Strong approximations of additive functionals of a planar Brownian motion

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Abstract: This paper is devoted to the study of the additive functional $t \to \int_0^t f(W(s))ds$, where f denotes a measurable function and W is a planar Brownian motion. Kasahara and Kotani [19] have obtained its second-order asymptotic behaviors, by using the skew-product representation of W and the ergodicity of the angular part. We prove that the vector $(\int_0^\cdot f_j(W(s))ds)_{1\le j\le n}$ can be strongly approximated by a multi-dimensional Brownian motion time changed by an independent inhomogeneous Lévy process. This strong approximation yields central limit theorems and almost sure behaviors for additive functionals. We also give their applications to winding numbers and to symmetric Cauchy process.

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1 Introduction

Let $W = (W_1, W_2)$ be a planar Brownian motion, where W_1 and W_2 are two independent one-dimensional Brownian motions. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a measurable locally integrable function. The additive functional, $t \to \int_0^t f(W(s))ds$, together with other functionals of planar Brownian motion such as windings and crossing numbers, have been a subject of many studies, see for instance Pitman and Yor [23], [25], Hu and Yor [15] for studies and references. Here, we will turn our attention to the additive functionals. The following two results describe respectively the first-order and the second-order asymptotic behaviors:

Theorem A (Kallianpur-Robbins [17]) *Let $f_1, f_2 \in L^1(\mathbb{R}^2)$ and $f_2 > 0$, both having compact supports. Then as $t \to \infty$,*

$$\frac{\int_0^t f_1(W(s))ds}{\int_0^t f_2(W(s))ds} \xrightarrow{\text{a.s.}} \frac{C_1(f_1)}{C_1(f_2)}, \tag{1.1}$$

$$\frac{1}{\log t}\int_0^t f_1(W(s))ds \xrightarrow{(d)} \frac{C_1(f_1)}{2\pi}\,\mathbf{e}, \tag{1.2}$$

where $\mathbf{e}$ denotes a standard exponential variable and $C_1(f) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} f(x)dx$.

Ergodic results similar to (1.1) hold for a large class of recurrent Markov process, see e.g. [2] for a general statement. The convergence in law (1.2) can be extended as the convergence in terms of processes, furthermore, the following result holds:

Theorem B (Kasahara and Kotani [19]) *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded function such that $\int |f(x)||x|^\nu dx < \infty$ for some $\nu > 2$. Then as $\lambda \to \infty$,*

$$\Big(\frac{1}{\lambda}\int_0^{e^{\lambda t}} f(W(s))ds,\, t \geq 0\Big) \xrightarrow{(f.d.)} \Big(\frac{C_1(f)}{2\pi}\mathbf{e}(t),\, t \geq 0\Big), \tag{1.3}$$

where "$\xrightarrow{(f.d.)}$" means the convergence in the finite marginal sense, and $(\mathbf{e}(t), t \geq 0)$ denotes an inhomogeneous Lévy process such that $\mathbf{e}(t)$ is an exponential variable with mean t. Moreover, if $C_1(f) = 0$, then

$$\Big(\frac{1}{\sqrt{\lambda}}\int_0^{e^{\lambda t}} f(W(s))ds,\, t \geq 0\Big) \xrightarrow{(f.d.)} \Big(\widetilde{C}_2(f)\,\widetilde{\beta}(\mathbf{e}(t)),\, t \geq 0\Big), \qquad \lambda \to \infty,$$

where $\widetilde{\beta}$ is a standard one-dimensional Brownian motion, independent of $\mathbf{e}(\cdot)$, and

$$\widetilde{C}_2(f) = \Big(-\frac{1}{\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| f(x)f(y)dxdy\Big)^{1/2}. \tag{1.4}$$

The constant $\widetilde{C}_2$ was given in Kasahara [18] by evaluating the asymptotics of the resolvent (see also Touati [29] for more general Markov process). Let us briefly describe the idea of

Kasahara and Kotani [19]: Identifying $\mathbb{C} = \mathbb{R}^2$ and assuming without loss of generality that $W(0) = 1$, we recall the following skew-product representation (cf. [16], pp. 270):

$$W(t) \;=\; R(t)\, e^{i\,\theta(t)} = \exp\Big(\beta(\Xi(t)) + i\gamma(\Xi(t))\Big), \tag{1.5}$$

$$\Xi(t) \;=\; \int_0^t \frac{ds}{R^2(s)}, \tag{1.6}$$

where β and γ denote two independent real-valued Brownian motions both starting from 0. Then the additive functionals of the planar Brownian motion can be transferred to that of the Brownian motion $(\beta_t, \dot{\gamma}_t)$ on the cylinder $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$, with $\dot{x} \stackrel{\text{def}}{=} x(\text{mod } 2\pi)$; Therefore, we can make use of the ergodicity of $\dot{\gamma}$ to solve the two-dimensional problem.

In this paper, our main goals are to unify Theorems A and B and to obtain the fluctuations in these convergences in law. This will be done by establishing a strong approximation of the vector of additive functionals $(\int_0^t f_j(W(s))ds, 1 \le j \le n)$. Before stating our results, we remark that the Lévy process $\mathbf{e}(\cdot)$ in Theorem B can be realized as

$$\mathbf{e}(t) \stackrel{\text{def}}{=} \ell(\sigma(t/2)), \qquad t \ge 0, \tag{1.7}$$

where $(\ell(t), t \ge 0)$ denotes the process of local times at 0 of the one-dimensional Brownian motion β and $\sigma(\cdot)$ is the first passage process of β:

$$\sigma(x) \stackrel{\text{def}}{=} \inf\{s > 0 : \beta(s) > x\}, \qquad x \ge 0. \tag{1.8}$$

The inverse process of $(\mathbf{e}(t))$ is called an extremal process, see Resnick [26] and Watanabe [31].

Theorem 1.1 *Fix $n \ge 1$. Let $f_1, ..., f_n : \mathbb{R}^2 \to \mathbb{R}$ be n measurable real-valued functions. Assume that there exists some constants $K > 0$ and $\nu > \frac{5}{2}$ such that for all $1 \le j \le n$,*

$$\sup_{|z|=r} |f_j(z)| \;\le\; \frac{K}{r^2(1 + |\log r|)^{\nu}}, \qquad r > 0. \tag{1.9}$$

Then, possibly in an enlarged probability space, we may define a version of the planar Brownian motion W, a $\mathbb{R}^n$-valued Brownian motion $Y = (Y_1, ..., Y_n)$ starting from 0 and a process $\widetilde{\mathbf{e}}$ such that $\widetilde{\mathbf{e}}$ has the same law as $\mathbf{e}$, Y and $\widetilde{\mathbf{e}}$ are independent and such that almost surely for any $1 \le j \le n$ and for all large t,

$$\int_0^t f_j(W(s))ds - \frac{C_1(f_j)}{2\pi}\mathbf{e}(\log t) - C_2(f_j) Y_j(\widetilde{\mathbf{e}}(\log t)) \;=\; o((\log t)^{\frac{1}{2}-\delta}), \tag{1.10}$$

$$|\mathbf{e}(\log t) - \widetilde{\mathbf{e}}(\log t)| \;=\; o((\log t)^{1-\delta}), \tag{1.11}$$

where $\delta > 10^{-5}$ denotes some constant, and the covariance matrix of the n-dimensional Brownian motion Y is given by

$\mathbb{E}(Y_j(1)\,Y_k(1)) = C_3(f_j, f_k)$, *with*

$$C_2(f) \stackrel{\text{def}}{=} \Big(-\frac{1}{\pi^2}\int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dy' f(y) f(y') \log|y-y'| + \frac{2}{\pi^2} C_1(f) \int_{\mathbb{R}^2} dy f(y) \log\max(|y|,1)\Big)^{1/2}, \tag{1.12}$$

$$C_3(f_j, f_k) \stackrel{\text{def}}{=} \frac{1}{4}\frac{(C_2(f_j+f_k))^2 - (C_2(f_j-f_k))^2}{C_2(f_j)\,C_2(f_k)}. \tag{1.13}$$

The constant $C_2(f)$ is well defined thanks to (3.27) and (3.8). Remark that $\mathbf{e}(r)$ is of order r and $\mathbf{e}(r) \gg r^{1-\delta}$ almost surely for all large r. This shows in particular the RHS of (1.11) is negligible with respect to $\mathbf{e}(\log t)$ or to $\tilde{\mathbf{e}}(\log t)$.

It is essential that the Brownian motion Y and the inhomogeneous Lévy process $\widetilde{\mathbf{e}}$ are independent. But Y is not independent of the process $\mathbf{e}$, which is defined from W in terms of (1.5) and (1.7). We also mention that it is impossible to choose Y independent of W, otherwise (1.10) would contradict the usual law of iterated logarithm.

Besides the unification of Theorems A and B, we deduce from (1.10) and (1.11) the central limit theorem for the ergodic result (1.1):

Corollary 1.2 *Under the assumptions of* Theorem 1.1 *and assuming* $C_1(f_2) \neq 0$, *we have*

$$\sqrt{\log t}\,\Big(\frac{\int_0^t f_1(W(s))ds}{\int_0^t f_2(W(s))ds} - \frac{C_1(f_1)}{C_1(f_2)}\Big) \xrightarrow{(d)} a(f_1,f_2)\,\frac{\mathcal{N}}{\sqrt{\mathbf{e}(1)}}, \qquad t\to\infty,$$

where $\mathcal{N}$ *denotes a standard Gaussian variable, independent of* $\mathbf{e}(1)$ *which is exponentially distributed with mean* 1, *and*

$$a(f_1,f_2) = \frac{2\pi}{C_1^2(f_2)}\sqrt{C_1^2(f_1)C_2^2(f_2) + C_1^2(f_2)C_2^2(f_1) - 2C_1(f_1)C_1(f_2)C_2(f_1)C_2(f_2)C_3(f_1,f_2)}.$$

It is also interesting to compare (1.10) with the logarithmic average of Kallianpur and Robbins' law obtained by Mörters ([22], Theorem 1.1).

Theorem 1.1 yields in particular the almost sure behaviors of the additive functionals, for instance, we can obtain the following laws of iterated logarithm:

Corollary 1.3 *Let* $f:\mathbb{R}^2\to\mathbb{R}$ *satisfying* (1.9) *and such that* $C_1(f)=0$ *and* $C_2(f)>0$. *We have*

$$\limsup_{t\to\infty} \frac{\int_0^t f(W(s))ds}{\sqrt{\log t}\,\log\log\log t} = \frac{C_2(f)}{\sqrt{2}}, \qquad \text{a.s.} \tag{1.14}$$

Let $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\sqrt{\log t}/\kappa(t)$ is a nondecreasing function. Then

$$\mathbb{P}\Big(\sup_{0\le s\le t}\int_0^s f(W(u))du < \frac{\sqrt{\log t}}{\kappa(t)}\Big) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t(\log t)\kappa(t)} \begin{cases} < \infty \\ = \infty \end{cases},$$

$$\mathbb{P}\Big(\sup_{0\le s\le t}|\int_0^s f(W(u))du| < \frac{\sqrt{\log t}}{\kappa(t)}\Big) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t(\log t)\kappa^2(t)} \begin{cases} < \infty \\ = \infty \end{cases}.$$

See also Touati [30] for Strassen-type limit theorems with random normalization.

Chen [7] and [8] obtained (1.14)-type results for a Harris' recurrent Markov chain, see [3] for an interesting application. However it is not clear how to reduce the problem for the planar Brownian motion to a recurrent random walk problem in our settings.

The strong approximations of additive functionals of a one-dimensional diffusion process or a recurrent Markov chain have been extensively studied, see [14] for a survey and references. Let us also mention that Csáki and Földes [10] developed a general principle when the underlying Markov process is point-recurrent, this principle can not be applied here because every single point is polar for a planar Brownian motion.

The rest of this paper is organized as follows: In Section 2, we present some exponential moments related to a one-dimensional Brownian motion and a martingale representation; In Section 3, we state the corresponding results (Propositions 3.1 and 3.2) for the additive martingales on the cylinder, which imply in particular Theorem 1.1. We prove Propositions 3.1 and 3.2 in Section 4. Finally, some applications to winding numbers and Cauchy process are given in Section 5.

