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On the supremum of iterated local time

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Dedicated to the memory of Professor B. Gyires

Abstract. We obtain upper and lower class integral tests for the space-wise supremum of the iterated local time of two independent Wiener processes. We then establish a strong invariance principle between this iterated local time and the local time process of the simple symmetric random walk on the two-dimensional comb lattice. The latter, in turn, enables us to conclude upper and lower class tests for the local time of simple symmetric random walk on the two-dimensional comb lattice as well.

1. Introduction and main results

Let $\{W(t);\, t \geq 0\}$ be a standard Wiener process (Brownian motion), i.e., a Gaussian process with

$$E(W(t)) = 0, \quad E(W(t_1)W(t_2)) = \min(t_1, t_2), \quad t, t_1, t_2 \geq 0.$$

The local time process $\{\eta(x,t);\, x \in \mathbb{R},\, t \geq 0\}$ is defined via

$$\int_A \eta(x,t)\, dx = \lambda\{s:\, 0 \leq s \leq t,\, W(s) \in A\} \tag{1.1}$$

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for any $t \geq 0$ and Borel set $A \subset \mathbb{R}$, where $\lambda(\cdot)$ is the Lebesgue measure, and $\eta(\cdot, \cdot)$ is frequently referred to as Wiener or Brownian local time.

Let $\eta_1(x,t)$ and $\eta_2(x,t)$ be two independent Brownian local times. The iterated local time is defined by

$$\Upsilon(x,t) := \eta_1(x, \eta_2(0,t)).$$

Denote

$$\Upsilon^*(t) := \sup_{x \in \mathbb{R}} \Upsilon(x,t). \tag{1.2}$$

First we give asymptotic values for the upper and lower tails of the distribution of $\Upsilon^*(t)$.

Theorem 1.1. *As $z \to \infty$*

$$P(\Upsilon^*(t) > zt^{1/4}) \sim \frac{2^{11/3} z^{2/3}}{(3\pi)^{1/2}} \exp\left(-\frac{3z^{4/3}}{2^{5/3}}\right) \tag{1.3}$$

and as $z \to 0$,

$$P(\Upsilon^*(t) < zt^{1/4}) \sim \frac{4z^2}{(2\pi)^{1/2}} \int_0^\infty \frac{G(s)}{s^3}\, ds, \tag{1.4}$$

for all $t > 0$, where

$$G(s) := P\left(\sup_{x \in \mathbb{R}} \eta(x,1) < s\right).$$

Note that an explicit formula for $G(s)$ in terms of Bessel functions is given in Csáki and Földes [9].

The following integral tests are obtained.

Theorem 1.2. *Let $f(t) > 0$ be a non-decreasing function and put*

$$I(f) := \int_1^\infty \frac{f^2(t)}{t} \exp\left(-\frac{3}{2^{5/3}} f^{4/3}(t)\right) dt.$$

Then

$$P(\Upsilon^*(t) > t^{1/4} f(t) \text{ i.o. as } t \to \infty) = 0 \text{ or } 1$$

according as $I(f)$ converges or diverges.

Theorem 1.3. *Let $g(t) > 0$ be a non-increasing function and put*

$$J(g) := \int_1^\infty \frac{g^2(t)}{t}\, dt.$$

Then

$$P(\Upsilon^*(t) < t^{1/4} g(t) \text{ i.o. as } t \to \infty) = 0 \text{ or } 1$$

according as $J(g)$ converges or diverges.

In particular, we have the following law of the iterated logarithm:

$$\limsup_{t\to\infty} \frac{\Upsilon^*(t)}{t^{1/4}(\log\log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.}$$

To compare the above results with similar integral tests for $\Upsilon(0,t)$, note that $\{\eta(0,t);\ t\geq 0\}$ has the same distribution as $\{\sup_{0\leq s\leq t} W(s);\ t\geq 0\}$. Consequently, $\{\Upsilon(0,t);\ t\geq 0\}$ has the same distribution as $\{\sup_{0\leq s\leq t} W_1(\eta_2(0,s));\ t\geq 0\}$, or, as easily seen, the same distribution as $\{\sup_{0\leq s\leq t} W_1(W_2(s)\vee 0);\ t\geq 0\}$. From Bertoin [2] we obtain the following integral tests.

