Finite lattice packings and the Wulff–shape

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Abstract

We consider finite lattice packings $C^n + K$ of n copies of some centrally symmetric convex body K in E^d for large n. Assume that C^n is a subset of a lattice Λ, and ϱ is at least the covering radius; namely, $\Lambda + \varrho K$ covers the space. The parametric density $\delta(C^n, \varrho)$ is defined by $\delta(C^n, \varrho) = n \cdot V(K)/V(\operatorname{conv} C^n + \varrho K)$. We show that if $\delta(C^n, \varrho)$ is minimal for n large then the shape of $\operatorname{conv} C^n$ is approximately given by Wulff's condition, well-known from crystallography. Thus maximizing parametric density is equivalent to optimizing a certain Gibbs–Curie energy. We also prove that in case of lattice packings of K (allowing any packing lattice), for large n the optimal shape with respect to the parametric density is approximately a Wulff–shape associated to some densest packing lattice of K.

1 Introduction

Throughout this paper, E^d denotes the d–dimensional Euclidean space with Euclidean norm $||\cdot||$, inner product $\langle\cdot,\cdot\rangle$ and unit sphere S^{d-1}. $\mathcal{K}^d$ denotes the set of all convex bodies in E^d with nonempty interior and $\mathcal{K}_0^d$ denotes the subset of $\mathcal{K}^d$ consisting of all bodies which are symmetric to the origin. For a set $M \subset E^d$ we write $\dim M$ for the dimension of its affine hull, $\operatorname{conv} M$ for its convex hull, $V(M)$ for its volume in E^d and $|M|$ for its volume with respect to its affine hull. If $M \subset E^d$ is finite, we write $\#M$ for its cardinality. For a k–dimensional lattice Λ in E^d, we denote by $\det \Lambda$ the k–dimensional volume of an elementary cell of Λ. We denote by $\mathcal{P}_n(\Lambda)$ the family of all subsets of Λ with exactly n elements.

For notions in convexity, our standard reference is [S] but the main ideas can be followed for example using the classic [BF] of Bonnesen and Fenchel.

We consider lattice arrangements of some $K \in \mathcal{K}^d$. Λ is a packing lattice for $K \in \mathcal{K}^d$ if $x + K$ and $y + K$ do not overlap for $x \neq y \in \Lambda$ and a finite lattice packing is a finite subset of a packing lattice. We write $\mathcal{P}_n(K)$ for the family of all lattice packings of n copies of K; namely

$$\mathcal{P}_n(K) = \bigcup_{\Lambda \text{ packing lattice}} \mathcal{P}_n(\Lambda).$$

By Mahler's selection theorem, there always exists a packing lattice Λ with minimal determinant. We call such a Λ a densest packing lattice, and $\delta(K) = V(K)/\det \Lambda$ is the optimal packing density of infinite lattice packings (see [CS] or [GL]).

We use the following parametric densities of finite lattice packings:

Definition 1 *For $K \in \mathcal{K}^d$, a packing lattice Λ of K, $n \in \mathbb{N}$, $\varrho > 0$ and $C \subset E^d$ finite let*

$$\delta(K, C, \varrho) = \frac{\#C \cdot V(K)}{V(\operatorname{conv} C + \varrho K)}, \tag{1}$$

$$\delta(K, \Lambda, n, \varrho) = \max \left\{ \delta(K, C, \varrho) : C \in \mathcal{P}_n(\Lambda) \right\}, \tag{2}$$

$$\delta(K, n, \varrho) = \max\{\delta(K, C, \varrho) : C \in \mathcal{P}_n(K)\}, \tag{3}$$

$$\delta(K, \Lambda, \varrho) = \limsup_{n \to \infty} \delta(K, \Lambda, n, \varrho), \tag{4}$$

$$\delta(K, \varrho) = \limsup_{n \to \infty} \delta(K, n, \varrho). \tag{5}$$

We write $C_o^n(K, \Lambda, \varrho)$ for a $C \in \mathcal{P}_n(\Lambda)$ which satisfies $\delta(K, C, \varrho) = \delta(K, \Lambda, n, \varrho)$, and $C_o^n(K, \varrho)$ for a $C \in \mathcal{P}_n(K)$ which satisfies $\delta(K, C, \varrho) = \delta(K, n, \varrho)$. We are interested in sequences $\{C_o^n(K, \Lambda, \varrho)\}$ and $\{C_o^n(K, \varrho)\}$.