Throughout this paper, $c, c', c'' > 0$ denote some generic constants whose values may change from one paragraph to another one, whereas $(C_j, 1 \le j \le 20)$ denote some more important constants which may depend on f_j and on ν. In the sequel, we write that f satisfies some condition, say (1.9), to mean that (1.9) holds for f in lieu of f_j, and the condition $\nu > \frac{5}{2}$ may be relaxed to $\nu > 1$ or $\nu > 2$, this will be stated explicitly in each case. For the sake of notational convenience, we shall sometimes write ξ_t instead of $\xi(t)$.

2 One-dimensional Brownian motion

Let $(\ell(t,x), t \ge 0, x \in \mathbb{R})$ be the family of local times of the one-dimensional Brownian motion β. Let us write $\ell(t) \equiv \ell(t,0)$ and define

$$\tau(r) \stackrel{\text{def}}{=} \inf\{t > 0 : \ell(t) > r\}, \qquad r \ge 0.$$

2.1 Exponential moments

For the next result see Kazamaki ([20], pp. 9):

Lemma 2.1 *Let $(N_t, t \geq 0)$ be a continuous real-valued local martingale with respect to the filtration $(\mathcal{G}_t)$. Denote by $(\langle N \rangle_t)$ its bracket. Then for any $(\mathcal{G}_t)$-stopping time T finite or not, we have*

$$\mathbb{E}\exp\left(|N_T|\right) \leq 2\sqrt{\mathbb{E}\exp\left(2\langle N \rangle_T\right)}.$$

Recall Borell's inequality for a Gaussian process (cf. [1], pp. 43, Theorem 2.1):

Lemma 2.2 *Let $\{\xi(t), t \in \Lambda\}$ be a centered Gaussian process with a.s. bounded sample paths, where Λ denotes some parameter set. Then $C_4 \stackrel{\text{def}}{=} \mathbb{E}\sup_{t\in\Lambda}\xi(t) < \infty$, and*

$$\mathbb{P}\Big(|\sup_{t\in\Lambda}\xi(t) - C_4| > \lambda\Big) \leq 2\exp\Big(-\frac{\lambda^2}{2C_5}\Big), \qquad \lambda > 0,$$

with $C_5 \stackrel{\text{def}}{=} \sup_{t\in\Lambda}\mathbb{E}\xi^2(t)$.

Denote in this section by $(B(x), x \in \mathbb{R})$ a standard one-dimensional Brownian motion defined on $\mathbb{R}$. We have

Lemma 2.3 *Let $h : \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$C_6(h) \stackrel{\text{def}}{=} \int_{\mathbb{R}} |h(x)|\,|x|\,\log\log(|x| + \frac{1}{|x|} + 16)dx < \infty.$$

Then there exists some universal constant $c \geq 1$ such that for all $0 \leq a \leq \frac{1}{4C_6}$, we have

$$\mathbb{E}\exp\Big(a\int_{\mathbb{R}}|h(x)|B^2(x)dx\Big) \leq \ c.$$

In $C_6(h)$, the term $\log\log(|x| + \frac{1}{|x|} + 16) > 1$ comes from the usual laws of iterated logarithm both for $|x| \to 0_+$ and for $|x| \to \infty$.

Proof: Applying Lemma 2.2 twice to the Gaussian processes $\{\frac{\pm B(x)}{\sqrt{|x|\,\log\log(|x|+\frac{1}{|x|}+16)}}, x \in \mathbb{R}\}$, we obtain:

$$C_7 \stackrel{\text{def}}{=} \mathbb{E}m^* < \infty,$$

where $m^* \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}}\frac{|B(x)|}{\sqrt{|x|\,\log\log(|x|+\frac{1}{|x|}+16)}} < \infty$, a.s. by the usual law of iterated logarithm at 0 and at ∞. Since $\mathbb{E}\frac{B^2(x)}{|x|\,\log\log(|x|+\frac{1}{|x|}+16)} = \frac{1}{\log\log(|x|+\frac{1}{|x|}+16)} < 1$ we have

$$\mathbb{P}\Big(m^* > C_7 + \lambda\Big) \leq 4\exp\Big(-\frac{\lambda^2}{2}\Big), \qquad \lambda > 0.$$

Remark that

$$\int_{\mathbb{R}}|h(x)|B^2(x)dx \leq (m^*)^2\int_{\mathbb{R}}|h(x)|\,|x|\,\log\log(|x| + \frac{1}{|x|} + 16)dx \ = C_6(h)\,(m^*)^2.$$

Hence $\mathbb{E}\exp\Big(a\int_{\mathbb{R}}|h(x)|B^2(x)dx\Big) \leq \mathbb{E}\exp\Big(a\,C_6\,(m^*)^2\Big) \leq \mathbb{E}\exp\Big(\frac{(m^*)^2}{4}\Big) = c < \infty.$ $\square$

Proposition 2.4 *Let $h : \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$\int_{\mathbb{R}} |h(x)| \, [1 + |x| \, \log\log(|x| + 16)] dx < \infty. \tag{2.1}$$

Then there exists some constant $C_8(h) > 1$ such that for all $0 \le a \le \frac{1}{C_8}$ and $r > 0$, we have

$$\mathbb{E}\exp\left(a \int_0^{\tau_r} |h(\beta_s)| ds\right) \le C_8 \, e^{C_8 \, a \, r}, \tag{2.2}$$

$$\mathbb{E}\exp\left(a \Big| \int_0^{\tau_r} h(\beta_s) ds - r \int_{-\infty}^{\infty} h(x) dx \Big|\right) \le C_8 \, e^{C_8 \, a^2 \, r}. \tag{2.3}$$

Proof: According to Ray-Knight's theorem (cf. [27], Chap. XI), $x \to \ell(\tau(r), x)$ is the square of a zero-dimensional Bessel process, which is the unique nonnegative solution of the stochastic equation

$$\ell(\tau(r), x) = r + 2 \int_0^x \sqrt{\ell(\tau(r), y)} \, dB(y), \qquad x \in \mathbb{R}, \tag{2.4}$$

for some one-dimensional Brownian motion $(B(x), x \in \mathbb{R})$. Remark that $x \to \ell(\tau(r), x)$ is stochastically smaller than $x \to (\sqrt{r} + B(x))^2$, by using the comparison theorem of diffusions with different drift terms (cf. [27], Theorem IX.3.7). It follows that for $0 \le a \le \frac{1}{8C_6(h)}$, we have from Lemma 2.3 that

$$\begin{aligned}
\mathbb{E}\exp\left(a \int_0^{\tau_r} |h(\beta_s)| ds\right) &= \mathbb{E}\exp\left(a \int_{\mathbb{R}} |h(x)| \ell(\tau(r), x) dx\right) \\
&\le \mathbb{E}\exp\left(a \int_{\mathbb{R}} |h(x)| (\sqrt{r} + B(x))^2 dx\right) \\
&\le \mathbb{E}\exp\left(a \int_{\mathbb{R}} |h(x)| 2(r + B^2(x)) dx\right) \\
&\le c \, \exp\left(2a \, r \int_{\mathbb{R}} |h(x)| dx\right),
\end{aligned} \tag{2.5}$$

yielding (2.2). Define

$$H(x) \stackrel{\text{def}}{=} \int_x^{\infty} h(y) dy, \ x > 0; \qquad H(x) \stackrel{\text{def}}{=} - \int_{-\infty}^x h(y) dy, \ x \le 0.$$

Using the equation (2.4), we get

$$\begin{aligned}
\int_0^{\tau_r} h(\beta_s) ds - r \int_{-\infty}^{\infty} h(x) dx &= \int_{-\infty}^{\infty} h(x) (\ell(\tau(r), x)) - r) dx \\
&= - \int_{-\infty}^{\infty} (\ell(\tau(r), x) - r) dH(x) \\
&= 2 \int_{-\infty}^{\infty} H(x) \, \sqrt{\ell(\tau(r), x)} \, dB(x),
\end{aligned}$$

by integration by parts. It is elementary to check that H^2 satisfies the condition of integrability:

$$\int_{\mathbb{R}} H^2(x)\,[1+|x|\,\log\log(|x|+16)]dx < \infty.$$

Using Lemma 2.1 with $N(t) \stackrel{\text{def}}{=} 2a\int_{-\infty}^t H(x)\,\sqrt{\ell(\tau_r,x)}\,dB(x)$, we obtain that

$$\mathbb{E}\exp\Big(a\,|\int_0^{\tau_r} h(\beta_s)ds - r\int_{-\infty}^{\infty} h(x)dx|\Big) \le 2\sqrt{\mathbb{E}\exp\Big(8\,a^2\int_{-\infty}^{\infty} H^2(x)\,\ell(\tau(r),x)\,dx\Big)}.$$

Applying (2.5) to $H^2(x)$ instead of h, we have that for $8a^2 \le \frac{1}{8C_6(H^2)}$ (the constant $C_6(H^2)$ has been defined in Lemma 2.3),

$$\mathbb{E}\exp\Big(a\,|\int_0^{\tau_r} h(\beta_s)ds - r\int_{-\infty}^{\infty} h(x)dx|\Big) \le 2\sqrt{c}\,\exp\Big(8a^2\,r\int_{\mathbb{R}} H^2(x)dx\Big),$$

implying (2.3) by choosing a sufficiently large constant C_8. Finally, we shall also make use of the following simple fact (for example, by using (2.4)):

$$\mathbb{E}\Big(\int_0^{\tau_1}\frac{ds}{(1+|\beta_s|)^\nu}\Big)^2 < \infty, \qquad \nu > \frac{3}{2}. \tag{2.6}$$

$\square$

2.2 Martingale representation

Define

$$\overline{\beta}(t) = \sup_{o\le s\le t}\beta(s), \qquad t \ge 0.$$

Let $(\mathcal{B}_t)$ be the natural filtration generated by the Brownian motion β.

Lemma 2.5 *Let $r \ge 2$ and $u \in \mathbb{R}$. We have*

$$\mathbb{E}\Big(e^{iu\overline{\beta}(\tau_r)}\,|\,\mathcal{B}_t\Big) = \mathbb{E}\Big(e^{iu\overline{\beta}(\tau_r)}\Big) + \int_0^{t\wedge\tau_r}\zeta_v(r,u)\,d\beta_v, \qquad t\ge 0,$$

for some $(\mathcal{B}_v)$-predictable process $\zeta_v(r,u)$. Furthermore

$$|\zeta_v(r,u)| \le 2\Big(\frac{1_{(\overline{\beta}_v\ge 1)}}{\overline{\beta}_v} + 1_{(\overline{\beta}_v<1)}(1+|u|\log(1/\overline{\beta}_v))\Big).$$

Proof: The two parameters r and u are fixed. Using the Markov property at t, we obtain

$$D_t \stackrel{\text{def}}{=} \mathbb{E}\Big(e^{iu\overline{\beta}(\tau_r)}\,|\,\mathcal{B}_t\Big) = e^{iu\overline{\beta}(\tau_r)}1_{(t\ge\tau_r)} + 1_{(t<\tau_r)}\phi(\beta_t,\overline{\beta}_t,r-\ell_t), \tag{2.7}$$

with

$$\phi(x,y,s) = \mathbb{E}_x\left(e^{i\,u(\,y\vee\overline{\beta}(\tau_s))}\right), \qquad y \ge 0\vee x, s>0,$$

where $y\vee a = \max(y,a)$ and $\mathbb{E}_x$ (resp: $\mathbb{P}_x$) denotes the expectation (resp: probability) with respect to the Brownian motion β starting from x. Write in this proof $\sigma_0 = \inf\{t\ge 0 : \beta_t = 0\}$ and define

$$\eta(a,s) = \mathbb{E}_0\left(e^{i\,u\,(a\vee\overline{\beta}(\tau_s))}\right), \qquad a\ge 0, s\ge 0.$$