Theorem A. *Put*

$$\hat{I}(f) := \int_1^\infty \frac{f^{2/3}(t)}{t} \exp\left(-\frac{3}{2^{5/3}} f^{4/3}(t)\right) dt,$$

$$\hat{J}(g) := \int_1^\infty \frac{g(t)}{t}\, dt.$$

Then

$$P(\Upsilon(0,t) > t^{1/4} f(t) \text{ i.o. as } t\to\infty) = 0 \text{ or } 1$$

according as $\hat{I}(f)$ converges or diverges. Moreover,

$$P(\Upsilon(0,t) < t^{1/4} g(t) \text{ i.o. as } t\to\infty) = 0 \text{ or } 1$$

according as $\hat{J}(g)$ converges or diverges.

In particular, we have the same law of the iterated logarithm as for $\Upsilon^*(t)$:

$$\limsup_{t\to\infty} \frac{\Upsilon(0,t)}{t^{1/4}(\log\log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.}$$

In the subsequent sections the proofs of Theorem 1.1, 1.2 and 1.3 will be given. In Section 5 we apply the results for the local time of the simple random walk on the 2-dimensional comb.

In the proofs unimportant constants of possibly different positive values will be denoted by c, c_0, c_1, c_2.

2. Proof of Theorem 1.1

Since

$$\frac{\Upsilon^*(t)}{t^{1/4}} = \frac{\eta_1^*(\eta_2(0,t))}{(\eta_2(0,t))^{1/2}} \sqrt{\frac{\eta_2(0,t)}{t^{1/2}}},$$

it has the same distribution as $\eta_1^*(1)\sqrt{|N|}$, where $\eta_1^*(s) = \sup_{x\in\mathbb{R}} \eta_1(x,s)$ and N is a standard normal random variable independent of $\eta_1^*(1)$. Hence, denoting by φ the standard normal density,

$$P(\Upsilon^*(t) > zt^{1/4}) = 2\int_0^\infty \left(1 - G\left(\frac{z}{\sqrt{u}}\right)\right)\varphi(u)\,du. \tag{2.1}$$

For the upper tail of G we have (see Csáki [5])

$$1 - G(z) \sim 4\sqrt{\frac{2}{\pi}}z\exp\left(-\frac{z^2}{2}\right), \quad z \to \infty. \tag{2.2}$$

Now split the integral in (2.1) into three parts:

$$\int_0^\infty = \int_0^{z^{2/3}/2} + \int_{z^{2/3}/2}^{2z^{2/3}} + \int_{2z^{2/3}}^\infty = I_1 + I_2 + I_3.$$

Using (2.2), it is easy to see that

$$I_1 \le c(1 - G(2^{1/2}z^{2/3})) \le cz^{2/3}\exp(-z^{4/3}),$$

$$I_3 \le c\int_{2z^{2/3}}^\infty \varphi(u)\,du \le c\exp(-2z^{4/3}),$$

so I_1 and I_3 are negligible compared to (1.3). For I_2 we can use (2.2) and hence

$$I_2 \sim \frac{8}{\pi}\int_{z^{2/3}/2}^{2z^{2/3}} \frac{z}{\sqrt{u}}\exp\left(-\frac{z^2}{2u} - \frac{u^2}{2}\right)du$$

$$= \frac{16z^{4/3}}{\pi}\int_{1/\sqrt{2}}^{\sqrt{2}} \exp\left(-\frac{z^{4/3}}{2}\left(\frac{1}{v^2} + v^4\right)\right)dv.$$

The asymptotic value of this integral can be obtained by Laplace's method (cf., e.g., de Bruijn [3])

$$\int_a^b \exp(-\lambda h(v))\,dv \sim \frac{\sqrt{2\pi}e^{-\lambda h(v_0)}}{\sqrt{\lambda h''(v_0)}}, \quad \lambda \to \infty,$$

where v_0 is the place of the minimum of h in (a,b), i.e., $h'(v_0) = 0$. Applying this, a straightforward calculation leads to (1.3).

To see (1.4), we have similarly

$$P(\Upsilon^*(t) < zt^{1/4}) = 2\int_0^\infty G\left(\frac{z}{\sqrt{u}}\right)\varphi(u)\,du = 4z^2\int_0^\infty \frac{G(s)}{s^3}\varphi\left(\frac{z^2}{s^2}\right)ds.$$

This integral is finite, since

$$G(s) \sim c \exp\left(-\frac{2j_1^2}{s^2}\right), \quad s \to 0,$$

where j_1 is the smallest positive zero of the Bessel function $J_0(\cdot)$ (cf. Csáki and Földes [9]).