Clearly we have $\delta(K, n, \varrho) = \max \{\delta(K, \Lambda, n, \varrho) : \Lambda \text{ packing lattice}\}$. Cluster like packings show that $\delta(K, \varrho) \geq \delta(K)$. The parametric packing densities were introduced in [BHW1] for general packings. An overview of results related to finite packings can be found in [GW].

In our context the following results are of interest: For small ϱ, $\delta(K, n, \varrho)$ is given by linear packings, the so–called "sausages" ([BHW1], [BHW2]), and the same holds for arrangements with respect to given Λ. On the other hand, denote by $\mu(\Lambda)$ the covering radius of Λ; namely, the smallest number such that $\Lambda + \mu K$ is a covering. Then $\delta(K, \Lambda, \varrho) = V(K)/\det \Lambda$ for $\varrho \geq \mu(\Lambda)$ (see [H]). In addition, for $\varrho \geq 3$ and $K \in \mathcal{K}_0^d$ we have $\delta(K, n, \varrho) \geq \delta(K)$, and for the case of the d–ball B^d we have $\delta(B^d, n, \varrho) \geq \delta(B^d)$ for $\varrho \geq \sqrt{21}/2$ ([H]).

In the greater part of this paper we are interested in the case where ϱ is not too small. In particular, let the critical radius $\varrho_c(K, \Lambda)$ ($\varrho_c(K)$) be the minimal ϱ satisfying $\delta(K, \Lambda, \varrho) = V(K)/\det \Lambda$ ($\delta(K, \varrho) = \delta(K)$). Note that $\varrho > \varrho_c$ implies that the inradius $r(\operatorname{conv} C_o^n(K, \Lambda, \varrho))$ ($r(\operatorname{conv} C_o^n(K, \varrho))$) tends to ∞ (see [ABB] or [BS]). For $d = 2$ and 3, ϱ_c is determined in [BHW1] and [ABB], respectively. For $d \geq 4$, we have only the bounds mentioned above; namely, $\varrho_c(K, \Lambda) \leq \mu(K, \Lambda)$ and $\varrho_c(K) \leq 3$.

We denote the support function of K by $h_K(\cdot)$ (see [S] or [BF]). Observe that if K is the unit ball then $h_K(u) = 1$ for $u \in S^{d-1}$.

Let $\varrho > \varrho_c(\Lambda, K)$, and set

$$P^*_{\varrho,\Lambda,K} = \bigcap_{u \in S^{d-1},\, \dim u^\perp \cap \Lambda = d-1} \left\{ x \,\middle|\, \langle u, x \rangle \leq \varrho \cdot h_K(u) - \frac{\det \Lambda}{2 \det(u^\perp \cap \Lambda)} \right\},$$

where $u^\perp$ denotes the linear $(d-1)$–plane normal to u. $P^*_{\varrho,\Lambda,K}$ is always a polyhedron (see Section 3). More specifically $P_{\varrho,\Lambda,K}$ is Wulff's shape minimizing the Gibbs–Curie–energy defined by the right hand side ([D]). It is convenient to define $P_{\varrho,\Lambda,K}$ as the dilatate of $P^*_{\varrho,\Lambda,K}$ having circumradius 1.

Our main result is that this Wulff–shape describes the shape of optimal large packings with respect to the parametric density. We understand limits of compact sets with respect to the Hausdorff metric. We remark that by the Blaschke selection theorem, every sequence of uniformly bounded convex bodies contains a convergent subsequence.

To associate with the sequence $\{C^n_o(K, \Lambda, \varrho)\}$ a uniformly bounded sequence of convex bodies we use the following concept: For a bounded set $M \subset E^d$ we denote by $R(M)$ its circumradius and by $m(M)$ the center of its circumball.

Definition 2 *The* shape $\widetilde{M}$ *of the bounded set* $M \subset E^d$ *is defined by*

$$\widetilde{M} = (\operatorname{conv} M - m(M))/R(M).$$

Note that [ABB] actually proves that $\widetilde{C}^n_o(K, \Lambda, \varrho)$ $(\widetilde{C}^n_o(K, \varrho))$ is uniformly bounded for $\varrho > \varrho_c(K, \Lambda)$ $(\varrho > \varrho_c(K))$. Then our main theorem becomes:

Theorem 3 *Let* $K \in \mathcal{K}^d_0$.

