

$$\lim_{N\to\infty} \sup_{\alpha_i \le x \le \beta_i} \left| \frac{\delta(r_N, i)|W(xt_{r_N})|}{(2t_{r_N} \log\log t_{r_N})^{1/2}} - \varepsilon_i |f(x)| \right| = 0, \quad i = 1, \ldots, m. \tag{2.7}$$

for $\mathbf{P}$-almost all $\omega \in \Omega$.

Also, as a consequence of (2.5), we have

$$\lim_{N\to\infty} \sup_{x: f(x)=0} \left| \frac{W(xt_{r_N})}{(2t_{r_N} \log\log t_{r_N})^{1/2}} \right| = 0 \tag{2.8}$$

$\mathbf{P}$-almost surely.

Consequently, on account of (2.7) and (2.8), we conclude

$$\lim_{N\to\infty} \sup_{0\le x\le 1} \left| \frac{W(xt_{r_N})}{(2t_{r_N}\log\log t_{r_N})^{1/2}} - f(x) \right| = 0.$$

P-almost surely.

This also concludes the proof of Proposition 2.1. □

We need also the following theorem of Lévy [22].

Theorem O *Let $W(\cdot)$ be a standard Wiener process with local time process $\eta(\cdot,\cdot)$. Put $M(t) = \max_{0\le s\le t} W(s)$. The following equality in distribution holds:*

$$\{(\eta(0,t), |W(t)|),\ t\ge 0\} =_d \{(M(t), M(t)-W(t),\ t\ge 0\}.$$

From Borodin-Salminen [6], 1.3.3 on p. 127, we obtain

Theorem P *For $\theta > 0$ we have*

$$\mathbf{E}\left(e^{-\theta\eta(0,t)}\right) = 2e^{\theta^2 t/2}(1-\Phi(\theta\sqrt{t})),$$

where Φ is the standard normal distribution function.

From this and the well-known asymptotic formula

$$(1-\Phi(z)) \sim \frac{c}{z}e^{-z^2/2}, \quad z\to\infty$$

we get for $\theta\sqrt{t}\to\infty$

$$\mathbf{E}\left(e^{-\theta\eta(0,t)}\right) \sim \frac{c}{\theta\sqrt{t}} \tag{2.9}$$

with some positive constant c.

3 Proof of Theorem 1.1

Obviously, on a suitable probability space we may have two independent random walks $S_1(n), S_2(n)$, with respective local times $\xi_1(x,n), \xi_2(x,n)$ both satisfying Theorem J with respective Wiener processes $W_1(t), W_2(t)$ and their local times $\eta_1(x,t), \eta_2(x,t)$. We may assume moreover, that on the same probability space we have an i.i.d. sequence $G_1, G_2, \ldots$ of geometric random variables with

$$\mathbf{P}(G_1 = k) = \frac{1}{2^{k+1}}, \quad k = 0,1,2,\ldots$$

On this probability space we may construct a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ as follows. Put $T_N = G_1 + G_2 + \ldots G_N$, $N = 1,2,\ldots$ For $n = 0,\ldots,T_1$, let $C_1(n) = S_1(n)$ and

$C_2(n) = 0$. For $n = T_1 + 1, \ldots, T_1 + \rho_2(1)$, let $C_1(n) = C_1(T_1)$, $C_2(n) = S_2(n - T_1)$. In general, for $T_N + \rho_2(N) < n \le T_{N+1} + \rho_2(N)$, let

$$C_1(n) = S_1(n - \rho_2(N)),$$

$$C_2(n) = 0,$$

and, for $T_{N+1} + \rho_2(N) < n \le T_{N+1} + \rho_2(N+1)$, let

$$C_1(n) = C_1(T_{N+1} + \rho_2(N)) = S_1(T_{N+1}),$$

$$C_2(n) = S_2(n - T_{N+1}).$$

Then it can be seen in terms of these definitions for $C_1(n)$ and $C_2(n)$ that $\mathbf{C}(n) = (C_1(n), C_2(n))$ is a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$.