Therefore applying the strong Markov property at σ_0, $\mathbb{E}_x(e^{i\,u\,(y\vee\overline{\beta}(\tau_s))}\,|\mathcal{B}_{\sigma_0}) = \eta(y\vee\overline{\beta}(\sigma_0),s)$, we get

$$\begin{aligned}\phi(x,y,s) &= \mathbb{E}_x\left(\eta(y\vee\overline{\beta}(\sigma_0),s)\right)\\ &= 1_{(x\le 0)}\eta(y,s) + 1_{(0<x\le y)}\left(\eta(y,s)(1-\frac{x}{y}) + x\int_y^\infty \frac{da}{a^2}\eta(a,s)\right), \qquad (2.8)\end{aligned}$$

by using the fact that if $x>0$, $\mathbb{P}_x\left(\overline{\beta}(\sigma_0)\in da\right) = \frac{x}{a^2}1_{(a\ge x)}da$. On the other hand, it is known (cf. [26], [31], [6] pp. 191) that

$$\mathbb{P}_0\left(\overline{\beta}(\tau_s)\le a\right) = e^{-s/(2a)}, \qquad a,s>0. \qquad (2.9)$$

Hence

$$\eta(a,s) = e^{iua-s/(2a)} + \int_a^\infty \frac{db}{2b^2}\,s\,e^{iub-s/(2b)}.$$

Observe from (2.8) that $\frac{\partial\phi}{\partial x} = 0$ when $x<0$. Elementary computations show that for $x>0$,

$$\begin{aligned}\frac{\partial\phi}{\partial x}(x,y,s) &= -\frac{\eta(y,s)}{y} + \int_y^\infty \frac{da}{a^2}\eta(a,s) \qquad (2.10)\\ &= \lim_{A\to\infty}\int_y^A \frac{da}{a}\frac{\partial\eta}{\partial a}(a,s)\\ &= i\,u\lim_{A\to\infty}\int_y^A\frac{da}{a}\,e^{iua-\frac{s}{2a}}. \qquad (2.11)\end{aligned}$$

Going back to (2.7) and applying Itô's formula to the RHS of (2.7), we obtain that

$$D_t = D_0 + \int_0^{t\wedge\tau_r}\frac{\partial\phi}{\partial x}(\beta_v,\overline{\beta}_v, r-\ell_v)\,d\beta_v \equiv D_0 + \int_0^{t\wedge\tau_r}\zeta_v(r,u)\,d\beta_v,$$

the other terms vanish since (D_t) is a martingale. This gives that $\zeta_v(r,u) = \frac{\partial\phi}{\partial x}(\beta_v,\overline{\beta}_v,r-\ell_v)$. It remains to bound $\frac{\partial\phi}{\partial x}$. Let $0<x\le y$. Using the fact that $|\eta|\le 1$ to (2.10) yields that

$$|\frac{\partial\phi}{\partial x}| \le \frac{2}{y}, \qquad y>0.$$

When $0 < y \le 1$, we deduce from (2.11) that $\frac{\partial\phi}{\partial x}(x,y,s) - \frac{\partial\phi}{\partial x}(x,1,s) = iu\int_y^1 \frac{da}{a} e^{iua - s/(2a)}$. Since $|\frac{\partial\phi}{\partial x}(x,1,s)| \le 2$, we have

$$|\frac{\partial\phi}{\partial x}(x,y,s)| \le 2 + |u|\int_y^1 \frac{da}{a} e^{-s/(2a)} \le 2 + |u|\log(1/y),$$

ending the proof. $\square$

Lemma 2.6 *Let $\nu > \frac{3}{2}$. There exists some constant $c > 0$ such that for all $r \ge 2$,*

$$\mathbb{E}\Big(\int_0^{\tau_r} \frac{ds}{(1+|\beta_s|)^\nu} \frac{1}{1+\overline{\beta}_s}\Big)^2 \le c(\log r)^2.$$

Proof: Observe that

$$\int_0^{\tau_r} \frac{ds}{(1+|\beta_s|)^\nu} \frac{1}{1+\overline{\beta}_s} \le \sum_{1\le j\le r} \int_{\tau_{j-1}}^{\tau_j} \frac{ds}{(1+|\beta_s|)^\nu} \frac{1}{1+\overline{\beta}(\tau_{j-1})} \stackrel{\text{def}}{=} \sum_{1\le j\le r} \frac{\xi_j}{1+\overline{\beta}(\tau_{j-1})},$$

with obvious definition of ξ_j. The sequence (ξ_j) are i.i.d. and we have

$$\mathbb{E}\Big(\xi_j^2\Big) = \mathbb{E}\Big(\int_0^{\tau_1} \frac{ds}{(1+|\beta_s|)^\nu}\Big)^2 < \infty,$$

by virtue of (2.6). Thanks to the independence of ξ_j and $\mathcal{B}_{\tau_{j-1}}$, the sequence $(\frac{(\xi_j - \mathbb{E}\xi_j)}{1+\overline{\beta}(\tau_{j-1})})_{j\ge 1}$ is a square-integrable martingale difference, hence

$$\begin{aligned}
\mathbb{E}\Big(\sum_{1\le j\le r} \frac{\xi_j}{1+\overline{\beta}(\tau_{j-1})}\Big)^2 &\le 2\mathbb{E}\Big(\sum_{1\le j\le r} \frac{(\xi_j - \mathbb{E}\xi_j)}{1+\overline{\beta}(\tau_{j-1})}\Big)^2 + 2(\mathbb{E}\xi_1)^2 \mathbb{E}\Big(\sum_{1\le j\le r} \frac{1}{1+\overline{\beta}(\tau_{j-1})}\Big)^2 \\
&= 2\mathrm{Var}(\xi_1)\mathbb{E}\sum_{1\le j\le r} \frac{1}{(1+\overline{\beta}(\tau_{j-1}))^2} + 2(\mathbb{E}\xi_1)^2 \mathbb{E}\Big(\sum_{1\le j\le r} \frac{1}{1+\overline{\beta}(\tau_{j-1})}\Big)^2 \\
&\le 4\mathbb{E}(\xi_1)^2\, \mathbb{E}\Big(\sum_{1\le j\le r} \frac{1}{1+\overline{\beta}(\tau_{j-1})}\Big)^2. \qquad (2.12)
\end{aligned}$$

Applying the strong Markov property at τ_{j-1}, we obtain that for $l > j$,

$$\mathbb{E}\Big(\frac{1}{1+\overline{\beta}(\tau_{l-1})} \,|\, \mathcal{B}_{\tau_{j-1}}\Big) \le \mathbb{E}\Big(\frac{1}{1+\overline{\beta}(\tau_{l-j})}\Big) \le \frac{c'}{l-j},$$

by using the law of $\overline{\beta}(\tau_{l-j})$ given in (2.9). This law also implies that $\mathbb{E}\frac{1}{(1+\overline{\beta}(\tau_j))^2} \le \frac{c'}{j^2}$ for $j \ge 1$. It follows that

$$\begin{aligned}
\mathbb{E}\Big(\sum_{1\le j\le r} \frac{1}{1+\overline{\beta}(\tau_{j-1})}\Big)^2 &\le \sum_{1\le j\le r} \mathbb{E}\frac{1}{(1+\overline{\beta}(\tau_{j-1}))^2} + 2\sum_{1\le j<l\le r} \mathbb{E}\frac{1}{1+\overline{\beta}(\tau_{j-1})}\,\mathbb{E}\frac{1}{1+\overline{\beta}(\tau_{l-j})} \\
&\le c'\sum_{1\le j\le r} j^{-2} + 2(c')^2 \sum_{1\le j<l\le r} \frac{1}{j(l-j)} \\
&\le c''(\log r)^2, \qquad r \ge 2,
\end{aligned}$$

which in view of (2.12) completes the proof. $\square$

3 Additive martingales and additive functionals of Brownian motion on the cylinder

Let G denote the cylinder $\mathbb{R}\times\mathbb{R}/(2\pi\mathbb{Z})$ endowed with the Haar measure $dz = dxd\theta$, where $z = (x,\theta) \in G$ denotes a generic element of G. A Brownian motion X on the cylinder is a Feller process taking values in G, with homogeneous probability transition $(p_X(t,(x,\theta))dxd\theta)$:

$$p_X(t,(x,\theta)) = \frac{1}{2\pi t} e^{-\frac{x^2}{2t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(\theta+2\pi k)^2}{2t}}, \qquad (x,\theta) \in \mathbb{R}\times[0,2\pi] \equiv G.$$

It is clear that X can be realized as $X = (\beta, \dot\gamma)$, where (β,γ) is a planar Brownian motion, i.e. β and γ are independent Brownian motions on the line, and $\dot y \overset{\text{def}}{=} y(\operatorname{mod} 2\pi)$. The main result in this section is a strong approximation of additive martingales on the cylinder.

Fix $n \geq 1$. For $1 \leq j \leq n$, let $F^{(j)} : z \in G \to (F_1^{(j)}(z), F_2^{(j)}(z)) \in \mathbb{R}^2$ be n measurable functions. Assume that there exist some constants $\nu > \frac{3}{2}$ and $K > 0$ such that for all $1 \leq j \leq n$,

$$\sup_{0\leq\theta\leq 2\pi} |F^{(j)}(x,\theta)| \leq \frac{K}{(1+|x|)^\nu}, \qquad x \in \mathbb{R}. \tag{3.1}$$

Define the martingales $N^{(j)}$ from X:

$$N^{(j)}(t) \overset{\text{def}}{=} \int_0^t F^{(j)}(X_s)dX_s = \int_0^t F_1^{(j)}(\beta_s,\dot\gamma_s)d\beta_s + \int_0^t F_2^{(j)}(\beta_s,\dot\gamma_s)d\gamma_s, \ t \geq 0.$$

Recall (1.7) and (1.8) and that $\ell(\cdot)$ denotes the local time at 0 of β. We have

Proposition 3.1 *Assume* (3.1) *for some* $\nu > \frac{3}{2}$. *Possibly in a larger probability space, we may define a version of* $X = (\beta,\dot\gamma)$ *a Brownian motion on the cylinder* G *and an* n*-dimensional Brownian motion* $Y = (Y_1, ..., Y_n)$ *starting from* 0 *and a process* L *such that* $L(\cdot)$ *has the same law as* $\ell(\cdot)$, Y *and* L *are independent and such that almost surely for all* $1 \leq j \leq n$ *and all large* t,

$$\begin{aligned} N_t^{(j)} - \sqrt{C_9(F^{(j)})}\, Y_j(L_t) &= o(t^{\frac14-\delta}), &(3.2)\\ |\ell_t - L_t| &= o(t^{\frac12-\delta}), &(3.3)\end{aligned}$$

where $\delta > 10^{-5}$ *denotes some constant and the covariance matrix of the* n*-dimensional Brownian motion* Y *is given by* $\mathbb{E}\Big(Y_j(1)Y_k(1)\Big) = C_{10}(F^{(j)}, F^{(k)})$, *with*

$$\begin{aligned} C_9(F^{(j)}) &\overset{\text{def}}{=} \frac{1}{2\pi}\int_{\mathbb{R}} dx \int_0^{2\pi} d\theta \Big((F_1^{(j)}(x,\theta))^2 + (F_2^{(j)}(x,\theta))^2 \Big) &(3.4)\\ C_{10}(F^{(j)}, F^{(k)}) &\overset{\text{def}}{=} \frac14 \frac{C_9(F^{(j)}+F^{(k)}) - C_9(F^{(j)}-F^{(k)})}{\sqrt{C_9(F^{(j)})\,C_9(F^{(k)})}}. \end{aligned}$$

Similarly to Theorem 1.1, it is essential that the process L is independent from Y. To obtain Theorem 1.1, we also need an analogue of (3.2) such that (Y_j) are independent of $(\mathbf{e}(t))$, where the process $(\mathbf{e}(t)) = (\ell(\sigma(t/2)))$ is defined in (1.7).