Since $\varphi(z^2/s^2) \le \varphi(0)$, we have

$$P(\Upsilon^*(t) < xt^{1/4}) \sim 4z^2\varphi(0) \int_0^\infty \frac{G(s)}{s^3}\, ds, \quad z \to 0$$

by the dominated convergence theorem. This completes the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

From Shi [13] we have the following result.

Lemma A. *Let f be a function as in* Theorem 1.2. *Put $T_1 = 1$,*

$$T_{k+1} = T_k\left(1 + \frac{1}{f_k^{4/3}}\right), \quad k = 1, 2, \ldots,$$

where $f_k = f(T_k)$. Then $I(f) < \infty$ if and only if

$$\sum_{k=1}^{\infty} f_k^{2/3} \exp\left(-\frac{3}{2^{5/3}} f_k^{4/3}\right) < \infty.$$

First we prove the convergence part of Theorem 1.2. Assume that $I(f) < \infty$ and define the events

$$A_k = \{\Upsilon^*(T_{k+1}) > T_k^{1/4} f_k\}.$$

It follows from Theorem 1.1 that

$$P(A_k) \le c f_k^{2/3} \exp\left(-\frac{3}{2^{5/3}}\left(1 + \frac{1}{f_k^{4/3}}\right)^{-1/3} f_k^{4/3}\right).$$

Using the inequality

$$(1+u)^{-1/3} \ge 1 - \frac{u}{3}, \quad 0 \le u \le 1,$$

with $u = f_k^{-4/3}$, we obtain further

$$P(A_k) \le c f_k^{2/3} \exp\left(-\frac{3}{2^{5/3}} f_k^{4/3} \right),$$

which is summable by Lemma A. Hence $P(A_k \text{ i.o.}) = 0$, i.e., for large k we have almost surely

$$\Upsilon^*(T_{k+1}) \le T_k^{1/4} f(T_k).$$

But for $T_k \le t \le T_{k+1}$, i.e., for large t

$$\Upsilon^*(t) \le \Upsilon(T_{k+1}) \le T_k^{1/4} f(T_k) \le t^{1/4} f(t),$$

proving the convergence part.

For the divergence part, we follow the proof in [5]. Without loss of generality we may assume

$$(\log\log t)^{3/4} \le f(t) \le (2\log\log t)^{3/4}$$

and, as easily seen,

$$(\log k/2)^{3/4} \le f_k \le (2\log k)^{3/4}.$$

In the proof we also use the inequality

$$\frac{T_k}{T_\ell} \le \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k)}, \quad k < \ell.$$

Now assume that $I(f) = \infty$, and define the events

$$B_k = \{T_k^{1/4} f_k \le \Upsilon^*(T_k) < T_{k+1}^{1/4} f_k\},$$

where $f_k = f(T_k)$. It follows from Theorem 1.1 that

$$P(B_k) \\ \ge c f_k^{2/3} \exp\left(-\frac{3 f_k^{4/3}}{2^{5/3}}\right)\left[1 - \left(\frac{T_{k+1}}{T_k}\right)^{1/6} \exp\left(-\frac{3 f_k^{4/3}}{2^{5/3}}\left(\left(\frac{T_{k+1}}{T_k}\right)^{1/3} - 1\right)\right)\right].$$

It is readily seen that $\lim_{k\to\infty} T_{k+1}/T_k = 1$, and

$$\lim_{k\to\infty} f_k^{4/3}\left(\left(\frac{T_{k+1}}{T_k}\right)^{1/3} - 1\right) = \frac{1}{3},$$

so there is a positive constant c such that

$$P(B_k) \geq c f_k^{2/3} \exp\left(-\frac{3f_k^{4/3}}{2^{5/3}}\right),$$

and hence by Lemma A we have $\sum_k P(B_k) = \infty$.

