# TRANSIENT NEAREST NEIGHBOR RANDOM WALK AND BESSEL PROCESS

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*Abstract:* We prove strong invariance principle between a transient Bessel process and a certain nearest neighbor (NN) random walk that is constructed from the former by using stopping times. It is also shown that their local times are close enough to share the same strong limit theorems. It is shown furthermore, that if the difference between the distributions of two NN random walks are small, then the walks themselves can be constructed so that they are close enough. Finally, some consequences concerning strong limit theorems are discussed.

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# 1. Introduction

In this paper we consider a nearest neighbor (NN) random walk, defined as follows: let $X_0 = 0,\ X_1, X_2, \ldots$ be a Markov chain with

$$\begin{aligned} E_i &:= \mathbf{P}(X_{n+1} = i+1 \mid X_n = i) = 1 - \mathbf{P}(X_{n+1} = i-1 \mid X_n = i) \\ &= \begin{cases} 1 & \text{if} \quad i = 0 \\ 1/2 + p_i & \text{if} \quad i = 1, 2, \ldots, \end{cases} \end{aligned} \tag{1.1}$$

where $-1/2 \le p_i \le 1/2,\ i = 1, 2, \ldots$. In case $0 < p_i \le 1/2$ the sequence $\{X_i\}$ describes the motion of a particle which starts at zero, moves over the nonnegative integers and going away from 0 with a larger probability than to the direction of 0. We will be interested in the case when $p_i \sim B/4i$ with $B > 0$ as $i \to \infty$. We want to show that in certain sense, this Markov chain is a discrete analogue of continuous Bessel process and establish a strong invariance principle between these two processes.

The properties of the discrete model, often called birth and death chain, connections with orthogonal polynomials in particular, has been treated extensively in the literature. See e.g. the classical paper by Karlin and McGregor [12], or more recent papers by Coolen-Schrijner and Van Doorn [6] and Dette [9]. In an earlier paper [7] we investigated the local time of this Markov chain in the transient case.

There is a well-known result in the literature (cf. e.g. Chung [5]) characterizing those sequences $\{p_i\}$ for which $\{X_i\}$ is transient (resp. recurrent).

**Theorem A:** ([5], page 74) *Let $X_n$ be a Markov chain with transition probabilities given in* (1.1) *with $-1/2 < p_i < 1/2$, $i = 1, 2, \ldots$ Define*

$$U_i := \frac{1 - E_i}{E_i} = \frac{1/2 - p_i}{1/2 + p_i} \tag{1.2}$$

*Then $X_n$ is transient if and only if*

$$\sum_{k=1}^{\infty} \prod_{i=1}^{k} U_i < \infty.$$

As a consequence, the Markov chain $(X_n)$ with $p_R \sim B/4R$, $R \to \infty$ is transient if $B > 1$ and recurrent if $B < 1$.

The Bessel process of order $\nu$, denoted by $Y_\nu(t)$, $t \ge 0$ is a diffusion process on the line with generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu + 1}{2x}\frac{d}{dx}.$$

$d = 2\nu + 2$ is the dimension of the Bessel process. If $d$ is a positive integer, then $Y_\nu(\cdot)$ is the absolute value of a $d$-dimensional Brownian motion. The Bessel process $Y_\nu(t)$ is transient if and only if $\nu > 0$.

The properties of the Bessel process were extensively studied in the literature. Cf. Borodin and Salminen [2], Revuz and Yor [19], Knight [14].

Lamperti [15] determined the limiting distribution of $X_n$ and also proved a weak convergence theorem in a more general setting. His result in our case reads as follows.

**Theorem B:** ([15]) *Let $X_n$ be a Markov chain with transition probabilities given in* (1.1) *with $-1/2 < p_i < 1/2$, $i = 1, 2, \ldots$ If $\lim_{R\to\infty} Rp_R = B/4 > -1/4$, then the following weak convergence holds*:

$$\frac{X_{[nt]}}{\sqrt{n}} \Longrightarrow Y_{(B-1)/2}(t)$$

*in the space $D[0,1]$. In particular,*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_n}{\sqrt{n}} < x\right) = \frac{1}{2^{B/2-1/2}\Gamma(B/2+1/2)} \int_0^x u^B e^{-u^2/2}\, du.$$

In Theorems A and B values of $p_i$ can be negative. In the sequel however we deal only with the case when $p_i$ are non-negative, and the chain is transient, which will be assumed throughout without mentioning it.

Let

$$D(R,\infty) := 1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} U_{R+i}, \tag{1.3}$$

and define

$$p_R^* := \frac{\frac{1}{2} + p_R}{D(R,\infty)} = 1 - q_R^* \tag{1.4}$$

Now let $\xi(R,\infty)$, $R = 0, 1, 2, \ldots$ be the total local time at $R$ of the Markov chain $\{X_n\}$, i.e.

$$\xi(R,\infty) := \#\{n \geq 0 : X_n = R\}. \tag{1.5}$$

**Theorem C:** ([7]) *For a transient NN random walk*

$$\mathbf{P}(\xi(R,\infty) = k) = p_R^*(q_R^*)^{k-1}, \qquad k = 1, 2, \ldots \tag{1.6}$$

Moreover, $\eta(R,t)$, $R > 0$ will denote the local time of the Bessel process, i.e.

$$\eta(R,t) := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t I\{Y_\nu(s) \in (R-\varepsilon, R+\varepsilon)\}\, ds, \qquad \eta(R,\infty) := \lim_{t\to\infty} \eta(R,t).$$

It is well-known that $\eta(R,\infty)$ has exponential distribution (see e.g. [2]).

$$\mathbf{P}(\eta(R,\infty)<x)=1-\exp\left(-\frac{\nu}{R}x\right). \tag{1.7}$$

For $0<a<b$ let

$$\tau:=\tau(a,b)=\min\{t\geq 0:\, Y_\nu(t)\notin(a,b)\}. \tag{1.8}$$

Then we have (cf. Borodin and Salminen [2], Section 6, 3.0.1 and 3.0.4).