Proposition 3.2 *Assume* (3.1) *for some* $\nu > \frac{3}{2}$. *On some suitable probability space, we may define a version of* $X = (\beta, \dot\gamma)$, *a* n*-dimensional Brownian motion* $Y = (Y_1, ..., Y_n)$ *starting from* 0 *with the covariance matrix* $(C_{10}(F^{(j)}, F^{(k)}))_{1\le j,k\le n}$ *and an inhomogeneous Lévy process* $\widetilde{\mathbf{e}}(\cdot)$ *such that* Y *and* $\widetilde{\mathbf{e}}(\cdot)$ *are independent,* $\widetilde{\mathbf{e}}$ *has the same law as* $\mathbf{e}$ *and such that almost surely for all large* r *and for all* $t \in [\sigma(r - 2\log r), \sigma(r + 2\log r)]$, *we have*

$$N_t^{(j)} - \sqrt{C_9(F^{(j)})}\, Y_j(\widetilde{\mathbf{e}}(2r)) \;=\; o(r^{\frac{1}{2}-\delta}), \tag{3.5}$$

$$|\mathbf{e}(r) - \widetilde{\mathbf{e}}(r)| \;=\; o(r^{1-\delta}). \tag{3.6}$$

for some positive constant $\delta > 10^{-5}$.

Let us postpone the proofs of Propositions 3.1 and 3.2 in Section 4. The rest of this section is devoted to a strong approximation of additive functionals on the cylinder and to the proof of Theorem 1.1.

3.1 Additive functionals of X

Let $g : G \to \mathbb{R}$ be a measurable function. First we define two constants related to g (when the integrals are well defined):

$$C_{11}(g) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta\, g(x,\theta), \tag{3.7}$$

$$C_{12}(g) \overset{\text{def}}{=} \Big(-\frac{1}{2\pi^2}\int_{G\times G} dx dx' d\theta d\theta' g(x,\theta) g(x',\theta') \log|e^{x+i\theta} - e^{x'+i\theta'}|^2 + \frac{4}{\pi} C_{11}(g) \int_0^{\infty}\int_0^{2\pi} x\, g(x,\theta)\, dx d\theta \Big)^{1/2}. \tag{3.8}$$

The constant C_{12} is well defined, see (3.15) below. Now we will prove the following consequence of Proposition 3.1:

Corollary 3.3 *Fix* $n \ge 1$. *Let* $g_1, g_2, ..., g_n : G \to \mathbb{R}$ *be* n *measurable functions. Assume that there exist some constants* $\nu > \frac{5}{2}$ *and* $K > 0$ *such that for all* $1 \le j \le n$,

$$\sup_{0\le\theta\le 2\pi} |g_j(x,\theta)| \le \frac{K}{(1+|x|)^\nu}, \qquad x \in \mathbb{R}. \tag{3.9}$$

Then, possibly in a larger probability space, we may define a version of $X = (\beta, \dot\gamma)$, *an* n*-dimensional Brownian motion* $Y = (Y_1, ..., Y_n)$ *starting from* 0 *with covariance matrix*

$(C_{13}(f_j,f_k))_{j,k\le n}$ *and a process* L *such that* $L(\cdot)$ *has the same law as* $\ell(\cdot)$*,* Y *and* L *are independent and such that almost surely for all* $1\le j\le n$ *and all large* t*,*

$$\begin{aligned} \int_0^t g_j(X(s))ds - C_{11}(g_j)\,\ell(t) - C_{12}(g_j)\,Y_j(L_t) &= o(t^{\frac14-\delta}), && (3.10)\\ |\ell_t - L_t| &= o(t^{\frac12-\delta}), && (3.11)\end{aligned}$$

where $\delta > 10^{-5}$ *denotes some constant and*

$$C_{13}(g_j,g_k) \stackrel{\text{def}}{=} \frac14 \frac{(C_{12}(g_j+g_k))^2 - (C_{12}(g_j-g_k))^2}{C_{12}(g_j)\,C_{12}(g_k)}.$$

First let us introduce some notations. Let g be any of the functions $g_1,\dots,g_n$ of Corollary 3.3 and define

$$\overline{g}(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} g(x,\theta)d\theta, \qquad h(x,\theta)\stackrel{\text{def}}{=} g(x,\theta)-\overline{g}(x), \qquad (x,\theta)\in G. \tag{3.12}$$

Note that $\int d\theta h(x,\theta)=0$ for any $x\in\mathbb{R}$ and h satisfies (3.9). According to Kasahara and Kotani ([19], formula (2.1), pp. 141), we define

$$F(x,\theta) = \Psi * h(x,\theta) = \int_{-\infty}^{\infty}\int_0^{2\pi} dx'd\theta'\Psi(x-x',\theta-\theta')h(x',\theta'), \tag{3.13}$$

where $\Psi * h$ denotes the convolution of h with the function Ψ under the Haar measure, and Ψ is defined by

$$\Psi(x,\theta) \stackrel{\text{def}}{=} -\frac{1}{2\pi}\log|e^{i\theta}-e^{-|x|}|^2 = \Psi(x,-\theta). \tag{3.14}$$

We need the following elementary estimates on the partial derivatives of F:

Lemma 3.4 *Assume that* g *satisfies* (3.9) *with some* $\nu>1$*. Recall* (3.8)*. We have*

$$\begin{aligned} \sup_{0\le\theta\le 2\pi}|\nabla F|(x,\theta) &\le \frac{c}{(1+|x|)^{\nu}}, \qquad x\in\mathbb{R}.\\ (C_{12}(g))^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_0^{2\pi}d\theta\Big((\frac{\partial F}{\partial x}(x,\theta)+q(x))^2+\frac{\partial F}{\partial\theta}(x,\theta)^2\Big), && (3.15)\end{aligned}$$

where

$$q(y)\stackrel{\text{def}}{=} 2\int_y^{\infty}dx\overline{g}(x),\ y>0; \qquad q(y)\stackrel{\text{def}}{=} -2\int_{-\infty}^{y}dx\overline{g}(x),\ y\le 0. \tag{3.16}$$

Proof: Elementary calculations (cf. [19], pp. 136) show that

$$\begin{aligned} \frac{\partial\Psi}{\partial x}(x,\theta) &= -\frac{\operatorname{sgn}(x)(\cos\theta - e^{-|x|})e^{-|x|}}{\pi|e^{i\theta}-e^{-|x|}|^2} \in L^1(G,dxd\theta),\\ \frac{\partial\Psi}{\partial\theta}(x,\theta) &= -\frac{e^{-|x|}\sin\theta}{\pi|e^{i\theta}-e^{-|x|}|^2} \in L^1(G,dxd\theta).\end{aligned}$$

It follows that

$$\begin{aligned}
|\frac{\partial F}{\partial x}(x,\theta)| &= |\int_G \frac{\partial \Psi}{\partial x}(x',\theta')h(x-x',\theta-\theta')dx'd\theta'| \\
&\le 2K\int_G \frac{|\frac{\partial \Psi}{\partial x}(x',\theta')|}{(1+|x-x'|)^\nu}dx'd\theta' \\
&\le \frac{c}{(1+|x|)^\nu}.
\end{aligned}$$

The same holds for $\frac{\partial F}{\partial \theta}$.

To show (3.15), we may assume without loss of generality that g is regular (for example $g\in \mathcal{C}^2$), the general case follows from the usual approximation argument. Therefore, we have

$$\frac{1}{2}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \theta^2}\Big)F=-h. \tag{3.17}$$

It follows from the periodicity on θ that the RHS of (3.15) equals

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{\mathbb{R}} dx\int_0^{2\pi} d\theta|\nabla F|^2(x,\theta)+\int_{\mathbb{R}} dxq^2(x) \\
&= \frac{1}{\pi}\int_{\mathbb{R}} dx\int_0^{2\pi} d\theta h(x,\theta)F(x,\theta)+\int_{\mathbb{R}} dxq^2(x) \qquad \textit{integration by parts (3.17)} \\
&= -\frac{1}{2\pi^2}\int_{G\times G} dxdx'd\theta d\theta' h(x,\theta)h(x',\theta')\log|e^{i(\theta-\theta')}-e^{-|x-x'|}|^2+\int_{\mathbb{R}} dxq^2(x) \\
&= -\frac{1}{2\pi^2}\int_{G\times G} dxdx'd\theta d\theta' g(x,\theta)g(x',\theta')\log|e^{i(\theta-\theta')}-e^{-|x-x'|}|^2+\int_{\mathbb{R}} dxq^2(x), \quad (3.18)
\end{aligned}$$

where the last equality follows from the elementary fact

$$\int_0^{2\pi} d\theta\,\log|e^{i\theta}-r|^2=0, \qquad \forall\, 0\le r<1.$$

Since $|e^{i(\theta-\theta')}-e^{-|x-x'|}|^2=e^{-2\max(x,x')}\,|e^{x+i\theta}-e^{x'+i\theta'}|^2$, elementary computations show that the sum in (3.18) coincides with the RHS of (3.8), which is C_{12}^2. □

The following result is a first-order approximation of the additive functionals $\int_0^t g(X_s)ds$:

Lemma 3.5 *Assume that g satisfies (3.9) with some $\nu>2$. For any $\epsilon>0$, we have*

$$\int_0^t g(\beta_s,\dot{\gamma}_s)ds = C_{11}(g)\,\ell(t)+o(t^{\frac{1}{4}+\epsilon}), \qquad t\to\infty, \text{ a.s.}$$

where $C_{11}(g)$ has been defined in (3.7).

Proof: We decompose the additive functionals $\int_0^t g(X_s)ds$ as

$$\int_0^t g(X_s)ds = \int_0^t \overline{g}(\beta_s)ds + \int_0^t h(\beta_s, \dot{\gamma}_s)ds. \tag{3.19}$$

According to Csáki and Földes [10] (here we need $\nu > 2$),

$$\int_0^t \overline{g}(\beta_s)ds = C_{11}\, \ell_t + o(t^{\frac{1}{4}+\epsilon}), \qquad \text{a.s.},$$

it remains to show that

$$\int_0^t h(\beta_s, \dot{\gamma}_s)ds = o(t^{\frac{1}{4}+\epsilon}), \qquad \text{a.s.}$$

If g is in $\mathcal{C}^2$, we apply Itô's formula with (3.17):

$$\begin{aligned} & F(\beta_t, \dot{\gamma}_t) - F(\beta_0, \dot{\gamma}_0) + \int_0^t h(\beta_s, \dot{\gamma}_s)ds \\ = \; & \int_0^t \frac{\partial F}{\partial x}(\beta_s, \dot{\gamma}_s)d\beta_s + \int_0^t \frac{\partial F}{\partial \theta}(\beta_s, \dot{\gamma}_s)d\gamma_s \overset{\text{def}}{=} M_t, \end{aligned} \tag{3.20}$$

is a martingale. Using the approximation of g by regular functions and Lemma 3.4, the equality in (3.20) also holds for all g satisfying (3.9). It turns out that

$$\begin{aligned} \mathbb{E}M_t^2 &= \mathbb{E}\int_0^t |\nabla F|^2(\beta_s, \dot{\gamma}_s)ds \\ &\le c^2 \mathbb{E}\int_0^t \frac{ds}{(1+|\beta_s|)^{2\nu}} \\ &= c^2 \int_{\mathbb{R}} dx (1+|x|)^{-2\nu}\, \mathbb{E}\ell(t,x) \\ &\le c'\sqrt{t}, \qquad t > 0, \end{aligned}$$

since $\mathbb{E}\ell(t,x) \le \mathbb{E}\ell(t,0) = \sqrt{\frac{2t}{\pi}}$. Using Doob's maximal inequality for martingales, we obtain that for $t_n \overset{\text{def}}{=} 2^n$,

$$\mathbb{P}\Big(\sup_{t \le t_n} |M_t| > t_n^{\frac{1+\epsilon}{4}} \Big) \le 2c' t_n^{-\epsilon/2},$$

whose sums on n converges. The Borel-Cantelli lemma together with the monotonicity imply that $M_t = o(t^{\frac{1}{4}+\epsilon})$, a.s. □