Next we estimate $P(B_k B_\ell)$. Let $k < \ell$ and

$$\Upsilon^*(T_k, T_\ell) = \sup_{x \in \mathbb{R}} \left(\eta_1(x, \eta_2(0, T_\ell)) - \eta_1(x, \eta_2(0, T_k))\right).$$

Then, similarly to the proof in [5],

$$\Upsilon^*(T_k, T_\ell) \leq \Upsilon^*(T_\ell) \leq \Upsilon^*(T_k) + \Upsilon^*(T_k, T_\ell)$$

and

$$\begin{aligned} P(B_k B_\ell) &\leq P(T_k^{1/4} f_k \leq \Upsilon^*(T_k) < T_{k+1}^{1/4} f_k, \Upsilon^*(T_\ell) - \Upsilon^*(T_k) \geq T_\ell^{1/4} f_\ell - T_{k+1}^{1/4} f_k) \\ &\leq P(B_k) P(T_\ell^{1/4} f_\ell - T_{k+1}^{1/4} f_k \leq \Upsilon^*(T_k, T_\ell) \leq T_{\ell+1}^{1/4} f_\ell). \end{aligned}$$

But $\Upsilon^*(T_k, T_\ell)$ has the same distribution as $\Upsilon^*(T_\ell - T_k)$, or $(T_\ell - T_k)^{1/4} \Upsilon^*(1)$, hence

$$\begin{aligned} P(B_k B_\ell) &\leq P(B_k) P\left(\Upsilon^*(1) \geq \frac{f_\ell T_\ell^{1/4} - f_k T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}}\right) \\ &\leq P(B_k) P\left(\Upsilon^*(1) \geq f_\ell \frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}}\right) \\ &\leq c P(B_k) f_\ell^{2/3} H_{k,\ell}^{2/3} \exp\left(-\frac{3 f_\ell^{4/3} H_{k,\ell}^{4/3}}{2^{5/3}}\right), \end{aligned} \tag{3.1}$$

where

$$H_{k,\ell} = \frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}}.$$

Using the inequality

$$\frac{(1-u)^{3/4}}{4} \leq \frac{1 - u^{1/4}}{(1-u)^{1/4}} \leq 1, \quad 0 < u < 1,$$

we get

$$\frac{1}{4}\left(1 - \frac{T_k}{T_\ell}\right)^{3/4} \frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{T_\ell^{1/4} - T_k^{1/4}} \leq H_{k,\ell} \leq 1.$$

For $k+2 \le \ell$ we have, by straightforward calculation,

$$\frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{T_\ell^{1/4} - T_k^{1/4}} \ge \frac{T_{k+2}^{1/4} - T_{k+1}^{1/4}}{T_{k+2}^{1/4} - T_k^{1/4}} \sim \frac{1}{1 + \left(\frac{f_{k+1}}{f_k}\right)^{4/3}},$$

from which

$$c\left(1 - \frac{T_k}{T_\ell}\right)^{3/4} \le H_{k,\ell} \le 1$$

with certain constant $c > 0$. Consequently,

$$P(B_k B_\ell) \le c P(B_k) f_\ell^{2/3} \exp\left(-c_1 f_\ell^{4/3}\left(1 - \frac{T_k}{T_\ell}\right)\right).$$

Now, for fixed k, let

$$\begin{aligned} L_1 &= \{\ell :\ k+2 \le \ell \le k + f_\ell^{4/3}\}, \\ L_2 &= \left\{\ell :\ k + f_\ell^{4/3} < \ell \le k + 4 f_\ell^{4/3} \log f_\ell^{4/3}\right\}, \\ L_3 &= \left\{\ell :\ k + 4 f_\ell^{4/3} \log f_\ell^{4/3} < \ell\right\}. \end{aligned}$$

If $\ell \in L_1$, then

$$1 - \frac{T_k}{T_\ell} \ge 1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k)} \ge \frac{\ell - k}{2 f_\ell^{4/3}},$$

i.e.,

$$P(B_k B_\ell) \le c P(B_k) f_\ell^{2/3} e^{-c_2(\ell-k)},$$

consequently

$$\sum_{\ell \in L_1} P(B_k B_\ell) \le K P(B_k). \tag{3.2}$$

If $\ell \in L_2$, then

$$1 - \frac{T_k}{T_\ell} \ge 1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k)} \ge c$$

with some $c > 0$. We have

$$P(B_k B_\ell) \le c P(B_k) f_\ell^{2/3} e^{-c_0 f_\ell^{4/3}} \le c P(B_k) (\log \ell)^{1/2} \ell^{-c_0/2}$$

$$\leq cP(B_k)(\log k)^{1/2}k^{-c_0/2}.$$

But

$$\ell - k \leq 4f_\ell^{4/3}\log f_\ell^{4/3} \leq \frac{\ell}{2},$$

i.e., $\ell \leq 2k$, hence

$$\ell - k \leq 4f_{2k}^{4/3}\log f_{2k}^{4/3}.$$

Consequently,

$$\sum_{\ell\in L_2} P(B_kB_\ell) \leq cP(B_k)(\log k)^{1/2}k^{-c_0/2}f_{2k}^{4/3}\log f_{2k}^{4/3} \leq cP(B_k). \tag{3.3}$$