Lemma 3.1 *If $T_N + \rho_2(N) \le n < T_{N+1} + \rho_2(N+1)$, then, as $n \to \infty$, we have for any $\varepsilon > 0$*

$$N = O(n^{1/2+\varepsilon}) \quad \text{a.s.}$$

and

$$\xi_2(0, n) = N + O(n^{1/4+\varepsilon}) \quad \text{a.s.}$$

Proof. If $\rho_2(N) + T_N \le n < T_{N+1} + \rho_2(N+1)$, then we have by Theorem K and the law of large numbers for $\{T_N\}_{N\ge1}$

$$(1-\varepsilon)\left(\frac{N^2}{2\log\log N} + N\right) \le n \le (1+\varepsilon)(N+1) + N^2(\log N)^{2+\varepsilon}.$$

Hence,

$$n^{1/2-\varepsilon} \le N \le n^{1/2+\varepsilon}.$$

Also, $T_N = N + O(N^{1/2+\varepsilon})$ a.s., and

$$N = \xi_2(0, \rho_2(N)) \le \xi_2(0, T_N + \rho_2(N)) \le \xi_2(0, n) \le \xi_2(0, T_{N+1} + \rho_2(N+1)).$$

Consequently, with $\varepsilon > 0$, by Corollary A we arrive at

$$\xi_2(0, T_{N+1} + \rho_2(N+1)) = \xi_2(0, \rho_2(N+1)) + O(T_{N+1}^{1/2+\varepsilon}) = N + O(N^{1/2+\varepsilon}) = N + O(n^{1/4+\varepsilon}).$$

This completes the proof of Lemma 3.1. $\Box$

Proof of Theorem 1.1. Using the above introduced definition for $C_1(n)$, in the case of $\rho_2(N) + T_N \le n < T_{N+1} + \rho_2(N)$, in combination with Theorem J, Lemma 3.1 implies that, for any $\varepsilon > 0$,

$$C_1(n) = S_1(n-\rho_2(N)) = W_1(n-\rho_2(N)) + O(T_N^{1/4+\varepsilon}) = W_1(T_N) + O(N^{1/4+\varepsilon}) = W_1(N) + O(N^{1/4+\varepsilon})$$

$$= W_1(\xi_2(0,n)) + O(n^{1/8+\varepsilon}) = W_1(\eta_2(0,n)) + O(n^{1/8+\varepsilon}) \quad \text{a.s.}$$

On the other hand, since $C_2(n) = 0$ in the interval $\rho_2(N) + T_N \le n \le \rho_2(N) + T_{N+1}$ under consideration, we only have to estimate $W_2(n)$ in that domain. In this regard we have

$$|W_2(n)| \le |W_2(\rho_2(N))| + |W_2(T_N + \rho_2(N)) - W_2(\rho_2(N))|$$

$$+ \sup_{T_N \le t \le T_{N+1}} |W_2(\rho_2(N) + t) - W_2(\rho_2(N))| = O(N^{1/2+\varepsilon}) = O(n^{1/4+\varepsilon}),$$

i.e.,

$$0 = C_2(n) = W_2(n) + O(n^{1/4+\varepsilon}).$$

In the case when $T_{N+1} + \rho_2(N) \le n < T_{N+1} + \rho_2(N+1)$, by Lemma 3.1, Theorem J and Corollary A, and using again that $T_N = N + O(N^{1/2+\varepsilon})$, for any $\varepsilon > 0$, we have almost surely

$$C_1(n) = S_1(T_{N+1}) = W_1(\xi_2(0,n)) + O(n^{1/8+\varepsilon}) = W_1(\eta_2(0,n)) + O(n^{1/8+\varepsilon}),$$

and

$$C_2(n) = S_2(n - T_{N+1}) = W_2(n - T_{N+1}) + O(N^{1/2+\varepsilon}) = W_2(n) + O(n^{1/4+\varepsilon}).$$

This completes the proof of Theorem 1.1. □

4 Proof of Theorems 1.2, 1.3

The relative compactness follows from that of the components. So we only deal with the set of limit points as $t \to \infty$.

First consider the a.s. limit points of

$$\left(\frac{W_1(xt)}{(2t \log\log t)^{1/2}}, \frac{|W_2(xt)|}{(2t \log\log t)^{1/2}}, \frac{\eta_2(0,xt)}{(2t \log\log t)^{1/2}}; \, 0 \le x \le 1 \right)_{t \ge 3} \tag{4.1}$$

and

$$\left(\frac{W_1(\eta_2(0,xt))}{2^{3/4} t^{1/4} (\log\log t)^{3/4}}, \frac{|W_2(xt)|}{(2t \log\log t)^{1/2}}; \, 0 \le x \le 1 \right)_{t \ge 3}. \tag{4.2}$$