Proof of Corollary 3.3: For the notational convenience, we only consider the case $n = 1$ and $g = g_1$. To obtain the second order approximation, we first deduce from Tanaka's formula that

$$\int_0^t \overline{g}(\beta_s)ds = C_{11}(g)\, \ell_t + \int_0^t q(\beta_s)d\beta_s - \int_0^{\beta_t} q(x)\, dx, \tag{3.21}$$

where $q(\cdot)$ is defined in (3.16). The term $\int_0^{\beta_t} q(x)\,dx$ is bounded due to the integrability: $\int_{\mathbb{R}} |q(x)|dx < \infty$. This together with (3.19) and (3.20) implies that

$$\int_0^t g(X_s)ds = C_{11}(g)\ell_t + Q_t + F(X_0) - F(X_t) - \int_0^{\beta_t} dx q(x) \tag{3.22}$$

with $F(X_0) - F(X_t) - \int_0^{\beta_t} q(x)dx = O(1)$ and

$$\begin{aligned} Q_t &\overset{\text{def}}{=} M_t + \int_0^t q(\beta_s)d\beta_s \\ &= \int_0^t (\frac{\partial F}{\partial x}(\beta_s, \dot\gamma_s) + q(\beta_s))d\beta_s + \int_0^t \frac{\partial F}{\partial \theta}(\beta_s, \dot\gamma_s)d\gamma_s. \end{aligned}$$

Note from Lemma 3.4:

$$\sup_{0\le\theta\le 2\pi} |\nabla F|(x,\theta) + |q(x)| \le \frac{c}{(1+|x|)^{\nu-1}},$$

then we can apply Proposition 3.1 to (Q_t) and obtain Corollary 3.3, the constant $C_{12}(g)$ follows from (3.15). □

3.2 Proof of Theorem 1.1

Just like as Corollary 3.3 was a consequence of Proposition 3.1, the next corollary follows from Proposition 3.2 in a similar way, hence we omit the details of its proof:

Corollary 3.6 *Keeping all notations and assumptions of* Corollary 3.3 *we may define on a possibly larger probability space a version of $X = (\beta, \dot\gamma)$, an n-dimensional Brownian motion $Y = (Y_1, ..., Y_n)$ starting from 0 with covariance matrix $(C_{13}(g_j, g_k))_{j,k\le n}$ and a process $\widetilde{\mathbf{e}}$ such that Y and $\widetilde{\mathbf{e}}$ are independent, $\widetilde{\mathbf{e}}(\cdot)$ has the same law as $\mathbf{e}(\cdot)$ and such that almost surely for all large r and any $t \in [\sigma(r - 2\log r), \sigma(r + 2\log r)]$, we have*

$$\begin{aligned} \int_0^t g_j(X(s))ds - C_{11}(g_j)\,\mathbf{e}(2r) - C_{12}(g_j)\,Y_j(\widetilde{\mathbf{e}}(2r)) &= o(r^{\frac{1}{2}-\delta}), && (3.23) \\ |\mathbf{e}(r) - \widetilde{\mathbf{e}}(r)| &= o(r^{1-\delta}). && (3.24) \end{aligned}$$

The factor 2 in $\mathbf{e}(\cdot)$ comes from the fact that $\ell(\sigma(r)) = \mathbf{e}(2r)$.

Proof of Theorem 1.1: We give the proof in the case $n = 1$. Write $f \equiv f_1$. Using the skew-product representation (1.5), we have

$$\int_0^t f(W(s))ds = \int_0^{\Xi(t)} dv e^{2\beta(v)} f(e^{\beta(v)+i\gamma(v)}) = \int_0^{\Xi(t)} g(X_v)dv,$$

where $\dot{\gamma}(s) \stackrel{\text{def}}{=} \gamma(s) (\text{mod } 2\pi)$, $g(x,\theta) \stackrel{\text{def}}{=} e^{2x} f(e^{x+i\theta})$ and

$$\Xi(t) = \inf\{u > 0 : \int_0^u ds e^{2\beta(s)} > t\}. \tag{3.25}$$

Now we need the following result;

Lemma 3.7 (Shi [28]) *For any $s, t > 0$, we have*

$$\begin{aligned} \mathbb{P}\Big(\Xi(t) \le \sigma(s)\Big) &\le 2 \exp\Big(-\frac{t}{4} e^{-2s}\Big) \\ \mathbb{P}\Big(\Xi(t) \ge \sigma(s)\Big) &\le 4 \exp\Big(-\frac{e^{2s}}{16t}\Big), \end{aligned}$$

where we recall that $\sigma(s) \stackrel{\text{def}}{=} \inf\{u > 0 : \beta_u > s\}$.

Using the above Lemma and Borel-Cantelli, it is standard to obtain that almost surely for all large t,

$$\sigma(\frac{\log t}{2} - \log\log t) \le \Xi(t) \le \sigma(\frac{\log t}{2} + \log\log t). \tag{3.26}$$

Now Theorem 1.1 follows from Corollary 3.6 since we may define W through X by (1.5) and (3.25). Finally,

$$\begin{aligned} C_1(f) &= 2\pi C_{11}(g), \\ C_2(f) &= C_{12}(g), \\ C_3(f_j, f_k) &= C_{13}(g_j, g_k), \end{aligned} \tag{3.27}$$

with obvious definitions of g_j, g_k from f_j, f_k. Finally, we obtain (1.12) and (1.13) by change of variables. □

Proof of Corollary 1.2: It immediately follows from Theorem 1.1. □

Proof of Corollary 1.3: The proof goes in the same way as in Theorem 4.2 of [11]. The details are omitted. □

4 Proof of Propositions 3.1 and 3.2

Let us only consider the case $n = 1$ and $\Phi = F^{(1)} : z \in G \to (\Phi_1(z), \Phi_2(z)) \in \mathbb{R}^2$, the general case follows exactly in the same way, and we shall explain how to compute the correlation matrix when $n \ge 2$. Assuming that Φ satisfies the condition (3.1), define

$$N_t \stackrel{\text{def}}{=} \int_0^t \Phi(X_s) dX_s = \int_0^t \Phi_1(\beta_s, \dot{\gamma}_s) d\beta_s + \int_0^t \Phi_2(\beta_s, \dot{\gamma}_s) d\gamma_s, \qquad t \ge 0. \tag{4.1}$$

The goal is to approximate the continuous martingale N_t by a Brownian motion time-changed at $\ell(t)$ such that this Brownian motion is either *independent* of $\ell(\cdot)$ (Proposition 3.1) or *independent* of $(\ell(\sigma(\cdot)))$ (Proposition 3.2):

Dubins-Schwarz' representation theorem of continuous martingale implies that

$$N_t = B(\langle N \rangle_t), \tag{4.2}$$

with some one-dimensional Brownian motion B. It follows that

$$\begin{aligned}\langle N \rangle_t &= \int_0^t |\Phi|^2(X_s)ds = \int_0^t (\Phi_1^2(X_s) + \Phi_2^2(X_s))ds \\ &= C_9(\Phi)\,\ell(t) + o(t^{\frac{1}{4}+\epsilon}), \qquad t \to \infty, \qquad \text{a.s.,} \end{aligned}\tag{4.3}$$

by using Lemma 3.5. But we can not choose a Brownian motion B independent of $\ell(\cdot)$ or independent of $\ell(\sigma(\cdot))$ at this stage. The independence will be obtained by using Berkes and Philipp's lemma:

Lemma 4.1 ([4]) *Let $(\eta_k, k \geq 1)$ be a sequence of random variables with values in $\mathbb{R}^d$, adapted with respect to some filtration $(\mathcal{F}_k)$. Let $\{\mathbf{g}_k, k \geq 1\}$ be a sequence of characteristic functions of probability distributions $\mathbf{G}_k$ on $\mathbb{R}^d$. Suppose that for some nonnegative numbers ϵ_k, δ_k and $\Theta_k \geq 10^8 d$,*

$$\mathbb{E}\Big|\mathbb{E}\Big(e^{iz\eta_k} \mid \mathcal{F}_{k-1}\Big) - \mathbf{g}_k(z)\Big| \leq \epsilon_k, \qquad \forall\, z \in \mathbb{R}^d,\ |z| \leq \Theta_k,$$

and

$$\mathbf{G}_k\Big(z : |z| > \frac{\Theta_k}{4}\Big) \leq \delta_k.$$

Then without changing its distribution we can redefine the sequence $\{\eta_k, k \geq 1\}$ on a richer probability space together with a sequence of $\{Y_k, k \geq 1\}$ of independent random variables such that Y_k has characteristic function $\mathbf{g}_k$ and

$$\mathbb{P}\Big(|\eta_k - Y_k| \geq \alpha_k\Big) \leq \alpha_k$$

and

$$\alpha_k = 16d\frac{\log \Theta_k}{\Theta_k} + 4\sqrt{\epsilon_k}\,\Theta_k^d + \delta_k.$$

Let $(\mathcal{F}_t)$ be the natural filtration generated by X and denote by $\mathbb{E}_{x,\theta}$ the expectation with respect to the Brownian motion X starting from $X_0 = (x, \theta) \in G$. Let us present an exponential moment estimation:

Lemma 4.2 *Fix $\nu > 2$. Assume that $g : G \to \mathbb{R}$ is a measurable function such that for some constant $b > 0$,*

$$\sup_{0 \leq \theta \leq 2\pi} |g(x,\theta)| \leq \frac{b}{(1+|x|)^\nu}, \qquad x \in \mathbb{R}.$$

There exists some positive constants $C_{14}(g) > 1$ such that for all $0 \le a \le \frac{1}{C_{14}}$ and $r > 0$, we have

$$\mathbb{E}_{0,\theta} \exp\Big(a\Big|\int_0^{\tau_r} g(\beta_s, \dot\gamma_s)ds - C_{11}(g)r\Big|\Big) \le C_{14}\, e^{C_{14}\, a^2 r}, \tag{4.4}$$

$$\mathbb{E}_{0,\theta}\Big(\int_0^{\tau_r} g(\beta_s, \dot\gamma_s)ds - C_{11}(g)r\Big)^2 \le C_{14}\, r, \qquad r \ge 1, \tag{4.5}$$

where C_{11} is defined in (3.7).

Proof of Lemma 4.2: Recall (3.12). Applying (2.3) to $\overline{g}(\cdot)$ implies that

$$\mathbb{E}\exp\Big(a\Big|\int_0^{\tau_r} \overline{g}(\beta_s)ds - C_{11}(g)r\Big|\Big) \le c'\, e^{c'\, a^2 r},\ r > 0,\ 0 \le a \le \frac{1}{c'},$$

for some constant $c' > 1$ depending on g. Recall the martingale (M_t) defined in (3.20). We have

$$\int_0^{\tau_r} g(\beta_s, \dot\gamma_s)ds - C_{11}(g)r = \int_0^{\tau_r} \overline{g}(\beta_s)ds - C_{11}(g)r + M(\tau_r) + F(0, \dot\gamma_0) - F(0, \dot\gamma_{\tau_r}).$$

Using successively Lemmas 2.1, 3.4 and (2.2), we obtain

$$\begin{aligned}
\mathbb{E}_{0,\theta} \exp\Big(a|M(\tau_r)|\Big) &\le 2\sqrt{\mathbb{E}_{0,\theta} \exp\Big(2a^2\langle M\rangle(\tau_r)\Big)} \\
&= 2\sqrt{\mathbb{E}_{0,\theta} \exp\Big(2a^2 \int_0^{\tau_r} |\nabla F|^2(\beta_s, \dot\gamma_s)ds\Big)} \\
&\le 2\sqrt{\mathbb{E}\exp\Big(2c^2a^2 \int_0^{\tau_r} (1+|\beta_s|)^{-2\nu}ds\Big)} \\
&\le 2\sqrt{C_8(\nu)}\, e^{C_8(\nu)c^2a^2r}, \qquad \text{if } 2c^2a^2 \le \frac{1}{C_8(\nu)},
\end{aligned}$$

where $C_8(\nu) \ge 1$ denotes the constant in (2.2) corresponding to the function $h(x) = (1+|x|)^{-2\nu}$. Let $c'' = \max(c', 2(c+1)^2C_8(\nu))$. Then $a \le \frac{1}{c''}$ implies that $2c^2a^2 \le \frac{1}{C_8(\nu)}$. Hence, we have shown that

$$\mathbb{E}_{0,\theta} \exp\Big(a|M(\tau_r)|\Big) \le c'' e^{c'' a^2 r}, \qquad 0 \le a \le \frac{1}{c''}.$$