**Theorem D:** *For $0<a<x<b$ we have*

$$\mathbf{P}_x(Y_\nu(\tau)=a)=1-\mathbf{P}_x(Y_\nu(\tau)=b)=\frac{x^{-2\nu}-b^{-2\nu}}{a^{-2\nu}-b^{-2\nu}}, \tag{1.9}$$

$$\mathbf{E}_x e^{-\alpha\tau}=\frac{S_\nu(b\sqrt{2\alpha},x\sqrt{2\alpha})+S_\nu(x\sqrt{2\alpha},a\sqrt{2\alpha})}{S_\nu(b\sqrt{2\alpha},a\sqrt{2\alpha})}, \tag{1.10}$$

*where*

$$S_\nu(u,v)=(uv)^{-\nu}(I_\nu(u)K_\nu(v)-K_\nu(u)I_\nu(v)), \tag{1.11}$$

*$I_\nu$ and $K_\nu$ being the modified Bessel functions of the first and second kind, resp.*

Here and in what follows $\mathbf{P}_x$ and $\mathbf{E}_x$ denote conditional probability, resp. expectation under $Y_\nu(0)=x$. For simplicity we will use $\mathbf{P}_0=\mathbf{P}$, and $\mathbf{E}_0=\mathbf{E}$.

Now consider $Y_\nu(t)$, $t\geq 0$, a Bessel process of order $\nu$, $Y_\nu(0)=0$, and let $X_n$, $n=0,1,2,\ldots$ be an NN random walk with $p_0=p_1=1/2$,

$$p_R=\frac{(R-1)^{-2\nu}-R^{-2\nu}}{(R-1)^{-2\nu}-(R+1)^{-2\nu}}-\frac{1}{2}, \qquad R=2,3,\ldots \tag{1.12}$$

Our main results are strong invariance principles concerning Bessel process, NN random walk and their local times.

**Theorem 1.1.** *On a suitable probability space we can construct a Bessel process $\{Y_\nu(t),\, t\geq 0\}$, $\nu>0$ and an NN random walk $\{X_n,\, n=0,1,2,\ldots\}$ with $p_R$ as in (1.12) such that for any $\varepsilon>0$, as $n\to\infty$ we have*

$$Y_\nu(n)-X_n=O(n^{1/4+\varepsilon}) \qquad \text{a.s.} \tag{1.13}$$

Our strong invariance principle for local times reads as follows.

**Theorem 1.2.** *Let $Y_\nu(t)$ and $X_n$ as in* Theorem 1.1 *and let $\eta$ and $\xi$ their respective local times. As $R \to \infty$, we have*

$$\xi(R,\infty) - \eta(R,\infty) = O(R^{1/2}\log R) \quad \text{a.s.} \tag{1.14}$$

We prove the following strong invariance principle between two NN random walks.

**Theorem 1.3.** *Let $\{X_n^{(1)}\}_{n=0}^{\infty}$ and $\{X_n^{(2)}\}_{n=0}^{\infty}$ be two NN random walk with $p_j^{(1)}$ and $p_j^{(2)}$, resp. Assume that*

$$\left| p_j^{(1)} - \frac{B}{4j} \right| \le \frac{C}{j^\gamma} \tag{1.15}$$

*and*

$$\left| p_j^{(2)} - \frac{B}{4j} \right| \le \frac{C}{j^\gamma} \tag{1.16}$$

*$j = 1, 2, \ldots$ with $B > 1$, $1 < \gamma \le 2$ and some non-negative constant $C$. Then on a suitable probability space one can construct $\{X_n^{(1)}\}$ and $\{X_n^{(2)}\}$ such that as $n \to \infty$*

$$|X_n^{(1)} - X_n^{(2)}| = O((X_n^{(1)} + X_n^{(2)})^{2-\gamma}) = O((n\log\log n)^{1-\gamma/2}) \quad \text{a.s.}$$

The organization of the paper is as follows. In Section 2 we will present some well-known facts and prove some preliminary results. Sections 3-5 contain the proofs of Theorems 1.1-1.3, respectively. In Section 6 we prove strong theorems (most of them are integral tests) which easily follow from Theorems 1.1 and 1.2 and the corresponding results for Bessel process. In Section 7, using our Theorem 1.3 in both directions, we prove an integral test for the local time of the NN-walk, and a strong theorem for the speed of escape of the Bessel process.

## 2. Preliminaries

**Lemma 2.1.** *Let $Y_\nu(\cdot)$ be a Bessel process starting from $x = R$ and let $\tau$ be the stopping time defined by* (1.8) *with $a = R-1$ and $b = R+1$. Let $p_R$ be defined by* (1.12). *Then as $R \to \infty$*

$$p_R = \frac{2\nu+1}{4R} + O\left(\frac{1}{R^2}\right), \tag{2.1}$$

$$\mathbf{E}_R(\tau) = 1 + O\left(\frac{1}{R}\right), \tag{2.2}$$

$$Var_R(\tau) = O(1). \tag{2.3}$$

**Proof:** For $\nu = 1/2$, i.e. for $d = 3$-dimensional Bessel process, in case $x = R$, $a = R - 1$, $b = R + 1$ we have

$$\mathbf{E}_R(e^{\lambda\tau}) = \frac{1}{\cos(\sqrt{2\lambda})}$$

which does not depend on $R$. We prove that this holds asymptotically in general, when $\nu > 0$.