If $\ell \in L_3$, then

$$\frac{T_\ell^{1/4} - T_{k+1}^{1/4}}{(T_\ell - T_k)^{1/4}} \geq 1 - \left(\frac{T_{k+1}}{T_\ell}\right)^{1/4} \geq 1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4}.$$

Hence, using (3.1),

$$P(B_kB_\ell) \leq cP(B_k)f_\ell^{2/3}\exp\left(-\frac{3f_\ell^{4/3}}{2^{5/3}}\left(1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4}\right)^{4/3}\right).$$

It can be seen that

$$\frac{3f_\ell^{4/3}}{2^{5/3}}\left(\left(1 - \left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4}\right)^{4/3} - 1\right)$$

$$\sim -2^{1/3}f_\ell^{4/3}\left(1 + \frac{1}{f_\ell^{4/3}}\right)^{-(\ell-k-1)/4}$$

$$= -2^{1/3}f_\ell^{4/3}\exp\left(-\frac{\ell-k-1}{4}\log\left(1 + \frac{1}{f_\ell^{4/3}}\right)\right)$$

$$\sim -2^{1/3}f_\ell^{4/3}\exp\left(-\frac{\ell-k-1}{4f_\ell^{4/3}}\right) \geq -2^{1/3}f_\ell^{4/3}\exp\left(-\log f_\ell^{4/3}\right) \geq -2^{1/3}.$$

It follows that

$$P(B_kB_\ell) \leq cP(B_k)f_\ell^{2/3}\exp\left(-\frac{3f_\ell^{4/3}}{2^{5/3}}\right) \leq cP(B_k)P(B_\ell). \tag{3.4}$$

On using (3.2), (3.3), (3.4) together with $P(B_kB_\ell) \le P(B_k)$ for $\ell = k, k+1$, we obtain

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n \sum_{\ell=1}^n P(B_kB_\ell)}{\left(\sum_{k=1}^n P(B_k)\right)^2} > 0,$$

hence from Borel-Cantelli lemma and 0-1 law we obtain $P(B_k \text{ i.o.}) = 1$, completing the proof of Theorem 1.2. $\square$

4. Proof of Theorem 1.3

First assume that $J(g) < \infty$. Let $t_k = e^k$ and define the events

$$B_k = \{\Upsilon^*(t_k) < t_{k+1}^{1/4} g(t_{k+1})\}.$$

Then

$$P(B_k) \le cg^2(t_{k+1}),$$

which is well-known to be summable if $J(g) < \infty$. Hence for large k we have almost surely

$$\Upsilon^*(t_k) \ge t_{k+1}^{1/4} g(t_{k+1}),$$

and for $t_k \le t < t_{k+1}$

$$\Upsilon^*(t) \ge \Upsilon^*(t_k) \ge t_{k+1}^{1/4} g(t_{k+1}) \ge t^{1/4} g(t),$$

proving the convergence part.

Now assume that $J(g) = \infty$. Put $t_k = 2^k$ and define the events

$$A_k = \{\eta_2(0, t_k) \le t_k^{1/2} g^2(t_k)\},$$

$$B_k = \{\eta_1^*(t_k^{1/2} g^2(t_k)) \le t_k^{1/4} g(t_k)\}.$$

Then $P(A_k \text{ i.o.}) = 1$ (cf. Csáki [4], the proof of the divergent part of Theorem 2.1 (i) on p. 211) and, by scaling property, $P(B_k) = p > 0$, independently of k. It follows from Lemma 3.1 of Csáki et al. [7] that $P(A_kB_k \text{ i.o.}) \ge p$. Consequently, $P(\Upsilon^*(t_k) \le t_k^{1/4} g(t_k) \text{ i.o.}) \ge p > 0$. Now the proof of the divergence part is complete by $0-1$ law. $\square$

5. Simple random walk on 2-dimensional comb

We consider a simple random walk $\mathbf{C}(n)$ on the 2-dimensional comb lattice $\mathbb{C}^2$ that is obtained from $\mathbb{Z}^2$ by removing all horizontal lines off the x-axis.