The continuous function $F(0, \cdot)$ is uniformly bounded by some constant, say $c_0(F)$. It follows from Cauchy-Schwarz' inequality that for all $0 \le a \le \frac{1}{2c''}$, we have

$$\begin{aligned}
\mathbb{E}_{0,\theta} e^{a\left|\int_0^{\tau_r} g(\beta_s,\dot\gamma_s)ds - C_{11}(g)r\right|} &\le e^{2c_0(F)a}\sqrt{\mathbb{E}_{0,\theta} e^{2a\left|\int_0^{\tau_r}\overline{g}(\beta_s)ds - C_{11}(g)r\right|}}\sqrt{\mathbb{E}_{0,\theta} e^{2a\left|M(\tau_r)\right|}} \\
&\le e^{c_0(F)/c''}\sqrt{c'c''}\, e^{2(c'+c'')a^2r} \\
&\le c''' e^{c''' a^2 r},
\end{aligned}$$

with $c''' \stackrel{\text{def}}{=} e^{c_0(F)/c''} 2(c'+c'') > 2c'' > 2$. Using the above estimate together with the elementary inequality: $x^2 \le 2\,(c''')^2\, r\, \exp\big(\frac{|x|}{c'''\sqrt{r}}\big)$, we obtain that for all $0 \le a \le \frac{1}{c'''}$,

$$\mathbb{E}_{0,\theta}\, |\int_0^{\tau_r} g(\beta_s, \dot\gamma_s) ds - C_{11}(g) r| \le 2(c''')^3\, e^{1/c'''}\, r < 4(c''')^3 r.$$

Finally, we choose $C_{14} = 4(c''')^3$ and both (4.4) and (4.5) are satisfied. □

We shall use several times the following estimates:

Lemma 4.3 *Assuming that Φ satisfies (3.1) for some $\nu > 1$. There exists some constant $C_{15}(\Phi) > 1$ such that for all $r \ge 1$, $|u| \le \frac{r^{1/4}}{C_{15}}$, we have*

$$\mathbb{E}_{0,\theta}\, e^{\frac{2u^2}{r}\left|\langle N,N\rangle_{\tau_r} - C_9(\Phi) r\right|} \le C_{15}, \tag{4.6}$$

$$\mathbb{E}_{0,\theta}\, |e^{\frac{u^2}{r}\left(\langle N,N\rangle_{\tau_r} - C_9(\Phi) r\right)} - 1| \le \frac{C_{15} u^2}{\sqrt{r}}, \tag{4.7}$$

for any $\theta \in [0, 2\pi]$, and $C_9(\Phi)$ has been defined in (3.4).

Proof: By the condition on Φ,

$$|\frac{d\langle N, \beta\rangle_s}{ds}| \le K\, (1 + |\beta_s|)^{-\nu}, \tag{4.8}$$

$$\frac{d\langle N, N\rangle_s}{ds} = |\Phi(X_s)|^2 \le K^2\, (1 + |\beta_s|)^{-2\nu}. \tag{4.9}$$

Applying Lemma 4.2 to $g = |\Phi|^2$ and $a = 2u^2/r \le 2/C_{15}^2 \le 1/C_{14}(|\Phi|^2)$ implies (4.6). Using the elementary fact that for $x \in \mathbb{R}$, $|e^x - 1| \le |x| e^{|x|}$, (4.6) and Cauchy-Schwarz' inequality, we have

$$\begin{aligned} & \mathbb{E}_{0,\theta}\, |e^{\frac{u^2}{r}\left(\langle N,N\rangle_{\tau_r} - C_9(\Phi) r\right)} - 1| \\ \le\ & \frac{u^2}{r} \sqrt{\mathbb{E}_{0,\theta} \Big(\langle N, N\rangle_{\tau_r} - C_9(\Phi) r\Big)^2\, \mathbb{E}_{0,\theta}\, e^{\frac{2u^2}{r}\left(\langle N,N\rangle_{\tau_r} - C_9(\Phi) r\right)}} \\ \le\ & \sqrt{C_{14} C_{15}}\, \frac{u^2}{\sqrt{r}}, \end{aligned}$$

by virtue of (4.5). □

The main technical lemmas are the following Lemmas 4.4 and 4.5:

Lemma 4.4 *Assume that Φ satisfies (3.9) with $\nu > \frac{3}{2}$. For $u, v \in \mathbb{R}$ with $|u| \le r^{1/4}/C_{15}$ and $|v| \le r$, we have that for any $\theta \in [0, 2\pi]$,*

$$\Big|\mathbb{E}_{0,\theta}\, e^{iu\frac{N(\tau_r)}{\sqrt{r}} + iv\frac{\overline{\beta}(\tau_r)}{r}} - e^{-C_9(\Phi) u^2/2}\, \mathbb{E} e^{iv\overline{\beta}(\tau_1)}\Big| \le \frac{c|u| \log r}{\sqrt{r}} + \frac{cu^2}{\sqrt{r}}.$$

Proof: Recall that $(\mathcal{F}_t)$ is the natural filtration generated by (β,γ). Lemma 2.5 implies that

$$D_t \overset{\text{def}}{=} \mathbb{E}\Big(e^{iv\frac{\overline{\beta}(\tau_r)}{r}} \mid \mathcal{F}_t\Big) = D_0 + \int_0^{t\wedge\tau_r} \zeta_s(r,\frac{v}{r})\, d\beta_s,$$

with $D_0 = \mathbb{E}e^{iv\,\frac{\overline{\beta}(\tau_r)}{r}} = \mathbb{E}e^{iv\overline{\beta}(\tau_1)}$ by the Brownian scaling property, and

$$|\zeta_s(r,\frac{v}{r})| \le 2\Big(\frac{1_{(\overline{\beta}_s\ge 1)}}{\overline{\beta}_s} + 1_{(\overline{\beta}_s<1)}(1+\frac{|v|}{r}\log(1/\overline{\beta}_s))\Big).$$

Let

$$R_t = \exp\Big(i\frac{u}{\sqrt{r}}N_t + \frac{u^2}{2r}(\langle N\rangle_t - C_9(\Phi)r)\Big) = e^{-C_9\,u^2/2} + \frac{iu}{\sqrt{r}}\int_0^t R_s dN_s.$$

Define $R_t^* \overset{\text{def}}{=} \sup_{0\le s\le t}|R_s|$. Then by Lemma 4.3,

$$\mathbb{E}(R_{\tau_r}^*)^4 \le \mathbb{E}e^{\frac{2u^2}{r}(\langle N,N\rangle_{\tau_r}-C_9 r)} \le C_{15}, \qquad |u|\le \frac{r^{1/4}}{C_{15}}. \tag{4.10}$$

We have $d\langle R,D\rangle_s = i\frac{u}{\sqrt{r}}R_s\zeta_s(r,\frac{v}{r})\Phi_1(X_s)ds$, hence

$$|\frac{d\langle R,D\rangle_s}{ds}| \le \frac{2c|u|}{\sqrt{r}}\,R^*_{\tau_r}\,(1+|\beta_s|)^{-\nu}\times\begin{cases}(1+\overline{\beta}_s)^{-1} & \text{if } \overline{\beta}_s>1,\\[2ex] 1+\frac{|v|}{r}\log(1/\overline{\beta}_s) & \text{if } \overline{\beta}_s\le 1.\end{cases} \tag{4.11}$$

First we prove that

$$|\mathbb{E}_{0,\theta}\Big(R_{\tau_r}\,D_{\tau_r}\Big) - R_0\,D_0| \le \frac{c|u|\log r}{\sqrt{r}}. \tag{4.12}$$

To this end, we remark that

$$\begin{aligned}
|\mathbb{E}_{0,\theta}\Big(R_{\tau_r}\,D_{\tau_r}\Big) - R_0D_0| &= |\mathbb{E}_{0,\theta}\langle R,D\rangle_{\tau_r}|\\
&\le \frac{2K|u|}{\sqrt{r}}\Big(\mathbb{E}_{0,\theta}R^*_{\tau_r}\int_0^{\tau_r}(1+|\beta_s|)^{-\nu}\frac{1_{(\overline{\beta}_s\ge1)}}{\overline{\beta}_s}ds\\
&\qquad +\mathbb{E}_{0,\theta}R^*_{\tau_r}\int_0^{\tau_r}(1+|\beta_s|)^{-\nu}1_{(\overline{\beta}_s<1)}(1+\frac{|v|}{r}\log(1/\overline{\beta}_s))ds\Big)\\
&\overset{\text{def}}{=} \frac{2K|u|}{\sqrt{r}}\Big(J_1+J_2\Big),
\end{aligned}$$

with obvious definitions of J_1 and J_2. By using Cauchy-Schwarz' inequality and Lemma 2.6, we have

$$J_1 \le \sqrt{\mathbb{E}_{0,\theta}(R^*_{\tau_r})^2}\sqrt{\mathbb{E}_{0,\theta}\Big(\int_0^{\tau_r}(1+|\beta_s|)^{-\nu}\frac{ds}{1+\overline{\beta}_s}\Big)^2} \le c'\log r, \qquad r\ge 2.$$

Recall (1.8) for $\sigma(x)$. Again from Cauchy-Schwarz' inequality,

$$J_2 \le (C_{15})^{1/4} \sqrt{\mathbb{E}\Big(\int_0^{\sigma(1)} \frac{ds}{(1+|\beta_s|)^\nu} (1+\log(1/\overline{\beta}_s))\Big)^2} = c'' < \infty,$$

where we may obtain the square integrability of $\int_0^{\sigma_1} \frac{ds}{(1+|\beta_s|)^\nu}(1+\log(1/\overline{\beta}_s))$ by using the following argument: for $n \ge 1$,

$$\int_{\sigma(e^{-n})}^{\sigma(1)} \frac{ds}{(1+|\beta_s|)^\nu}(1+\log(1/\overline{\beta}_s)) \le \sum_{k=1}^n (1+k) \int_{\sigma(e^{-k})}^{\sigma(e^{-(k-1)})} \frac{ds}{(1+|\beta_s|)^\nu}.$$

It is easy to obtain that the second moment of the above sum (of independent variables) is uniformly bounded with respect to n, hence c'' is finite and (4.12) follows. Finally, we have

$$\Big|\mathbb{E}_{0,\theta} e^{iu\frac{N(\tau_r)}{\sqrt{r}} + iv\frac{\overline{\beta}(\tau_r)}{r}} - \mathbb{E}_{0,\theta} R_{\tau_r} D_{\tau_r}\Big| \le \mathbb{E}_{0,\theta}\Big| e^{\frac{u^2}{2r}(\langle N\rangle_{\tau_r} - C_9 r)} - 1\Big| \le \frac{cu^2}{r^{1/2}},$$

by (4.7). This together with (4.12) completes the proof. □

Recall the Fourier transform for the stable law τ_1: $\mathbb{E}e^{iv\tau_1} = \exp(-\sqrt{|v|}(1-i\,\mathrm{sgn}(v))), v \in \mathbb{R}$.