Using the identity (cf. [2], page 449 and [21], page 78)

$$K_\nu(x) = \begin{cases} \dfrac{\pi}{2\sin(\nu\pi)}(I_{-\nu}(x) - I_\nu(x)) & \text{if } \nu \text{ is not an integer} \\ \\ \lim_{\mu\to\nu} K_\mu(x) & \text{if } \nu \text{ is an integer} \end{cases}$$

and the series expansion

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)},$$

one can see that the coefficient of $-\alpha$ in the Taylor series expansion of the Laplace transform (1.10) is

$$\mathbf{E}_x(\tau) = \frac{1}{2(\nu+1)} \frac{(b^2 - x^2)a^{-2\nu} + (x^2 - a^2)b^{-2\nu} - (b^2 - a^2)x^{-2\nu}}{a^{-2\nu} - b^{-2\nu}}$$

from which by putting $x = R$, $a = R - 1$, $b = R + 1$, we obtain

$$\mathbf{E}_R(\tau) = \frac{1}{2(\nu+1)} \frac{(2R+1)(R-1)^{-2\nu} + (2R-1)(R+1)^{-2\nu} - 4R^{1-2\nu}}{(R-1)^{-2\nu} - (R+1)^{-2\nu}}$$

giving (2.2) after some calculations.

(2.3) can also be obtained similarly, but it seems quite complicated. A simpler argument is to use moment generating function and expansion of the Bessel functions for imaginary arguments near infinity. Put $\alpha = -\lambda$ into (1.10) to obtain

$$\mathbf{E}_x(e^{\lambda\tau}) = \frac{S_\nu(ib\sqrt{2\lambda}, ix\sqrt{2\lambda}) + S_\nu(ix\sqrt{2\lambda}, ia\sqrt{2\lambda})}{S_\nu(ib\sqrt{2\lambda}, ia\sqrt{2\lambda})}, \tag{2.4}$$

where $i = \sqrt{-1}$. We use the following asymptotic expansions (cf. Erdélyi et al. [11], page 86, or Watson [21], pages 202, 219)

$$I_\nu(z) = (2\pi z)^{-1/2}\left(e^z + ie^{-z+i\nu\pi} + O(|z|^{-1})\right),$$

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} \left(e^{-z} + O(|z|^{-1})\right).$$

Hence one obtains for $\lambda > 0$ fixed, and $x < b$,

$$S_\nu(ib\sqrt{2\lambda}, ix\sqrt{2\lambda}) = (-2\lambda bx)^{-\nu}(I_\nu(ib\sqrt{2\lambda})K_\nu(ix\sqrt{2\lambda}) - I_\nu(ix\sqrt{2\lambda})K_\nu(ib\sqrt{2\lambda}))$$

$$= \frac{1}{2}(-2\lambda bx)^{-\nu-1/2}\left(e^{i(b-x)\sqrt{2\lambda}} - e^{-i(b-x)\sqrt{2\lambda}} + O\left(\frac{1}{x}\right)\right), \quad x \to \infty.$$

One can obtain asymptotic expansions similarly for $S_\nu(ix\sqrt{2\lambda}, ia\sqrt{2\lambda})$, $S_\nu(ib\sqrt{2\lambda}, ia\sqrt{2\lambda})$. Putting these into (2.4), with $x = R$, $a = R-1$, $b = R+1$, we get as $R \to \infty$

$$\mathbf{E}_R(e^{\lambda\tau}) = \frac{(R^2+R)^{-\nu-1/2} + (R^2-R)^{-\nu-1/2}}{(R^2-1)^{-\nu-1/2}} \, \frac{e^{i\sqrt{2\lambda}} - e^{-i\sqrt{2\lambda}} + O\left(\frac{1}{R}\right)}{e^{2i\sqrt{2\lambda}} - e^{-2i\sqrt{2\lambda}} + O\left(\frac{1}{R}\right)}$$

$$= \frac{1}{\cos(\sqrt{2\lambda})} + O\left(\frac{1}{R}\right).$$

Hence putting $\lambda = 1$, there exists a constant $C$ such that $\mathbf{E}_R(e^\tau) \le C$ for all $R = 1, 2, \ldots$ By Markov's inequality we have

$$\mathbf{P}_R(\tau > t) = \mathbf{P}_R(e^\tau > e^t) \le Ce^{-t},$$

from which $\mathbf{E}_R(\tau^2) \le 2C$, implying (2.3). □

Here and throughout $C, C_1, C_2, \ldots$ denotes unimportant positive (possibly random) constants whose values may change from line to line.

Recall the definition of the upper and lower classes for a stochastic process $Z(t)$, $t \ge 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$ (cf. Révész [18], p. 33).

The function $a_1(t)$ belongs to the *upper-upper class* of $Z(t)$ ($a_1(t) \in \mathrm{UUC}(Z(t))$ if for almost all $\omega \in \Omega$ there exists a $t_0(\omega) > 0$ such that $Z(t) < a_1(t)$ if $t > t_0(\omega)$.

The function $a_2(t)$ belongs to the *upper-lower class* of $Z(t)$ ($a_1(t) \in \mathrm{ULC}(Z(t))$ if for almost all $\omega \in \Omega$ there exists a sequence of positive numbers $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \ldots$ with $\lim_{i\to\infty} t_i = \infty$ such that $Z(t_i) \ge a_2(t_i)$, $(i = 1, 2, \ldots)$.

The function $a_3(t)$ belongs to the *lower-upper class* of $Z(t)$ ($a_3(t) \in \mathrm{LUC}(Z(t))$ if for almost all $\omega \in \Omega$ there exists a sequence of positive numbers $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \ldots$ with $\lim_{i\to\infty} t_i = \infty$ such that $Z(t_i) \le a_3(t_i)$, $(i = 1, 2, \ldots)$.

The function $a_4(t)$ belongs to the *lower-lower class* of $Z(t)$ ($a_4(t) \in \mathrm{LLC}(Z(t))$ if for almost all $\omega \in \Omega$ there exists a $t_0(\omega) > 0$ such that $Z(t) > a_4(t)$ if $t > t_0(\omega)$.

The following lower class results are due to Dvoretzky and Erdős [10] for integer $d = 2\nu+2$. In the general case when $\nu > 0$, the proof is similar (cf. also Knight [14] and Chaumont and Pardo [4] in the case of positive self-similar Markov processes).