A formal way of describing a simple random walk $\mathbf{C}(n)$ on the above 2-dimensional comb lattice $\mathbb{C}^2$ can be formulated via its transition probabilities as follows: for $(x,y)\in\mathbb{Z}^2$

$$P(\mathbf{C}(n+1)=(x,y\pm 1)\mid \mathbf{C}(n)=(x,y))=\frac{1}{2},\quad \text{if } y\neq 0, \tag{5.1}$$

$$P(\mathbf{C}(n+1)=(x\pm 1,0)\mid \mathbf{C}(n)=(x,0))$$
$$=P(\mathbf{C}(n+1)=(x,\pm 1)\mid \mathbf{C}(n)=(x,0))=\frac{1}{4}. \tag{5.2}$$

Unless otherwise stated, we assume that $\mathbf{C}(0)=\mathbf{0}=(0,0)$. The coordinates of the just defined vector valued simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ will be denoted by $C_1(n), C_2(n)$, i.e., $\mathbf{C}(n):=(C_1(n),C_2(n))$.

For a recent review of some related literature concerning this simple random walk we refer to Bertacchi [1] and Csáki et al. [8]. In the latter paper we established a strong approximation for the random walk $\mathbf{C}(n)=(C_1(n),C_2(n))$ that reads as follows.

Theorem B. *On an appropriate probability space for the random walk*

$$\{\mathbf{C}(n)=(C_1(n),C_2(n));n=0,1,2,\ldots\}$$

on $\mathbb{C}^2$*, one can construct two independent standard Wiener processes* $\{W_1(t);\,t\geq 0\}$*,* $\{W_2(t);\,t\geq 0\}$ *so that, as* $n\to\infty$*, we have with any* $\varepsilon>0$

$$n^{-1/4}|C_1(n)-W_1(\eta_2(0,n))|+n^{-1/2}|C_2(n)-W_2(n)|=O(n^{-1/8+\varepsilon})\quad a.s.,$$

where $\eta_2(0,\cdot)$ *is the local time process at zero of* $W_2(\cdot)$*.*

Define now the local time process $\Xi(\cdot,\cdot)$ of the random walk $\{\mathbf{C}(n);n=0,1,\ldots\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$ by

$$\Xi(\mathbf{x},n):=\#\{0<k\leq n:\,\mathbf{C}(k)=\mathbf{x}\},\quad \mathbf{x}\in\mathbb{C}^2,\ n=1,2,\ldots \tag{5.3}$$

We now introduce our next result that concludes a strong approximation of the just introduced local time process $\Xi((x,0),n)$.

Theorem 5.1. *On a suitable probability space we can define a simple random walk on $\mathbb{C}^2$ and two independent Wiener local times $\eta_1(\cdot,\cdot)$, $\eta_2(\cdot,\cdot)$ such that as $n\to\infty$, we have for any $\varepsilon>0$*

$$\sup_{x\in\mathbb{Z}}|\Xi((x,0),n)-2\eta_1\left(x,\eta_2(0,n)\right)|=O(n^{1/8+\varepsilon})\quad a.s. \tag{5.4}$$

Proof. As in [8], start with two independent simple symmetric random walks on the line

$$\{S_1(n),S_2(n);\,n=0,1,\ldots\}$$

with respective local times

$$\xi_i(x,n):=\#\{j:\,1\le j\le n,S_i(j)=x\},\quad i=1,2,\quad x\in\mathbb{Z},\quad n=1,2,\ldots$$

and inverse local times

$$\rho_i(N):=\min\{j>\rho_{N-1}:\,S_i(j)=0\},\quad i=1,2,\quad N=1,2,\ldots$$

with $\rho_i(0)=0$. Assume that on the same probability space we have an i.i.d. sequence of random variables $G_1,G_2,\ldots$ with geometric distribution,

$$P(G_1=k)=\frac{1}{2^{k+1}},\quad k=0,1,2,\ldots,$$

that is independent of $S_1(\cdot),S_2(\cdot)$. We may construct a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ as follows. Put $T_N=G_1+G_2+\ldots G_N$, $N=1,2,\ldots$ For $n=0,\ldots,T_1$, let $C_1(n)=S_1(n)$ and $C_2(n)=0$. For $n=T_1+1,\ldots,T_1+\rho_2(1)$, let $C_1(n)=C_1(T_1)$, $C_2(n)=S_2(n-T_1)$. In general, for $T_N+\rho_2(N)<n\le T_{N+1}+\rho_2(N)$, let