Lemma 4.5 *Assume that Φ satisfies (3.9) with $\nu > \frac{3}{2}$. There exists some constant $C_{16}(g) > 0$ such that for any $\theta \in [0, 2\pi]$, $|u| \le \frac{r^{1/4}}{C_{15}}$, $v \in \mathbb{R}$ and $r \ge 2$, we have*

$$\Big|\mathbb{E}_{0,\theta} e^{i\frac{u}{\sqrt{r}}N(\tau_r) + i\frac{v}{r^2}\tau_r} - e^{-\frac{C_9(\Phi)u^2}{2}} \exp(-\sqrt{|v|}(1-i\,\mathrm{sgn}(v)))\Big| \le C_{16}\Big(\frac{u^2}{\sqrt{r}} + \frac{|u||v|^{1/2}}{\sqrt{r}}\Big). \quad (4.13)$$

Proof: Let

$$w \stackrel{\text{def}}{=} (1 - i\ \mathrm{sgn}(v))\frac{\sqrt{|v|}}{r}.$$

Observe that the process

$$S_t \stackrel{\text{def}}{=} \exp\Big(i\frac{v}{r^2}t - w(|\beta_t| - \ell_t + r)\Big) = S_0 - w\int_0^t S_s\ \mathrm{sgn}(\beta_s) d\beta_s, \qquad t \ge 0,$$

is a martingale such that $S_{\tau_r} = e^{iv\tau_r/r^2}$. Furthermore, $\sup_{0\le s\le \tau_r}|S_s| \le 1$. Using the martingale (R_t) introduced in the proof of Lemma 4.4 and (4.8), we obtain that

$$\begin{aligned}
\left|\mathbb{E}_{0,\theta}\Big(R_{\tau_r}S_{\tau_r}\Big) - R_0 S_0\right| &= \left|\mathbb{E}_{0,\theta}\langle R, S\rangle_{\tau_r}\right| \\
&\le K\frac{|u|}{\sqrt{r}}|w|\,\mathbb{E}_{0,\theta}\Big(R^*_{\tau_r}\int_0^{\tau_r}(1+|\beta_s|)^{-\nu}\,ds\Big) \\
&\le K\frac{|u|}{\sqrt{r}}|w|\sqrt{\mathbb{E}_{0,\theta}(R^*_{\tau_r})^2}\sqrt{\mathbb{E}\Big(\int_0^{\tau_r}(1+|\beta_s|)^{-\nu}\,ds\Big)^2} \\
&\le c\frac{|u|\sqrt{|v|}}{\sqrt{r}},
\end{aligned}$$

by means of (4.10) and (2.6). Therefore we have shown that

$$\Big|\mathbb{E}_{0,\theta}\left(e^{iu\frac{N(\tau_r)}{\sqrt{r}}+\frac{u^2}{2r}(\langle N,N\rangle_r-C_9 r)}e^{iv\frac{\tau_r}{r^2}}\right)-e^{-C_9\frac{u^2}{2}}\exp(-(1-i\,\mathrm{sgn}(v))\sqrt{|v|})\Big|\le c\,\frac{|u|\sqrt{|v|}}{\sqrt{r}},$$

which together with (4.7) implies Lemma 4.5. □

Proof of Proposition 3.2: Let $t_k=k^\rho$ with $\rho>20$ and $k\ge 100$. Define

$$\eta_k=\Big(\frac{N(\tau_{t_k})-N(\tau_{t_{k-1}})}{\sqrt{t_k-t_{k-1}}},\frac{\sup_{\tau_{t_{k-1}}\le s\le\tau_{t_k}}\beta_s}{t_k-t_{k-1}}\Big)\equiv(\eta_k^{(1)},\eta_k^{(2)}).$$

The strong Markov property implies that for $z=(u,v)\in\mathbb{R}^2$, we have

$$\mathbb{E}\Big(e^{iz\cdot\eta_k}\,|\,\mathcal{F}_{\tau_{t_{k-1}}};\,\dot\gamma(\tau_{t_{k-1}})=\theta\Big)=\mathbb{E}_{0,\theta}\Big(e^{i\frac{u}{\sqrt{r}}N(\tau_r)+i\frac{v}{r}\overline{\beta}_{\tau_r}}\Big),$$

with $r=t_k-t_{k-1}$. Let

$$\mathbf{g}(u,v)=e^{-C_9\frac{u^2}{2}}\,\mathbb{E}e^{iv\overline{\beta}(\tau_1)},\qquad u,v\in\mathbb{R},$$

be the joint Fourier transform of a Gaussian variable $\mathcal{N}(0,C_9)$ and an independent copy of $\overline{\beta}(\tau_1)$, whose law has been given in (2.9).

Applying Lemmas 4.4 and 4.1 with $\Theta_k=k^{\frac{\rho-1}{16}}$, we get that for all $|u|,|v|\le\Theta_k$,

$$\mathbb{E}\Big|\mathbb{E}\Big(e^{iz\cdot\eta_k}\,|\,\mathcal{F}_{\tau_{t_{k-1}}}\Big)-\mathbf{g}(z)\Big|\le Kk^{-\frac{3(\rho-1)}{8}}.$$

It follows from the Gaussian tail and the distribution of $\overline{\beta}(\tau_1)$ given in (2.9) that

$$\mathbf{G}_k\Big(z:|z|\ge\frac{\Theta_k}{4}\Big)\le K\,k^{-(\rho-1)/16}.$$

Hence we may construct a sequence $\{Z_k=(Z_k^{(1)},Z_k^{(2)}),k\ge1\}$ of i.i.d. variables and a version of $\{\eta_k=(\eta_k^{(1)},\eta_k^{(2)}),k\ge1\}$ in a sufficiently large probability space such that the two sequences $(Z_k^{(1)})$ and $(Z_k^{(2)})$ are *independent* and that

$$\begin{aligned}Z_k^{(1)}&\overset{\text{law}}{=}\mathcal{N}(0,C_9(\Phi)),\qquad Z_k^{(2)}\overset{\text{law}}{=}\overline{\beta}(\tau_1),\\ \mathbb{P}\Big(|\eta_k-Z_k|\ge\alpha_k\Big)&\le\alpha_k,\\ \alpha_k&\le c\,k^{-(\rho-1)/16}\log k.\end{aligned}$$

The Borel-Cantelli lemma yields that almost surely for all large k, $|\eta_k-Z_k|\le\alpha_k$. Hence

$$N(\tau_{t_n})=\sum_{k=1}^n\eta_k^{(1)}\sqrt{t_k-t_{k-1}}=\sum_{k=1}^n Z_k^{(1)}\sqrt{t_k-t_{k-1}}+\Phi_n^{(1)},\tag{4.14}$$

$$\overline{\beta}(\tau_{t_n})=\max_{1\le k\le n}(\eta_k^{(2)}(t_k-t_{k-1}))=\max_{1\le k\le n}(Z_k^{(2)}(t_k-t_{k-1}))+\Phi_n^{(2)},\tag{4.15}$$

where the error terms $\Phi_n^{(1)}$ and $\Phi_n^{(2)}$ can be estimated as follows: almost surely as $n\to\infty$,

$$
\begin{aligned}
|\Phi_n^{(1)}| &\le \sum_1^n \alpha_k\sqrt{t_k-t_{k-1}}+O(1)\le O(n^{\frac{7\rho+9}{16}}\log n)\le O(t_n^{\frac{7}{16}+\frac{9}{16\rho}}\log t_n),\\
|\Phi_n^{(2)}| &\le \max_{1\le k\le n}(\alpha_k(t_k-t_{k-1}))+O(1)\le O(t_n^{\frac{15}{16}-\frac{15}{16\rho}}\log t_n).
\end{aligned}
$$

Lemma 4.6 ([4]) *Let $S_i, i=1,2,3$ be separable Banach spaces. Let F be a distribution on $S_1\times S_2$ and let G be a distribution on $S_2\times S_3$ such that the second marginal of F equals the first marginal of G. Then there exist three random variables Z_1, Z_2 and Z_3 defined on some probability space such that*

$$(Z_1,Z_2)\overset{\text{law}}{=}F,\qquad (Z_2,Z_3)\overset{\text{law}}{=}G.$$

Using repeatedly Lemma 4.6, we may rewrite (4.14) and (4.15) as follows: Possibly in an enlarged probability space, we may define a version of $(N(\tau_r),\overline{\beta}(\tau_r))$ and a Brownian motion Y and a process T_r such that Y and T are *independent*, T has the same law as $\overline{\beta}(\tau_\cdot)$ and

$$
\begin{aligned}
|N(\tau_{t_n})-\sqrt{C_9(\Phi)}\,Y(t_n)| &\le O(t_n^{\frac{7}{16}+\frac{9}{16\rho}}\log t_n), &(4.16)\\
|\overline{\beta}(\tau_{t_n})-T(t_n)| &\le O(t_n^{\frac{15}{16}-\frac{15}{16\rho}}\log t_n). &(4.17)
\end{aligned}
$$

Recall the following result on the increments of a standard Brownian motion (cf. [12], Theorem 1.2.1): For a non-decreasing function $0<a_t\le t$ such that $t/a_t\uparrow+\infty$, we have

$$\limsup_{t\to\infty}\frac{1}{\sqrt{2a_t(\log(t/a_t)+\log\log t)}}\sup_{0\le s\le t-a_t}\sup_{0\le v\le a_t}|Y(s+v)-Y(s)|=1,\qquad \text{a.s.}\quad (4.18)$$

Using (4.2), (4.3) and (4.18), we obtain that almost surely for all large k ($t_k=k^\rho$), we have

$$
\begin{aligned}
\sup_{\tau(t_k)\le s\le\tau(t_{k+1})}|N(s)-N(\tau(t_k))| &= \sup_{\langle N\rangle(\tau(t_k))\le u\le\langle N\rangle(\tau(t_{k+1}))}|B(u)-B(\langle N\rangle(\tau(t_k)))|\\
&\le 3\sqrt{C_9\,(t_{k+1}-t_k)\,\log\log k}\\
&\le t_k^{\frac12-\frac{1}{2\rho}}\log k.
\end{aligned}
$$

The same holds for the Brownian motion Y:

$$\sup_{\tau(t_k)\le s\le\tau(t_{k+1})}|Y(\ell_s)-Y(t_k)|\le t_k^{\frac12-\frac{1}{2\rho}}\log k.$$

It follows that almost surely for all large r, $\tau_{t_k}\le r<\tau_{t_{k+1}}$, we have $t_k\le\ell_r\le 2\sqrt{r\log\log r}$ and

$$
\begin{aligned}
N(r) &= N(\tau_{t_k})+O(t_k^{\frac12-\frac{1}{2\rho}}\log k)=\sqrt{C_9(\Phi)}\,Y(t_k)+O(t_k^{\frac12-\epsilon})\\
&= \sqrt{C_9(\Phi)}\,Y(\ell(r))+O(r^{\frac14-\frac{\epsilon}{2}}),\ \text{a.s.}, &(4.19)
\end{aligned}
$$

with $\epsilon < \min(\frac{1}{16} - \frac{9}{\rho}, \frac{1}{2\rho})$.

We also need the increments of the process $\mathbf{e}(t) = \ell(\sigma(t/2))$. Remark that for $r > 10$,

$$\mathbb{P}\Big(\ell(\sigma(r + 2r^{1-\epsilon/2})) - \ell(\sigma(r - 2r^{1-\epsilon/2})) \geq r^{1-\epsilon/3}\Big) \leq \mathbb{P}\Big(\ell(\sigma(4r^{1-\epsilon/2})) \geq r^{1-\epsilon/3}\Big) = e^{-r^{1/6}/8},$$

since $\ell(\sigma(t))$ is exponentially distributed with mean $2t$. It is routine to apply the Borel-Cantelli lemma and the monotonicity and obtain that almost surely for all large r,

$$\ell(\sigma(r + r^{1-\epsilon/2})) - \ell(\sigma(r - r^{1-\epsilon/2})) \leq r^{1-\epsilon/3}. \tag{4.20}$$

Similarly, we have that almost surely for all large r,

$$\ell(\sigma(r + 2\log r)) - \ell(\sigma(r - 2\log r)) \leq 16\log^2 r. \tag{4.21}$$

Let $\Gamma \overset{\text{def}}{=} T^{-1}$ denote the inverse process of T and define $\widetilde{\mathbf{e}}(r) = \Gamma(r/2)$. Hence $\widetilde{\mathbf{e}}(\cdot)$ has the same law as $\mathbf{e}(\cdot)$. Using (4.17) and (4.20), we can show that for all large r,

$$|\Gamma_r - \ell(\sigma_r)| \leq r^{1-\epsilon/6}.$$

In fact, for large r, there exists a n such that $\overline{\beta}(\tau_{t_{n-1}}) \leq r < \overline{\beta}(\tau_{t_n})$, then $t_{n-1} \leq \ell(\sigma_r) < t_n$. By (4.17), $r < \overline{\beta}(\tau_{t_n}) < T(t_n) + t_n^{1-\epsilon}$. Since T^{-1} has the same law as $\ell(\sigma(\cdot))$, we deduce from (4.20) that $T_r^{-1} \leq T^{-1}(T_{t_n} + t_n^{1-\epsilon}) < t_n + (T(t_n))^{1-\epsilon/3} < t_n + t_n^{1-\epsilon/4}$ since $t_n^{1-\epsilon} < (T_{t_n})^{1-\epsilon/2}$. Similarily, we have $T_r^{-1} > t_{n-1} - t_{n-1}^{1-\epsilon/4}$. Hence

$$|\Gamma_r - \ell(\sigma_r)| \leq t_n - t_{n-1} + 2t_n^{1-\epsilon/4} \leq 3t_n^{1-\epsilon/4} \leq r^{1-\epsilon/6}.$$