**Theorem E:** *Let $\nu > 0$ and let $b(t)$ be a non-increasing, non-negative function.*

- $t^{1/2}b(t) \in \mathrm{LLC}(Y_\nu(t))$ *if and only if* $\int_1^\infty (b(2^t))^{2\nu}\,dt < \infty.$

It follows e.g. that in case $\nu > 0$, for any $\varepsilon > 0$ we have

$$Y_\nu(t) \geq t^{1/2-\varepsilon} \tag{2.5}$$

almost surely for all sufficiently large $t$.

In fact, from our invariance principle it will follow that the integral test in Theorem E holds also for our Markov chain $(X_n)$. In the proof however we need an analogue of (2.5) for $X_n$.

One can easily calculate the exact distribution of $\xi(R,\infty)$, the total local time of $X_n$ of Theorem 1.1 according to Theorem C.

**Lemma A:** *If $p_R$ is given by* (1.12)*, then $\xi(R,\infty)$ has geometric distribution* (1.6) *with*

$$p_R^* = \frac{\frac{1}{2}+p_R}{D(R,\infty)} = \frac{(\frac{1}{2}+p_R)((R+1)^{2\nu}-R^{2\nu})}{(R+1)^{2\nu}} = \frac{\nu}{R} + O\left(\frac{1}{R}\right). \tag{2.6}$$

**Lemma 2.2.** *For any $\delta > 0$ we have*

$$X_n \geq n^{1/2-\delta}$$

*almost surely for all large enough $n$.*

**Proof:** From Lemma A it is easy to conclude that almost surely for some $R_0 > 0$

$$\xi(R,\infty) \leq CR\log R$$

if $R \geq R_0$, with some random positive constant $C$. Hence the time $\sum_{R=1}^{S}\xi(R,\infty)$ which the particle spent up to $\infty$ in $[1,S]$ is less than

$$\sum_{R=1}^{R_0-1}\xi(R,\infty) + C\sum_{R=R_0}^{S} R\log R \leq C_1 S^{2+\delta}$$

with some (random) $C_1 > 0$. Consequently, after $C_1 S^{2+\delta}$ steps the particle will be farther away from the origin than $S$. Let

$$n = [C_1 S^{2+\delta}],$$

then

$$S \geq \left(\frac{n}{C_1}\right)^{1/(2+\delta)}$$

and hence

$$X_n \geq \left(\frac{n}{C_1}\right)^{1/(2+\delta)} \geq n^{1/2-\delta}$$

for $n$ large enough. This proves the Lemma. □

# 3. Proof of Theorem 1.1

Define the sequences $(\tau_n)$, $t_0 = 0$, $t_n := \tau_1 + \ldots + \tau_n$ as follows:

$$\begin{aligned}
\tau_1 &:= \min\{t: \ t > 0, \ Y_\nu(t) = 1\}, \\
\tau_2 &:= \min\{t: \ t > 0, \ Y_\nu(t + t_1) = 2\}, \\
\tau_n &:= \min\{t: \ t > 0, \ |Y_\nu(t + t_{n-1}) - Y_\nu(t_{n-1})| = 1\} \quad \text{for} \quad n = 3, 4, \ldots
\end{aligned}$$

Let $X_n = Y_\nu(t_n)$. Then (cf. (1.12)) it is an NN random walk with $p_0 = p_1 = 1/2$,

$$p_R = \frac{(R-1)^{-2\nu} - R^{-2\nu}}{(R-1)^{-2\nu} - (R+1)^{-2\nu}} - \frac{1}{2}, \qquad R = 2, 3, \ldots$$

Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $(\tau_k, \ Y_\nu(\tau_k))_{k=1}^n$ and consider

$$M_n := \sum_{i=1}^n (\tau_i - \mathbf{E}(\tau_i \mid \mathcal{F}_{i-1})).$$

Then the sequence $(M_n)_{n\geq 1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 1}$. It follows from (2.2) of Lemma 2.1 that for $i = 2, 3, \ldots$ we have

$$\mathbf{E}(\tau_i \mid \mathcal{F}_{i-1}) = \mathbf{E}(\tau_i \mid Y_\nu(t_{i-1})) = 1 + O\left(\frac{1}{Y_\nu(t_{i-1})}\right).$$

Hence

$$|t_n - n| \leq |M_n| + |\tau_1 - 1| + C_1 \sum_{i=2}^n \frac{1}{Y_\nu(t_{i-1})} = |M_n| + |\tau_1 - 1| + C_1 \sum_{i=2}^n \frac{1}{X_{i-1}}$$

with some (random) constant $C_1$. By (2.3) of Lemma 2.1 we have $\mathbf{E}M_n^2 \leq Cn$. Let $\varepsilon > 0$ be arbitrary and define $n_k = [k^{1/\varepsilon}]$. From the martingale inequality we get

$$\mathbf{P}\left(\max_{n_{k-1}\leq n\leq n_k} |M_n| \geq C_1 n_{k-1}^{1/2+\varepsilon}\right) \leq \frac{C_2}{n_k^{2\varepsilon}},$$

hence we obtain by Borel-Cantelli lemma

$$\max_{n_{k-1}\leq n\leq n_k} |M_n| \leq C_1 n_{k-1}^{1/2+\varepsilon}$$

almost surely for large $k$. Hence we also have

$$|M_n| = O(n^{1/2+\varepsilon}) \qquad \text{a.s.}$$

By Lemma 2.2

$$\sum_{i=2}^{n} \frac{1}{X_{i-1}} = O(n^{1/2+\varepsilon}) \qquad \text{a.s.},$$

consequently

$$|t_n - n| = O(n^{1/2+\varepsilon}) \qquad \text{a.s.} \tag{3.1}$$

It is well-known (cf. [2], p. 69) that $Y_\nu(t)$ satisfies the stochastic differential equation

$$dY_\nu(t) = dW(t) + \frac{2\nu+1}{2Y_\nu(t)}dt, \tag{3.2}$$

where $W(t)$ is a standard Wiener process. Hence

$$X_n - Y_\nu(n) = Y_\nu(t_n) - Y_\nu(n) = W(t_n) - W(n) + \int_{t_n}^{n} \frac{2\nu+1}{2Y_\nu(s)}\,ds,$$

consequently,

$$|X_n - Y_\nu(n)| \leq |W(t_n) - W(n)| + \frac{(2\nu+1)|t_n - n|}{2} \max_{\min(n,t_n)\leq t\leq \max(n,t_n)} \frac{1}{Y_\nu(t)}.$$