In view of Theorem O the set of a.s. limit points of (4.1) is the same as that of

$$\left(\frac{W_1(xt)}{(2t \log\log t)^{1/2}}, \frac{M(xt) - W(xt)}{(2t \log\log t)^{1/2}}, \frac{M(xt)}{(2t \log\log t)^{1/2}}; \, 0 \le x \le 1 \right)_{t \ge 3}, \tag{4.3}$$

and the set of a.s. limit points of (4.2) is the same as that of

$$\left(\frac{W_1(M(xt))}{2^{3/4} t^{1/4} (\log\log t)^{3/4}}, \frac{M(xt) - W(xt)}{(2t \log\log t)^{1/2}}; \, 0 \le x \le 1 \right)_{t \ge 3}, \tag{4.4}$$

where $W(\cdot)$ is a standard Wiener process, independent of $W_1(\cdot)$ and $M(t) := \max_{0\le s\le t} W(s)$.

By Theorem N, the set of a.s. limit points of (4.3), and hence also that of (4.1), is

$$\{(f(x), h(x) - \ell(x), h(x)) : (f, \ell) \in \mathcal{S}^2\}, \tag{4.5}$$

where

$$h(x) = \max_{0\le u\le x} \ell(u).$$

Moreover, applying Theorem 3.1 of [12], we get that the set of a.s. limit points of (4.4), hence also that of (4.2), is

$$\{(f(h(x)), h(x) - \ell(x)) : (f, \ell) \in \mathcal{S}^2\}.$$

It is easy to see that $\dot{h}(x)(h(x) - \ell(x)) = \dot{h}(x)(\dot{h}(x) - \dot{\ell}(x)) = 0$ and

$$\int_0^1 ((\dot{h}(x) - \dot{\ell}(x))^2 + \dot{h}^2(x))\,dx = \int_0^1 \dot{\ell}^2(x)\,dx + 2\int_0^1 \dot{h}(x)(\dot{h}(x) - \dot{\ell}(x))\,dx = \int_0^1 \dot{\ell}^2(x)\,dx.$$

Since $(f, \ell) \in \mathcal{S}^2$, we have

$$\int_0^1 (\dot{f}^2(x) + (\dot{h}(x) - \dot{\ell}(x))^2 + \dot{h}^2(x))\,dx \le 1.$$

On denoting the function $h(\cdot) - \ell(\cdot)$ in (4.5) by $|g(\cdot)|$, we can now conclude that the set of a.s. limit points of the net in (4.1) is the set of functions $(f, |g|, h)$, where $(f, g, h) \in \mathcal{S}^{(3)}$. Consequently, via Proposition 2.1, the set of functions $(f, g, h) \in \mathcal{S}^{(3)}$ is seen to be the almost sure set of limit points of the net of random vectors in (1.18), as $t \to \infty$, on repeating the proof of Proposition 2.1 in the context of the net of random vectors as in (1.18) and (4.1).

This also completes the proof of Theorem 1.2. □

To finish the proof of Theorem 1.3, it remains to show that $\mathcal{S}_1^{(2)} = \mathcal{S}_2^{(2)}$, where

$$\mathcal{S}_1^{(2)} := \Big\{(f(h(x)), g(x)) : (f, g) \in \mathcal{S}^2, h \in \mathcal{S}_M, \\ \int_0^1 (\dot{f}^2(x) + \dot{g}^2(x) + \dot{h}^2(x))\,dx \le 1, g(x)\dot{h}(x) = 0 \ a.e.\Big\}$$

$$\mathcal{S}_2^{(2)} := \Big\{(k(x), g(x)) : k(0) = g(0) = 0,\ k, g \in \dot{C}([0,1], \mathbb{R}) \\ \int_0^1 (|3^{3/4}2^{-1/2}\dot{k}(x)|^{4/3} + \dot{g}^2(x))\,dx \le 1,\ \dot{k}(x)g(x) = 0 \ a.e.\Big\}.$$

Assume first that $(f(h), g) \in \mathcal{S}_1^{(2)}$. Let $k(x) = f(h(x))$. Obviously $k(0) = g(0) = 0$, $k, g \in \dot{C}([0,1], \mathbb{R})$, and $\dot{k}(x)g(x) = 0$ a.e. Using Hölder's inequality, the simple inequality $A^{2/3}B^{1/3} \le 2^{2/3}3^{-1}(A + B)$ and $h(1) \le 1$ (cf. the proof of Lemma 2.1 in [14]) we get

$$\int_0^1 (3^{3/4}2^{-1/2}|\dot{k}(x)|)^{4/3}\,dx \le 3/2^{2/3}\left(\int_0^1 \dot{f}^2(x)\,dx\right)^{2/3}\left(\int_0^1 \dot{h}^2(x)\,dx\right)^{1/3} \le \int_0^1 (\dot{f}^2(x) + \dot{h}^2(x))\,dx,$$

showing that $(k, g) \in \mathcal{S}_2^{(2)}$.