Assembling (4.18) and (4.21) and applying (4.19), we have that almost surely for all large r and for all $\sigma(r - 2\log r) \leq t \leq \sigma(r + 2\log r)$,

$$N_t \;=\; \sqrt{C_9(\Phi)}\, Y(\ell(\sigma_r)) + O(r^{(1-\epsilon)/2}) = \sqrt{C_9(\Phi)}\, Y(\Gamma_r) + o(r^{\frac{1}{2} - \frac{\epsilon}{14}}),$$

proving Proposition 3.6 for the case $n = 1$, and we may choose $\rho = 160$, $\epsilon = \frac{1}{400}$ and $\delta = \epsilon/16 > 10^{-5}$. $\square$

Proof of Proposition 3.1: The proof goes in the same way as Proposition 3.2, by considering the sequence of vectors

$$\Big(\frac{N(\tau_{t_k}) - N(\tau_{t_{k-1}})}{\sqrt{t_k - t_{k-1}}}, \frac{\tau_{t_k} - \tau_{t_{k-1}}}{(t_k - t_{k-1})^2}\Big), \qquad t_k = k^\rho,\; \rho > 10,$$

and by applying Lemma 4.5. The details are omitted.

Let us compute the correlation matrix. Assume (3.1) for $F^{(1)}, ..., F^{(n)}$. Define N_j the martingale from $F^{(j)}$ in the same way as N was defined from Φ. Then by using Lemma 3.5, we have

$$\langle N_j, N_k\rangle(t) = C_{17}(F^{(j)}, F^{(k)})\; \ell(t) + o(t^{1/4+\epsilon}), \qquad \text{a.s.},$$

where

$$\begin{aligned} C_{17}(F^{(j)}, F^{(k)}) &\stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{\mathbb{R}} dx \int_0^{2\pi} d\theta \left[F_1^{(j)}(X_s)F_1^{(k)}(X_s) + F_2^{(j)}(X_s)F_2^{(k)}(X_s)\right] \\ &= \frac{1}{4}\Big[C_9(F^{(j)}+F^{(k)}) - C_9(F^{(j)}-F^{(k)})\Big], \end{aligned}$$

Therefore, the correlation matrix is given by

$$C_{10}(F^{(j)}, F^{(k)}) \stackrel{\text{def}}{=} \frac{C_{17}(F^{(j)}, F^{(k)})}{\sqrt{C_9(F^{(j)})C_9(F^{(k)})}} = \frac{1}{4}\frac{C_9(F^{(j)}+F^{(k)}) - C_9(F^{(j)}-F^{(k)})}{\sqrt{C_9(F^{(j)})C_9(F^{(k)})}}.$$

□

5 Some applications

5.1 Winding numbers

Recall (1.5). The process $\theta(\cdot)$ describes the total winding angle of W around the origin. Define

$$\Pi(t) \stackrel{\text{def}}{=} \int_0^t f(|W(s)|)\, d\theta(s), \qquad t \geq 0,$$

Where $f : \mathbb{R}^+ \to \mathbb{R}$ is a function satisfying

$$|f(x)| \leq \frac{K}{(1+|\log x|)^\nu}, \qquad x > 0$$

with some $K > 0$ and $\nu > \frac{3}{2}$.

Using the skew-product representation (1.5) and (1.6), we have

$$\Pi(t) = \int_0^{\Xi(t)} f(e^{\beta_s})\ d\gamma_s.$$

In view of (3.26), we deduce from Proposition 3.2 that in a suitable probability space

$$\Pi(t) = \left(\int_0^\infty \frac{f^2(r)}{r}\, dr\right)^{1/2} Y(\widetilde{\mathbf{e}}(\log t)) + o((\log t)^{\frac{1}{2}-\delta}, \qquad \text{a.s.},$$

for a one-dimensional Brownian motion Y, independent of the process $\widetilde{\mathbf{e}}$. This allows us to obtain the upper and lower functions for $\Pi(t)$ as in Corollary 1.3, the details are omitted.

Let $0 < r_1 < r_2 < \infty$, then the particular function $f(x) = 1_{(r_1 \leq x \leq r_2)}$ gives winding in a ring. See Messulam and Yor [21] for studies of convergence in law, Shi [28] (case $r_1 = 0$ or $r_2 = \infty$) and Dorofeev [13] for upper and lower functions.

5.2 Additive functionals of a Cauchy process

Let $(C(t), t \geq 0)$ be a symmetric Cauchy process on $\mathbb{R}$, which means a Lévy process with marginal distribution

$$\mathbb{P}\Big(C(t) \in dx\Big) = \frac{t\,dx}{\pi(t^2 + x^2)}, \qquad x \in \mathbb{R}, \qquad t > 0.$$

Several interesting geometric quantities such as level crossings have been studied in [24] and [5]. Let $f : \mathbb{R} \to \mathbb{R}$. Kasahara [18] studied the additive functional $\int_0^\cdot f(C(s))ds$ and obtained its second-order behavior similar to Theorem B. The goal of this paragraph is to obtain an analogue Theorem 1.1 for Cauchy process, by using Spitzer's representation of $C(\cdot)$: Let $W = (W_1, W_2)$ be the planar Brownian motion starting from $(1, 0)$. Denote by $L_2(\cdot)$ the local time at 0 of W_2 and $\tau_2(\cdot)$ the inverse process of $L_2(\cdot)$. Then the process

$$C(t) \stackrel{\text{def}}{=} W_1(\tau_2(t)), \qquad t \geq 0,$$

is a symmetric Cauchy process starting from 1 (the starting point does not influence our result). Recall that X is the Brownian motion on the cylinder: $X_s = (\beta_s, \dot{\gamma}_s), s \geq 0$.

Lemma 5.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that for some constants $K > 0$ and $\nu > 2$, we have*

$$|f(x)| \leq \frac{K}{|x|\,(1 + |\log|x||)^\nu}, \qquad x \in \mathbb{R}\backslash\{0\}. \tag{5.1}$$

Then

$$\begin{aligned}
&\int_0^t f(C(s))ds \\
&= C_{18}(f)\,\ell(\Xi(\tau_2(t))) + \int_0^{\Xi(\tau_2(t))} \Big((\frac{\partial F_0}{\partial x}(X_s) + q_0(\beta_s))d\beta_s + \frac{\partial F_0}{\partial \theta}(X_s)d\gamma_s \Big) \\
&\qquad + F_0(X_0) - F_0(X(\Xi(\tau_2(t)))) - \int_0^{\beta(\Xi(\tau_2(t)))} dx q_0(x),
\end{aligned}$$

where, using the function Ψ given in (3.14), we define

$$\begin{aligned}
C_{18}(f) &\stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{\mathbb{R}} f(x)\,dx \\
F_0(x,\theta) &\stackrel{\text{def}}{=} \int_{\mathbb{R}} dx' e^{x'} \Big(f(e^{x'})\Psi(x - x', \theta) + f(-e^{x'})\Psi(x - x', \theta - \pi)\Big), \ x \in \mathbb{R}, \theta \in [0, 2\pi] \\
q_0(x) &\stackrel{\text{def}}{=} \frac{1}{\pi}\int_{|y| \geq e^x} f(y)dy, \ \ \textit{if } x > 0; \ q_0(x) \stackrel{\text{def}}{=} -\frac{1}{\pi}\int_{|y| < e^x} f(y)dy, \ \ \textit{if } x \leq 0.
\end{aligned}$$

Proof: Assume without loss of generality that $f \in \mathcal{C}^2(\mathbb{R} \to \mathbb{R})$ has compact support. Let $f_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_\epsilon(x_1, x_2) \stackrel{\text{def}}{=} f(x_1)\, \frac{1}{\epsilon}\, 1_{(0<x_2<\epsilon)}, \qquad x_1, x_2 \in \mathbb{R}.$$

Define $g_\epsilon : G \to \mathbb{R}$ by

$$g_\epsilon(x,\theta) \stackrel{\text{def}}{=} e^{2x} f_\epsilon(e^{x+i\theta}) \equiv e^{2x} f_\epsilon(e^x \cos\theta, e^x \sin\theta), \qquad (x,\theta) \in G.$$

Therefore using Spitzer's representation and change of variable,

$$\begin{aligned} \int_0^t f(C(s))\, ds &= \int_0^{\tau_2(t)} f(W_1(u))\, dL_2(u) \\ &= \lim_{\epsilon \to 0} \int_0^{\tau_2(t)} f_\epsilon(W_1(u), W_2(u))\, du \qquad \text{(approximation of local time)} \\ &= \lim_{\epsilon \to 0} \int_0^{\Xi(\tau_2(t))} g_\epsilon(X_s)\, ds. \qquad \text{(1.5) and (1.6)} \end{aligned}$$

Recall (3.12), (3.13), (3.16), (3.19) and (3.20). We define h_ϵ, F_ϵ, q_ϵ and M_ϵ from g_ϵ in the same way as h, F, q, M was defined from g. Then (3.22) holds for g_ϵ instead of g, which implies the Lemma by letting $\epsilon \to 0$. $\Box$

Applying (3.26) gives that almost surely for all large t, $\sigma(\log t - 3\log\log t) \le \Xi(\tau_2(t)) \le \sigma(\log t + 3\log\log t)$, since $\frac{t^2}{\log t} \le \tau_2(t) \le t^2 \log^3 t$. Applying Proposition 3.2 to the above lemma and using (4.21), we obtain

Proposition 5.2 *Assume that $f_1, ...f_n : \mathbb{R} \to \mathbb{R}$ are n measurable functions such that for some constants $K > 0$ and $\nu > \frac{5}{2}$, we have*

$$|f_j(x)| \le \frac{K}{|x|\,(1 + |\log|x||)^\nu}, \qquad 1 \le j \le n,\ x \in \mathbb{R}\backslash\{0\}.$$

Then we may define a version of a Cauchy process $C(\cdot)$, an n-dimensional Brownian motion $Y = (Y_1, ..., Y_n)$ starting from 0 with covariance matrix $(C_{20}(f_j, f_k))_{1\le j,k\le n}$ and two inhomogeneous Lévy process $\mathbf{e}$ and $\widetilde{\mathbf{e}}$ such that Y and $\widetilde{\mathbf{e}}$ are independent, $\widetilde{\mathbf{e}}$ has the same law as $\mathbf{e}$ and such that almost surely for all large t, we have

$$\begin{aligned} \int_0^t f_j(C(s))ds - C_{18}(f_j)\,\mathbf{e}(2\log t) - C_{19}(f_j)\, Y_j(\widetilde{\mathbf{e}}(2\log t)) &= o((\log t)^{\frac{1}{2}-\delta}), \\ |\mathbf{e}(\log t) - \widetilde{\mathbf{e}}(\log t)| &= o((\log t)^{1-\delta}), \end{aligned}$$

for some positive constant $\delta > 0$ and

$$C_{19}(f) \quad \stackrel{\text{def}}{=} \quad \Big(- \frac{1}{2\pi^2} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx' f(x) f(x') \log|x - x'|^2$$

$$+\frac{2}{\pi^2} C_{18}(f) \int_{|x|\geq 1} f(x) \log|x| dx \Big)^{1/2},$$

$$C_{20}(f_j, f_k) \stackrel{\text{def}}{=} \frac{1}{4} \frac{(C_{19}(f_j + f_k))^2 - (C_{19}(f_j - f_k))^2}{C_{19}(f_j)\, C_{19}(f_k)}$$

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