Now by (3.1) and (2.5) the last term is $O(n^{2\varepsilon})$ almost surely and since for the increments of the Wiener process (cf. [8], page 30)

$$|W(t_n) - W(n)| = O(n^{1/4+\varepsilon}) \qquad \text{a.s.}$$

as $n \to \infty$, we have (1.13) of Theorem 1.1. □

# 4. Proof of Theorem 1.2

For $R > 0$ integer define

$$\begin{aligned}
\kappa_1 &:= \min\{t \geq 0 : Y_\nu(t) = R\}, \\
\delta_1 &:= \min\{t \geq \kappa_1 : Y_\nu(t) \notin (R-1, R+1)\}, \\
\kappa_i &:= \min\{t \geq \delta_{i-1} : Y_\nu(t) = R\}, \\
\delta_i &:= \min\{t \geq \kappa_i : Y_\nu(t) \notin (R-1, R+1)\}, \\
\kappa^* &:= \max\{t \geq 0 \ : Y_\nu(t) = R\},
\end{aligned}$$

$i = 2, 3, \ldots$

Consider the local times at $R$ of the Bessel process during excursions around $R$, i.e. let

$$\zeta_i := \eta(R, \delta_i) - \eta(R, \kappa_i), \quad i = 1, 2, \ldots,$$

$$\tilde{\zeta} := \eta(R, \infty) - \eta(R, \kappa^*).$$

We have

$$\eta(R, \infty) = \sum_{i=1}^{\xi(R,\infty)-1} \zeta_i + \tilde{\zeta}.$$

**Lemma 4.1.**

$$\mathbf{E}\left(e^{\lambda \eta(R,\infty)}\right) = \frac{p_R^* \varphi(\lambda)}{1 - q_R^* \varphi(\lambda)}, \tag{4.1}$$

*where*

$$p_R^* = \frac{A_R}{A_R + B_R} \frac{(R+1)^{2\nu} - R^{2\nu}}{(R+1)^{2\nu}}, \quad q_R^* = 1 - p_R^*, \tag{4.2}$$

$$\varphi(\lambda) = \frac{\nu(A_R + B_R)}{\nu(A_R + B_R) - \lambda R^{2\nu+1} A_R B_R}, \tag{4.3}$$

*and*

$$A_R = (R-1)^{-2\nu} - R^{-2\nu}, \qquad B_R = R^{-2\nu} - (R+1)^{-2\nu}. \tag{4.4}$$

**Proof:** By ([2], p. 395, 3.3.2) $\zeta_i$ are i.i.d. random variables having exponential distribution with moment generating function $\varphi(\lambda)$ given in (4.3). Moreover, it is obvious that $\tilde{\zeta}$ is independent from $\sum_{i=1}^{\xi(R,\infty)-1} \zeta_i$. Furthermore, $\tilde{\zeta}$ is the local time of $R$ under the condition that starting from $R$, $Y_\nu(t)$ will reach $R+1$ before $R-1$. Hence its distribution can be calculated from formula 3.3.5(b) of [2], and its moment generating function happens to be equal to $\varphi(\lambda)$ of (4.3). □

We can see

$$\theta := \mathbf{E}(\zeta_i) = \mathbf{E}(\tilde{\zeta}) = \frac{\nu(A_R + B_R)}{R^{2\nu+1} A_R B_R} = 1 + O\left(\frac{1}{R}\right), \quad R \to \infty.$$

$$\begin{aligned}
\mathbf{P}(|\eta(R,\infty) - \xi(R,\infty)| \geq u) &= \mathbf{P}\left(\left|\sum_{i=1}^{\xi(R,\infty)-1} (\zeta_i - \theta) + \tilde{\zeta} - \theta\right| \geq u\right) \\
&\leq \mathbf{P}(\xi(R.\infty) > N) + \mathbf{P}\left(\max_{k \leq N}\left|\sum_{i=1}^{k}(\zeta_i - \theta)\right| \geq u\right) \\
&\leq (q_R^*)^N + e^{-\lambda u}\left(\left(\frac{e^{\lambda\theta}}{1+\lambda\theta}\right)^N + \left(\frac{e^{-\lambda\theta}}{1-\lambda\theta}\right)^N\right).
\end{aligned}$$

In the above calculation we used the common moment generating function (4.3) of $\zeta_i$ and $\tilde{\zeta}$, the exact distribution of $\xi(R,\infty)$ (see (1.6)) and the exponential Kolmogorov inequality. Estimating the above expression with standard methods and selecting

$$N = CR\log R, \quad u = CR^{1/2}\log R, \quad \lambda = \frac{u}{\theta^2 N}$$

we conclude that

$$\mathbf{P}(|\eta(R,\infty) - \xi(R,\infty)| \geq CR^{1/2}\log R) \leq C_1 \exp\left(-\frac{C\log R}{2\theta}\right).$$

With a big enough $C$ the right hand side of the above inequality is summable in $R$, hence Theorem 1.2 follows by the Borel-Cantelli lemma. □

# 5. Proof of Theorem 1.3

Let $p_j^{(1)}$ and $p_j^{(2)}$ as in Theorem 1.3. Define the two-dimensional Markov chain $(X_n^{(1)}, X_n^{(2)})$ as follows. If $p_j^{(1)} \geq p_k^{(2)}$, then let

$$\begin{aligned}
\mathbf{P}\left((X_{n+1}^{(1)}, X_{n+1}^{(2)}) = (j+1, k+1) \mid (X_n^{(1)}, X_n^{(2)}) = (j,k)\right) &= \frac{1}{2} + p_k^{(2)} \\
\mathbf{P}\left((X_{n+1}^{(1)}, X_{n+1}^{(2)}) = (j+1, k-1) \mid (X_n^{(1)}, X_n^{(2)}) = (j,k)\right) &= p_j^{(1)} - p_k^{(2)} \\
\mathbf{P}\left((X_{n+1}^{(1)}, X_{n+1}^{(2)}) = (j-1, k-1) \mid (X_n^{(1)}, X_n^{(2)}) = (j,k)\right) &= \frac{1}{2} - p_j^{(1)}.
\end{aligned}$$