Now assume that $(k, g) \in \mathcal{S}_2^{(2)}$. Define

$$h(x) = \frac{1}{2^{1/3}} \int_0^x |\dot{k}(u)|^{2/3}\, du$$

and

$$f(u) = \begin{cases} k(h^{-1}(u)) & \text{for} \quad 0 \le u \le h(1), \\ k(1) & \text{for} \quad h(1) \le u \le 1. \end{cases}$$

Then (cf. [14])

$$\begin{aligned} \int_0^1 \dot{f}^2(u)\, du + \int_0^1 \dot{h}^2(x)\, dx &= \int_0^1 |\dot{f}(h(x))|^2 \dot{h}(x)\, dx + \int_0^1 \frac{1}{2^{2/3}} |\dot{k}(x)|^{4/3}\, dx \\ &= \frac{3}{2^{2/3}} \int_0^1 |\dot{k}(x)|^{4/3}\, dx, \end{aligned}$$

from which $(f(h(x)), g(x)) \in \mathcal{S}_1^{(2)}$ follows. This completes the proof of Theorem 1.3. □

5 Proof of Theorem 1.5

Recall the definitions (1.25)-(1.31), and put

$$k(x, K) := k(x, B, K), \qquad g(x, K) := g(x, A, K).$$

It is easy to see that

$$\int_0^1 (|3^{3/4} 2^{-1/2} \dot{k}(x, K)|^{4/3} + (\dot{g}(x, K))^2) dx = \frac{3B^{4/3}}{2^{2/3} K^{1/3}} + \frac{A^2}{1 - K} = F(|B|, |A|, K).$$

Hence, if

$$F(|B|, |A|, K) \le 1,$$

then

$$(k(x, K), g(x, K)) \in \mathcal{S}^{(2)}$$

and

$$D_2 \subseteq D.$$

Now we have to show that $D \subseteq D_2$. On assuming that $(k_0(\cdot), g_0(\cdot)) \in \mathcal{S}^{(2)}$, we show that $(k_0(1), g_0(1)) \in D_2$. Let

$$\begin{aligned} L &= \{x : \ \dot{k}_0(x) = 0\}, \qquad & \lambda(L) &= \kappa, \\ M &= \{x : \ g_0(x) = 0\}, \qquad & \lambda(M) &= \mu, \end{aligned}$$

where λ is the Lebesgue measure. Clearly $\mu+\kappa \geq 1$ and there exist monotone, measure preserving, one to one transformations $m(x)$ resp. $n(x)$ defined on the complements of the above sets $\overline{L}$ resp. $\overline{M}$ such that $m(x)$ maps $\overline{L}$ onto $[0,1-\kappa]$ and $n(x)$ maps $\overline{M}$ onto $[\mu,1]$:

$$\begin{aligned} m(x) &\in [0,1-\kappa] \quad (x \in \overline{L}), \\ n(x) &\in [\mu,1] \quad (x \in \overline{M}). \end{aligned}$$

Define the funtion $k_1(y)$ resp. $g_1(y)$ by

$$k_1(y) = \begin{cases} k_0(m^{-1}(y)) \text{ for } y \in [0,1-\kappa] \\ k_1(1-\kappa) \text{ for } y \in (1-\kappa,1], \end{cases}$$

$$g_1(y) = \begin{cases} 0 & \text{for } y \in [0,\mu] \\ g_0(n^{-1}(y)) & \text{for } \ y \in (\mu,1]. \end{cases}$$

Note that

$$\begin{aligned} \int_0^1 |\dot{k}_1(y)|^{4/3} dy &= \int_0^1 |\dot{k}_0(x)|^{4/3} dx, \\ \int_0^1 (\dot{g}_1(y))^2 dy &= \int_0^1 (\dot{g}_0(x))^2 dx, \\ (k_1(y), g_1(y)) &\in \mathcal{S}^{(2)}. \end{aligned}$$