$$\begin{aligned}C_1(n)&=S_1(n-\rho_2(N)),\\C_2(n)&=0,\end{aligned}$$

and, for $T_{N+1}+\rho_2(N)<n\le T_{N+1}+\rho_2(N+1)$, let

$$\begin{aligned}C_1(n)&=C_1(T_{N+1}+\rho_2(N))=S_1(T_{N+1}),\\C_2(n)&=S_2(n-T_{N+1}).\end{aligned}$$

Then it can be seen that, in terms of these definitions for $C_1(n)$ and $C_2(n)$, $\mathbf{C}(n)=(C_1(n),C_2(n))$ is a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$.

First we approximate the local time $\Xi((x,0),n)$ by iterated simple symmetric random walk local time.

Proposition 5.1. *On a suitable probability space we can define a simple random walk* $\mathbf{C}$ *on* $\mathbb{C}^2$ *with local time* Ξ *and two simple random walks* S_1, S_2 *on* $\mathbb{Z}$ *with local times* ξ_1, ξ_2 *such that as* $n \to \infty$*, we have for any* $\varepsilon > 0$

$$\sup_{x\in\mathbb{Z}} |\Xi((x,0),n) - 2\xi_1\,(x, \xi_2(0,n))| = O(n^{1/8+\varepsilon}) \quad a.s. \tag{5.5}$$

Proof. Introduce the following notations. For the random walk $\mathbf{C}(\cdot)$ let $H(n)$ be the horizontal steps on the x-axis up to time n and let $V(n)$ be the number of vertical steps up to time n. Moreover, let $B(n)$ be the number of vertical visits to the x-axis up to time n. Put

$$\begin{aligned}&\Xi^{(h)}((x,0),n) \;:= \\ &\#\{0 < k \le n:\, \mathbf{C}(k) = (x,0),\, |C_1(k) - C_1(k-1)| > 0,\, C_2(k-1) = 0\}\end{aligned}$$

and

$$\Xi^{(v)}((x,0),n) = \Xi((x,0),n) - \Xi^{(h)}((x,0),n),$$

i.e., the horizontal, resp. vertical, visits to the point $(x,0)$ up to time n. Then, we have clearly

$$\Xi^{(h)}((x,0),n) = \xi_1(x, H(n)),$$

$$B(n) = \xi_2(0, V(n)) = \xi_2(0, n - H(n)) = O(n^{1/2+\varepsilon}) \quad \text{a.s.},$$

$$H(n) = G_1 + G_2 + \ldots + G_{B(n)} = O(B(n)) = O(n^{1/2+\varepsilon}) \quad \text{a.s.},$$

$$\begin{aligned}|H(n) - B(n)| &= |G_1 + G_2 + \ldots + G_{B(n)} - B(n)| \\ &= O((B(n))^{1/2+\varepsilon}) = O(n^{1/4+\varepsilon}) \quad \text{a.s.},\end{aligned}$$

as $n \to \infty$. Using the increment property of simple symmetric random walk local time (cf. Révész [12], Theorem 11.15), we get

$$\xi_2(0,n) - \xi_2(0, n - H(n)) = O((H(n))^{1/2+\varepsilon}) \quad \text{a.s.}, \quad n \to \infty,$$

and

$$\Xi^{(h)}((x,0),n) = \xi_1(x, H(n)) = \xi_1(x, B(n) + O(B(n)^{1/2+\varepsilon}))$$

$$= \xi_1(x, B(n)) + O(B(n)^{1/4+\varepsilon}) = \xi_1(x, \xi_2(0, n - H(n))) + O((\xi_2(0, n - H(n))^{1/4+\varepsilon})$$

$$= \xi_1(x, \xi_2(0,n)) + O((H(n))^{1/4+\varepsilon}) = \xi_1(x, \xi_2(0,n)) + O(n^{1/8+\varepsilon}),$$

almost surely, where we used that $H(n) = O(n^{1/2+\varepsilon})$ a.s., $n \to \infty$.

Now we show that $\Xi^{(h)}$ and $\Xi^{(v)}$ are close to each other.