If, however $p_j^{(1)} \le p_k^{(2)}$, then let

$$\mathbf{P}\left((X_{n+1}^{(1)}, X_{n+1}^{(2)}) = (j+1, k+1) \mid (X_n^{(1)}, X_n^{(2)}) = (j,k)\right) = \frac{1}{2} + p_j^{(1)}$$
$$\mathbf{P}\left((X_{n+1}^{(1)}, X_{n+1}^{(2)}) = (j-1, k+1) \mid (X_n^{(1)}, X_n^{(2)}) = (j,k)\right) = p_k^{(2)} - p_j^{(1)}$$
$$\mathbf{P}\left((X_{n+1}^{(1)}, X_{n+1}^{(2)}) = (j-1, k-1) \mid (X_n^{(1)}, X_n^{(2)}) = (j,k)\right) = \frac{1}{2} - p_k^{(2)}.$$

Then it can be easily seen that $X_n^{(1)}$ and $X_n^{(2)}$ are two NN random walks as desired. Consider the following 4 cases.

- (i) $p_j^{(1)} \le p_k^{(2)}$, $j \le k$,
- (ii) $p_j^{(1)} \le p_k^{(2)}$, $j \ge k$,
- (iii) $p_j^{(1)} \ge p_k^{(2)}$, $j \le k$,
- (iv) $p_j^{(1)} \ge p_k^{(2)}$, $j \ge k$.

In case (i) from (1.15) and (1.16) we obtain

$$\frac{B}{4j} - \frac{C}{j^\gamma} \le \frac{B}{4k} + \frac{C}{k^\gamma} \le \frac{B}{4k} + \frac{C}{kj^{\gamma-1}},$$

implying

$$k - j \le \frac{2Cj^{2-\gamma}}{B/4 - Cj^{1-\gamma}} = O(j^{2-\gamma})$$

if $j \to \infty$. So in this case if $X_n^{(1)} = j$ and $X_n^{(2)} = k$, then we have

$$0 \le X_n^{(2)} - X_n^{(1)} = O((X_n^{(1)})^{2-\gamma})$$

if $n \to \infty$.

In case (ii) either $X_{n+1}^{(1)} - X_{n+1}^{(2)} = X_n^{(1)} - X_n^{(2)}$, or $X_{n+1}^{(1)} - X_{n+1}^{(2)} = X_n^{(1)} - X_n^{(2)} - 2$, so that we have

$$-2 \le X_{n+1}^{(1)} - X_{n+1}^{(2)} \le X_n^{(1)} - X_n^{(2)}.$$

Similar procedure shows that in case (iii)

$$-2 \le X_{n+1}^{(2)} - X_{n+1}^{(1)} \le X_n^{(2)} - X_n^{(1)}$$

and in case (iv)

$$0 \le X_n^{(1)} - X_n^{(2)} = O((X_n^{(2)})^{2-\gamma}).$$

Hence Theorem 1.3 follows from the law of the iterated logarithm for $X_n^{(i)}$ (cf. [3]). □

## 6. Strong theorems

As usual, applying Theorem 1.1 and Theorem 1.3, we can give limit results valid for one of the processes to the other process involved.

In this section we denote $Y_\nu(t)$ by $Y(t)$ and define the following related processes.

$$M(t) := \max_{0\le s\le t} Y(s), \qquad Q_n := \max_{1\le k\le n} X_k.$$

The future infimums are defined as

$$I(t) := \inf_{s\ge t} Y(s), \qquad J_n := \inf_{k\ge n} X_k.$$

Escape processes are defined by

$$A(t) := \sup\{s : \ Y(s) \le t\}, \qquad G_n := \sup\{k : \ X_k \le n\}.$$

Laws of the iterated logarithm are known for Bessel processes (cf. [2]) and NN random walks (cf. [3]) as well. Upper class results for Bessel process read as follows (cf. Orey and Pruitt [16] for integral $d$, and Pardo [17] for the case of positive self-similar Markov processes).

**Theorem F**: *Let $a(t)$ be a non-decreasing non-negative continuous function. Then for $\nu \ge 0$*

$$t^{1/2}a(t) \in \mathrm{UUC}(Y(t)) \qquad \textit{if and only if} \qquad \int_1^\infty \frac{(a(x))^{2\nu+2}}{x} e^{-a^2(x)/2}\, dx < \infty.$$

Now Theorems 1.1, 1.3 and Theorems E and F together imply the following result.

**Theorem 6.1.** *Let $\{X_n\}$ be an NN random walk with $p_R$ satisfying*

$$p_R = \frac{B}{4R} + O\left(\frac{1}{R^{1+\delta}}\right), \quad R \to \infty$$

*with $B > 1$ and for some $\delta > 0$. Let furthermore $a(t)$ be a non-decreasing non-negative function. Then*

$$n^{1/2}a(n) \in \mathrm{UUC}(X_n) \qquad \textit{if and only if} \qquad \sum_{k=1}^\infty \frac{(a(k))^{B+1}}{k} e^{-a^2(k)/2} < \infty.$$

*If $b(t)$ is a non-increasing non-negative function, then*

$$n^{1/2}b(n) \in \mathrm{LLC}(X_n) \qquad \textit{if and only if} \qquad \sum_{k=1}^\infty (b(2^k))^{B-1} < \infty.$$

Next we prove the following invariance principles for the processes defined above.