Taking into account that $1-\kappa \leq \mu$, we define the following linear approximations k_2 resp. g_2 of k_1 resp. g_1 :

$$k_2(x) = k(x, k_1(1), 1-\kappa) = \begin{cases} \dfrac{x}{\mu} k_1(1) & \text{if} \quad 0 \leq x \leq \mu, \\ k_1(1) & \text{if} \quad \mu \leq x \leq 1, \end{cases}$$

$$g_2(x) = g(x, g_1(1), 1-\mu) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \mu, \\ \dfrac{x-\mu}{1-\mu} g_1(1) & \text{if} \quad \mu \leq x \leq 1. \end{cases}$$

It follows from Hölder's inequality (cf., e.g. Riesz and Sz.-Nagy [28] p. 75) that

$$\begin{aligned} &\int_0^1 (\, |3^{3/4} 2^{-1/2} \dot{k}_2(x)|^{4/3} + (\dot{g}_2(x))^2 \,)\, dx \\ \leq &\int_0^1 (\, |3^{3/4} 2^{-1/2} \dot{k}_1(x)|^{4/3} + (\dot{g}_1(x))^2 \,)\, dx = F(|k_1(1)|, |g_1(1)|, \mu) \leq 1, \end{aligned}$$

implying that $(k_1(1), g_1(1)) \in D_2$. Taking into account that $|k_0(1)| \leq |k_1(1)|$ and $|g_0(1)| \leq |g_1(1)|$ by our construction, $(k_0(1), g_0(1)) \in D_2$ as well, which implies that $D \subseteq D_2$. $\square$

6 Proof of Theorem 1.6

First assume that $\int_1^\infty \beta^2(t)/t\,dt < \infty$. Put $t_n = e^n$. Then we also have $\sum_n \beta^2(t_n) < \infty$. Indeed, it is well known that the integral and series in hand are equiconvergent. For arbitrary $\varepsilon > 0$ consider the events

$$A_n = \left\{ \sup_{0 \le s \le t_n} |W_1(\eta_2(0,s))| \le \frac{1}{\varepsilon} t_{n+1}^{1/4} \beta(t_n) \right\},$$

$n = 1, 2, \ldots$ It follows from (1.35) of Theorem G that

$$\mathbf{P}(A_n) \le \frac{c_2}{\varepsilon^2} \left(\frac{t_{n+1}}{t_n} \right)^{1/2} \beta^2(t_n) = c_3 \beta^2(t_n),$$

which is summable, hence $\mathbf{P}(A_n \text{ i.o.}) = 0$. Consequently, for large n, we have

$$\sup_{0 \le s \le t_n} |W_1(\eta_2(0,s))| \ge \frac{1}{\varepsilon} t_{n+1}^{1/4} \beta(t_n),$$

and for $t_n \le t < t_{n+1}$, we have as well

$$\sup_{0 \le s \le t} |W_1(\eta_2(0,s))| \ge \frac{1}{\varepsilon} t^{1/4} \beta(t) \quad \text{a.s.}$$

Since the latter inequality is true for t large enough and $\varepsilon > 0$ is arbitrary, we arrive at

$$\liminf_{t \to \infty} \frac{\sup_{0 \le s \le t} |W_1(\eta_2(0,s))|}{t^{1/4} \beta(t)} = \infty \quad \text{a.s.}$$

Now assume that $\int_1^\infty \beta^2(t)/t\,dt = \infty$. Put $t_n = e^n$. Hence we have also $\sum_n \beta^2(t_n) = \infty$. Let $W^*(t) = \sup_{0 \le s \le t} |W_1(\eta_2(0,s))|$. Consider the events

$$A_n = \left\{ W^*(t_n) \le t_n^{1/4} \beta(t_n) \right\},$$

$n = 1, 2, \ldots$ It follows from (1.35) of Theorem G that

$$\mathbf{P}(A_n) \ge c\beta^2(t_n),$$

consequently $\sum_n \mathbf{P}(A_n) = \infty$.