[a] Let Λ *be a packing lattice of* K *and let* $\varrho > \varrho_c(K, \Lambda)$. *Then*

[i]

$$\lim_{n \to \infty} \widetilde{C}^n_o(K, \Lambda, \varrho) = P_{\varrho,\Lambda,K}.$$

[ii]

$$\delta(K, \Lambda, n, \varrho) = \frac{V(K)}{\det \Lambda} + n^{-1/d} \cdot \frac{V(K)\Omega(P_{\varrho,\Lambda,K})}{(\det \Lambda)^{1/d}} + o(n^{-1/d}),$$

where the constant $\Omega(P_{\varrho,\Lambda,K})$ *is defined just before Lemma 10.*

[b] Let $\varrho > \varrho_c(K)$.

[i] Then every convergent subsequence $\{\widetilde{C}_o^{n'}(K,\varrho)\}$ of $\{\widetilde{C}_o^{n}(K,\varrho)\}$ satisfies

$$\lim_{n\to\infty} \widetilde{C}_o^{n'}(K,\varrho) = P_{\varrho,\Lambda,K}$$

where Λ is a densest packing lattice of K.

[ii]

$$\delta(K,n,\varrho) = \frac{V(K)}{\det\Lambda} + n^{-1/d}\cdot\frac{V(K)\Omega(P_{\varrho,\Lambda,K})}{(\det\Lambda)^{1/d}} + o(n^{-1/d}),$$

where Λ is a suitable densest packing lattice.

We remark that it was first suggested by P. Gruber to use the Wulff-shape to describe the limit shape of large optimal packings (private communication).

Further at this point it is worth while to mention that, for a fixed lattice and if K is the unit ball B^d, large lattice packings were also studied by Wills [W1], [W2]. He introduced the concept of a density deviation and showed that with respect to this deviation the P_{ϱ,Λ,B^d} are optimal. More precisely, the density deviation measures the first order term (as a function of $1/\lambda$) of the parametric density of the lattice points in λQ, λ large, where Q is a given lattice polytope. The results of Wills show that lattice polytopes whose shape is closer to the shape of P_{ϱ,Λ,B^d} induce packings with a smaller density deviation. Using the Ehrhart polynomial it is easy to construct series of lattice polytopes $\{Q^n\}$ with $r(Q^n)\to\infty$, such that the size of the second order term of $1/\lambda$ is large compared to the first order term (see [W3]). Thus the results on density deviation have no immediate implication on the shape of the $C_o^n(B^d,\Lambda,\varrho)$. For the densities $\delta(B^d,\Lambda,n,\varrho)$ these results yield in our context

$$\liminf n^{1/d}\left(\frac{V(K)}{\det\Lambda} - \delta(B^d,\Lambda,n,\varrho)\right) \le \frac{V(K)\Omega(P_{\varrho,\Lambda,B^d})}{(\det\Lambda)^{1/d}}.$$

Schnell [Sc] generalizes the notion of parametric density to periodic arrangements, and considers packings of different sizes of balls (atoms).

Following the concept introduced in [ABB] we also consider lower dimensional packings. For $M\subset E^d$ and a plane H we denote by $\pi_H(M)$ the orthogonal projection of M on H and by $H^\perp$ the orthogonal complement of H.

Definition 4 *Let $K \in \mathcal{K}^d$, L a k–dimensional packing lattice of K contained in a k–dimensional linear subspace H of E^d, Λ a d–dimensional packing lattice of K, and $\varrho > 0$. Then*

1. $\delta(K, L, \varrho) = \dfrac{V(K)}{\varrho^{d-k} \det L |\pi_{H^\perp}(K)|}$,
2. $\delta_k(K, \Lambda, \varrho) = \max\{\delta(K, L, \varrho) : L$ *k–dimensional sublattice of* $\Lambda\}$,
3. $\delta_k(K, \varrho) = \max\{\delta(K, L, \varrho) : L$ *k–dimensional packing lattice of* $K\}$.

In the special case $K = B^d$ and $\varrho = 1$ the quantities $\delta_k(B^d, 1)$ were mentioned in [FGW]. We observe that for fixed k the lattice L which gives $\delta_k(K, \varrho)$ $(\delta_k(K, \Lambda, \varrho))$ is independent of ϱ.