**Theorem 6.2.** *Let $Y(t)$ and $X_n$ as in* Theorem 1.1. *Then for any $\varepsilon > 0$ we have*

$$|M(n) - Q_n| = O(n^{1/4+\varepsilon}) \quad \text{a.s.} \tag{6.1}$$

*and*

$$|I(n) - J_n| = O(n^{1/4+\varepsilon}) \quad \text{a.s.} \tag{6.2}$$

**Proof:** Define $\tilde{s}, s^*, \tilde{k}, k^*$ by

$$Y(\tilde{s}) = M(n), \quad Y(s^*) = I(n), \quad X_{\tilde{k}} = Q_n, \quad X_{k^*} = J_n.$$

Then as $n \to \infty$, we have almost surely

$$Q_n - M(n) = X_{\tilde{k}} - Y(\tilde{s}) \leq X_{\tilde{k}} - Y(\tilde{k}) = O(n^{1/4+\varepsilon})$$

and

$$\begin{aligned} M(n) - Q_n &= Y(\tilde{s}) - X_{\tilde{k}} = Y(\tilde{s}) - Y([\tilde{s}]) - (X_{[\tilde{s}]} - Y([\tilde{s}])) + X_{[\tilde{s}]} - X_{\tilde{k}} \\ &\leq Y(\tilde{s}) - Y([\tilde{s}]) - (X_{[\tilde{s}]} - Y([\tilde{s}]) = Y(\tilde{s}) - Y([\tilde{s}]) + O(n^{1/4+\varepsilon}) \end{aligned}$$

By (3.2) and recalling the results on the increments of the Wiener process (see [8] page 30) we get

$$Y(\tilde{s}) - Y([\tilde{s}]) = W(\tilde{s}) - W([\tilde{s}]) + \int_{[\tilde{s}]}^{\tilde{s}} \frac{2\nu + 1}{2Y(s)}\, ds$$

$$\leq \sup_{0 \leq t \leq n} \sup_{0 \leq s \leq 1} |W(t+s) - W(t)| + \frac{2\nu+1}{2} \max_{[\tilde{s}] \leq t \leq \tilde{s}} \frac{1}{Y(t)} = O(\log n) \quad \text{a.s.,}$$

since $Y(t)$ in the interval $([\tilde{s}], \tilde{s})$ is bounded away from zero. Hence (6.1) follows.

To show (6.2), note that $n \leq s^* \leq n^{1+\alpha}$ and $n \leq k^* \leq n^{1+\alpha}$ for any $\alpha > 0$ almost surely for all large $n$. Then as $n \to \infty$

$$I(n) - J_n \leq Y(k^*) - X_{k^*} = O((k^*)^{1/4+\varepsilon}) = O(n^{(1+\alpha)(1/4+\varepsilon)}) \quad \text{a.s.}$$

On the other hand,

$$J_n - I(n) \leq X_{k^*} - Y([s^*]) + Y([s^*]) - Y(s^*) = O(n^{(1+\alpha)(1/4+\varepsilon)}) + Y([s^*]) - Y(s^*).$$

By (3.2), taking into account that when applying this formula the integral contribution is negative, and recalling again the results on the increments of the Wiener process, we get

$$Y([s^*]) - Y(s^*) \leq W([s^*]) - W(s^*) \leq \sup_{0 \leq t \leq n^{1+\alpha}} \sup_{0 \leq s \leq 1} |W(t+s) - W(t)| = O(\log n) \quad \text{a.s.}$$

as $n \to \infty$. Hence

$$|I(n) - J_n| = O(n^{(1+\alpha)(1/4+\varepsilon)}) \quad \text{a.s.}$$

Since $\alpha > 0$ and $\varepsilon > 0$ are arbitrary, (6.2) follows. This completes the proof of Theorem 6.2. $\square$

**Theorem 6.3.** *Let $X_n^{(1)}$ and $X_n^{(2)}$ as in* Theorem 1.3 *and let $Q_n^{(1)}$ and $Q_n^{(2)}$ be the corresponding maximums, while let $J_n^{(1)}$ and $J_n^{(2)}$ be the corresponding future infimum processes. Then for any $\varepsilon > 0$, as $n \to \infty$ we have*

$$|Q_n^{(1)} - Q_n^{(2)}| = O(n^{1-\gamma/2+\varepsilon}) \quad \text{a.s.} \tag{6.3}$$

*and*

$$|J_n^{(1)} - J_n^{(2)}| = O(n^{1-\gamma/2+\varepsilon}) \quad \text{a.s.} \tag{6.4}$$

**Proof:** Define $\tilde{k}_i, k_i^*$, $i = 1, 2$ by

$$X_{\tilde{k}_i}^{(i)} = Q_n^{(i)}, \quad X_{k_i^*}^{(i)} = J_n^{(i)}.$$

Then

$$|Q_n^{(1)} - Q_n^{(2)}| \le \max(X_{\tilde{k}_1}^{(1)} - X_{\tilde{k}_1}^{(2)}, X_{\tilde{k}_2}^{(1)} - X_{\tilde{k}_2}^{(2)}) = O((n \log\log n)^{1-\gamma/2}) \quad \text{a.s.},$$

proving (6.3).

Moreover, for any $\alpha > 0$, $n \le k_i^* \le n^{1+\alpha}$ almost surely for large $n$, hence we have

$$|J_n^{(1)} - J_n^{(2)}| \le \max(X_{k_1^*}^{(1)} - X_{k_1^*}^{(2)}, X_{k_2^*}^{(1)} - X_{k_2^*}^{(2)}) = O((n \log\log n)^{(1+\alpha)(1-\gamma/2)}) \quad \text{a.s.}$$

Since $\alpha$ is arbitrary, (6.4) follows.

This completes the proof of Theorem 6.3. $\square$

Khoshnevisan et al. [13] (for $I(t)$ and $A(t)$), Adelman and Shi [1], and Shi [20] (for $Y(t) - I(t)$) proved the following upper and lower class results.