Now we are to estimate $\mathbf{P}(A_m A_n)$. In fact, we have to estimate the probability $\mathbf{P}(W^*(s) < a,\, W^*(t) < b)$ for $s = t_m$, $t = t_n$, with $a = t_m^{1/4} \beta(t_m)$, $b = t_n^{1/4} \beta(t_n)$. Applying Lemma 1 of Shi [29], we have for $0 < s < t$, $0 < a \le b$,

$$\mathbf{P}(W^*(s) < a,\, W^*(t) < b) \le \frac{16}{\pi^2} \mathbf{E} \left(\exp \left(-\frac{\pi^2}{8a^2} \eta_2(0,s) - \frac{\pi^2}{8b^2} (\eta_2(0,t) - \eta_2(0,s)) \right) \right).$$

Next we wish to estimate the expected value on the right-hand side of the latter inequality. For the sake of our calculations, we write $\eta(0,s)$ instead of $\eta_2(0,s)$ to stand for the local time at zero of a standard Wiener process $W(\cdot)$, i.e., we also write W instead of W_2. With this convenient notation, we now let

$$\alpha(s) = \max\{u < s : W(u) = 0\} \qquad \gamma(s) = \min\{v > s : W(v) = 0\},$$

and let $g(u,v)$, $0 < u < s < v$ denote the joint density function of these two random variables. Recall that the marginal distribution of $\alpha(s)$ is the arcsine law with density function

$$g_1(u) = \frac{1}{\pi\sqrt{u(s-u)}}, \quad 0 < u < s.$$

Putting $\lambda_1 = \pi^2/(8a^2)$, $\lambda_2 = \pi^2/(8b^2)$, a straightforward calculation yields

$$\mathbf{E}\left(\exp\left(-\lambda_1\eta(0,s) - \lambda_2(\eta(0,t) - \eta(0,s))\right)\right)$$

$$= \iint_{0<u<s<v} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u)=0) g(u,v) \mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v)=0)\, dudv = I_1 + I_2,$$

where $I_1 = \iint_{0<u<s<v<t/2}$ and $I_2 = \iint_{0<u<s,\, t/2<v}$. The first part is not void if $s = e^m$, $t = e^n$, $n < m$, since obviously $e^m < e^n/2$. Estimating them now, in the first case we use the inequality

$$\mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v)=0) \le \mathbf{E}(e^{-\lambda_2\eta(0,t/2)}),$$

while in the second case we simply estimate this expectation by 1. Thus

$$I_1 = \iint_{0<u<s<v<t/2} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u)=0) g(u,v) \mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v)=0)\, dudv$$

$$\le \mathbf{E}(e^{-\lambda_2\eta(0,t/2)}) \iint_{0<u<s<v} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u)=0) g(u,v)\, dudv$$

$$= \mathbf{E}(e^{-\lambda_1\eta(0,s)}) \mathbf{E}(e^{-\lambda_2\eta(0,t/2)}).$$

In the second case we have

$$\left(\int_{t/2}^{\infty} g(u,v)\, dv\right) du = \mathbf{P}(\alpha(t/2) \in du).$$

But

$$\frac{\mathbf{P}(\alpha(t/2) \in du)}{\mathbf{P}(\alpha(s) \in du)} \le c\frac{\sqrt{s-u}}{\sqrt{t/2-u}} \le c\sqrt{\frac{2s}{t}}.$$

Hence

$$I_2 = \iint_{0<u<s,\, v>t/2} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u)=0) g(u,v) \mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v)=0)\, dudv$$

$$\leq c\sqrt{\frac{s}{t}}\int_0^s \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u)=0)g_1(u)\,du = c\sqrt{\frac{s}{t}}\mathbf{E}\left(e^{-\lambda_1\eta(0,s)}\right).$$

On using (2.9) now, we arrive at

$$I_1 + I_2 \leq \frac{c}{\lambda_1\lambda_2\sqrt{st}} + \frac{c}{\lambda_1\sqrt{t}},$$

with some positive constant c. To estimate $\mathbf{P}(A_m\, A_n)$, put $s = t_m = e^m$, $t = t_n = e^n$. Then, on recalling the definitions of a and b, respectively in λ_1 and λ_2, we get

$$\lambda_1 = \frac{\pi^2}{8t_m^{1/2}\beta^2(t_m)}, \quad \lambda_2 = \frac{\pi^2}{8t_n^{1/2}\beta^2(t_n)},$$

which in turn implies

$$\mathbf{P}(A_m\, A_n) \leq c\beta^2(t_m)\beta^2(t_n) + c\frac{t_m^{1/2}}{t_n^{1/2}}\beta^2(t_m) \leq c\mathbf{P}(A_m)\mathbf{P}(A_n) + ce^{(m-n)/2}\mathbf{P}(A_m).$$

Since $e^{(m-n)/2}$ is summable for fixed m, by the Borel-Cantelli lemma we get $\mathbf{P}(A_n \text{ i.o.}) > 0$. Also, by 0-1 law, this probability is equal to 1. This completes the proof of Theorem 1.6. $\square$

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