Lemma 5.1. *As $n \to \infty$, we have almost surely*

$$\sup_{x \in \mathbb{Z}} |\Xi^{(h)}((x,0),n) - \Xi^{(v)}((x,0),n)| = O(n^{1/8+\varepsilon}). \tag{5.6}$$

Proof. By the law of the iterated logarithm we have $C_1(n) = O(n^{1/4+\varepsilon})$ almost surely, as $n \to \infty$, and hence it suffices to show

$$\sup_{|x| \le n^{1/4+\varepsilon}} |\Xi^{(h)}((x,0),n) - \Xi^{(v)}((x,0),n)| = O(n^{1/8+\varepsilon}) \quad \text{a.s.} \tag{5.7}$$

as $n \to \infty$.

Let $\kappa(x,0)$ be the time of the first horizontal visit of $\mathbf{C}(\cdot)$ to $(x,0)$, and for $\ell \ge 1$ let $\kappa(x,\ell)$ denote the time of the ℓ-th horizontal return of $\mathbf{C}(\cdot)$ to $(x,0)$. Then

$$\Xi^{(v)}((x,0),\kappa(x,\ell)) = \sum_{j=1}^{\ell} \left(\Xi^{(v)}((x,0),\kappa(x,j)) - \Xi^{(v)}((x,0),\kappa(x,j-1)) \right),$$

which is a sum of i.i.d. random variables with geometric distribution

$$P(\Xi^{(v)}((x,0),\kappa(x,j)) - \Xi^{(v)}((x,0),\kappa(x,j-1)) = i) = \frac{1}{2^{i+1}}, \quad i = 0,1,2,\dots$$

By exponential Kolmogorov inequality (see Tóth [14])

$$P(\max_{\ell \le m} |\Xi^{(v)}((x,0),\kappa(x,\ell) - \ell| > u) \le 2 \exp\left(-\frac{u^2}{8m} \right).$$

Hence, we have also

$$P(\max_{|x| \le m} \max_{\ell \le m} |\Xi^{(v)}((x,0),\kappa(x,\ell) - \ell| > u) \le 2m \exp\left(-\frac{u^2}{8m} \right).$$

Putting $u = m^{1/2+\varepsilon}$, Borel-Cantelli lemma implies

$$\max_{|x| \le m} \max_{\ell \le m} |\Xi^{(v)}((x,0),\kappa(x,\ell)) - \ell| = O(m^{1/2+\varepsilon}) \quad \text{a.s.}$$

as $m \to \infty$.

Since

$$\Xi^{(h)}((x,0),n) = O(n^{1/4+\varepsilon}) \quad \text{a.s.}, \quad n \to \infty,$$

with $m = n^{1/4+\varepsilon}$, we have the Lemma. □

This also completes the proof of the Proposition. □

Now Theorem 5.1 follows from strong invariance principle for local time (cf. Révész [11]) quoted as Theorem C below, and increment results for Wiener local time (cf. Révész [12], Theorem 11.11).

Theorem C. *On a suitable probability space one can define a Wiener process with local time η and a simple symmetric random walk on $\mathbb{Z}$ with local time ξ such that as $n \to \infty$, for any $\varepsilon > 0$ we have almost surely*

$$\sup_{x\in\mathbb{Z}} |\xi(x,n) - \eta(x,n)| = O(n^{1/4+\varepsilon}).$$

The proof of Theorem 5.1 is complete. □

Theorems 1.2, 1.3 and 5.1 imply the following Corollary.

Corollary 5.1. *Let $a(n)$ be a non-decreasing sequence of positive numbers. Then*

$$P(\sup_{x\in\mathbb{Z}} \Xi((x,0),n) > n^{1/4}a(n) \ i.o.) = 0 \ \text{or} \ 1$$

according as

$$\sum_{n=1}^{\infty} \frac{a^2(n)}{n} \exp\left(-\frac{3a^{4/3}(n)}{2^{5/3}}\right) < \infty \ \text{or} \ = \infty.$$

Let $b(n)$ be a non-increasing sequence of positive numbers. Then

$$P(\sup_{x\in\mathbb{Z}} \Xi((x,0),n) < n^{1/4}b(n) \ i.o.) = 0 \ \text{or} \ 1$$

according as

$$\sum_{n=1}^{\infty} \frac{b^2(n)}{n} < \infty \ \text{or} \ = \infty.$$

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