**Theorem G**: *Let $\varphi(t)$ be a non-increasing, and $\psi(t)$ be a non-decreasing function, both non-negative. Then for $\nu > 0$*

- $t^{1/2}\psi(t) \in \mathrm{UUC}(I(t))$ *if and only if* $\displaystyle\int_1^\infty \frac{(\psi(x))^{2\nu}}{x} e^{-\psi^2(x)/2}\, dx < \infty,$
- $t^2\varphi(t) \in \mathrm{LLC}(A(t))$ *if and only if* $\displaystyle\int_1^\infty \frac{1}{x\varphi^\nu(x)} e^{-1/2\varphi(x)}\, dx < \infty.$

- $t^{1/2}\psi(t) \in \mathrm{UUC}(Y(t) - I(t))$ *if and only if* $\displaystyle\int_1^\infty \frac{1}{x\psi^{2\nu-2}(x)} e^{-\psi^2(x)/2}\, dx < \infty,$

**Theorem H**: *Let $\rho(t) > 0$ be such that $(\log \rho(t))/\log t$ is non-decreasing.* Then

- $1/\rho(t) \in \mathrm{LLC}(M(t) - I(t))$ *if and only if* $\displaystyle\int_1^\infty \frac{dx}{x \log \rho(x)} < \infty.$

Taking into account that $J_n$ and $G_n$ are inverses of each other, immediate consequences of Theorems F, G, H, Theorems 6.2 and 6.3 are the following upper and lower class results.

**Theorem 6.4.** *Let $X_n$ be as in* Theorem 6.1 *and let $\varphi(t)$ be a non-increasing and $\psi(t)$ be a non-decresing function, both non-negative. Then*

- $n^{1/2}\psi(n) \in \mathrm{UUC}(J_n)$ *if and only if* $\displaystyle\sum_{k=1}^\infty \frac{(\psi(k))^{B-1}}{k} e^{-\psi^2(k)/2} < \infty,$

- $n^2\varphi(n) \in \mathrm{LLC}(G_n)$ *if and only if* $\displaystyle\sum_{k=1}^\infty \frac{1}{k\varphi^{(B-1)/2}(k)} e^{-1/2\varphi(k)} < \infty.$

- $n^{1/2}\psi(n) \in \mathrm{UUC}(X_n - J_n)$ *if and only if* $\displaystyle\sum_{k=1}^\infty \frac{1}{k\psi^{B-3}(k)} e^{-\psi^2(k)/2} < \infty,$

**Theorem 6.5.** *Let $\rho(t) > 0$ be such that $(\log \rho(t))/\log t$ is non-decreasing.*

- $1/\rho(n) \in \mathrm{LLC}(Q_n - J_n)$ *if and only if* $\displaystyle\sum_{k=2}^\infty \frac{1}{k \log \rho(k)} < \infty.$

# 7. Local time

We will need the following result from Yor [22], page 52.
**Theorem J:** *For the local time of a Bessel process of order $\nu$ we have*

$$\eta(R, \infty) \stackrel{\mathcal{D}}{=} (2\nu)^{-1} R^{1-2\nu} Y_0^2(R^{2\nu}),$$

*where $Y_0$ is a two-dimensional Bessel process and $\stackrel{\mathcal{D}}{=}$ means equality in distribution.*

Hence applying Theorem F for $\nu = 0$, we get

**Theorem K**: *If $f(x)$ is non-decreasing, non-negative function, then*

- $Rf(R) \in \mathrm{UUC}(\eta(R,\infty))$ *if and only if* $\displaystyle\int_1^\infty \frac{f(x)}{x} e^{-\nu f(x)}\, dx < \infty.$

From this and Theorem 1.2 we get the following result.

**Theorem 7.1.** *If $f(x)$ is non-decreasing, non-negative function, then*

- $Rf(R) \in \mathrm{UUC}(\xi(R,\infty))$ *if and only if* $\displaystyle\sum_{k=1}^{\infty} \frac{f(k)}{k} e^{-\nu f(k)} < \infty.$

In [7] we proved the following result.

**Theorem L:** *Let* $p_R = \dfrac{B}{4R} + O\left(\dfrac{1}{R^\gamma}\right)$ *with* $B > 1$*, and* $\gamma > 1$*. Then with probability* 1 *there exist infinitely many* $R$ *for which*

$$\xi(R+j,\infty) = 1$$

*for each* $j = 0, 1, 2, \ldots, [\log\log R/\log 2]$*. Moreover, with probability* 1 *for each* $R$ *large enough and* $\varepsilon > 0$ *there exists an*

$$R \le S \le \frac{(1+\varepsilon)\log\log R}{\log 2}$$

*such that*

$$\xi(S,\infty) > 1.$$

**Remark 1:** In fact in [7] we proved this result in the case when $p_R = B/4R$ but the same proof works also in the case of Theorem L.

This theorem applies e.g. for the case when $p_R$ is given by (1.12), which in turn, gives the following result for the Bessel process.

Let

(i) $\kappa(R) := \inf\{t : \; Y_\nu(t) = R\}$,

(ii) $\kappa^*(R) := \sup\{t : \; Y_\nu(t) = R\}$,

(iii) $\Psi(R)$ be the largest integer for which the event

$$A(R) = \bigcap_{j=-1}^{\Psi(R)} \{\kappa^*(R+j) < \kappa(R+j+1)\}$$

occurs.

$A(R)$ means that $Y_\nu(t)$ moves from $R$ to $R+1$ before returning to $R-1$, it goes from $R+1$ to $R+2$ before returning to $R$, $\ldots$ and also from $R+\Psi(R)$ to $R+\Psi(R)+1$ and it never returns to $R+\Psi(R)-1$. We say that the process $Y_\nu(t)$ escapes through $(R, R+\Psi(R))$ with large velocity.

**Theorem 7.2.**
$$\limsup_{R\to\infty} \frac{\Psi(R)}{\log\log R} = \frac{1}{\log 2} \quad \text{a.s.}$$

**Remark 2:** The statement of Theorem 7.2 (for integral $d = 2\nu + 2$) was formulated in [18], p. 291 as a Conjecture.

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