By means of the $\delta_k(K, \Lambda, \varrho)$ we can distinguish certain parameters. Let $1 \le k_1 < k_2 \le d$. Then we have for $\delta_{k_1}(K, \Lambda, \bar{\varrho}) \ge \delta_{k_2}(K, \Lambda, \bar{\varrho})$ that $\delta_{k_1}(K, \Lambda, \varrho) > \delta_{k_2}(K, \Lambda, \varrho)$ for all $\varrho < \bar{\varrho}$ and for $\delta_{k_1}(K, \Lambda, \widetilde{\varrho}) \le \delta_{k_2}(K, \Lambda, \widetilde{\varrho})$ that $\delta_{k_1}(K, \Lambda, \varrho) < \delta_{k_2}(K, \Lambda, \varrho)$ for all $\varrho > \widetilde{\varrho}$. Thus we obtain a sequence of parameters

$$0 = \varrho_0(K, \Lambda) < \varrho_1(K, \Lambda) \le \cdots \le \varrho_{d-1}(K, \Lambda) < \varrho_d(K, \Lambda) = \infty$$

such that for $\varrho \in (\varrho_{k-1}(K, \Lambda), \varrho_k(K, \Lambda))$ we have $\delta_k(K, \Lambda, \varrho) = \max\{\delta_1(K, \Lambda, \varrho), \ldots, \delta_d(K, \Lambda, \varrho)\}$.

In the same way we obtain a sequence of parameters

$$0 = \varrho_0(K) < \varrho_1(K) \le \cdots \le \varrho_{d-1}(K) < \varrho_d(K) = \infty$$

such that for $\varrho \in (\varrho_{k-1}(K), \varrho_k(K))$ we have $\delta_k(K, \varrho) = \max\{\delta_1(K, \varrho), \ldots, \delta_d(K, \varrho)\}$. We showed in [ABB] that in a certain sense these numbers which are defined by lower dimensional packings govern the behaviour of large optimal packings. One clearly has $\varrho_c(K) \ge \varrho_{d-1}(K)$ and $\varrho_c(K, \Lambda) \ge \varrho_{d-1}(K, \Lambda)$. Further one conjectures that $\varrho_c(K) = \varrho_{d-1}(K)$ and $\varrho_c(K, \Lambda) = \varrho_{d-1}(K, \Lambda)$ but it is known only for $d \le 3$ (see [BHW1] and [ABB]).

We want to study the behaviour of $\operatorname{conv} C^n$ for certain sequences $\{C^n\}$ with $\# C^n = n$. For $M \subset E^d$ and $1 \le i \le d$ we define the i–th inradius $r_i(M)$ as the radius of the largest i–dimensional ball contained in M (see [P] or [BH]). For $k > d$ we set $r_k(M) = 0$. Thus $r_d(M)$ is the usual inradius and $r_1(M)$ half of the diameter $\operatorname{diam}(M)$. We are going to describe the asymptotic properties of optimal packings by the following concepts:

Definition 5 *Let $\{C^n\}$ be a sequence with $\# C^n = n$.*

1. *$\{C^n\}$ has index of unboundedness k, if $\limsup_{n\to\infty} r_{k+1}(\operatorname{conv} C^n) < \infty$ but $\lim_{n\to\infty} r_k(\operatorname{conv} C^n) = \infty$.*

2. *$\{C^n\}$ is k–flat, if $\{C^n\}$ has index of unboundedness k and there exists an $n_0 \in \mathbb{N}$ such that $r_{k+1}(\operatorname{conv} C^n) = 0$ for all $n \geq n_0$.*

3. *$\{C^n\}$ is k–uniform, if $\{C^n\}$ is k–flat and*

$$\limsup_{n\to\infty} \frac{R(\operatorname{conv} C^n)}{r_k(\operatorname{conv} C^n)} < \infty.$$

In other words the sequence $\{C^n\}$ is k–flat, if the $\operatorname{conv} C^n$ become large in k independent directions and the dimension of $\operatorname{conv} C^n$ is k for all large n, and $\{C^n\}$ is k–uniform, if $\operatorname{conv} C^n$ grows in all possible directions in a k-plane with essentially the same speed. Observe that if C^n is a subset of a given lattice or $C^n + K$ is a packing then some subsequence of C^n has index of unboundedness k for certain $1 \leq k \leq d$. In this notation the earlier results give for all $K \in \mathcal{K}_0^d$ the existence of a $\bar{\varrho}(K) > 0$ such that for $\varrho \leq \bar{\varrho}(K)$ $\{C_o^n(K,\varrho)\}$ is 1–flat (a series of sausages), and for $\varrho \geq 3$, $\{C_o^n(K,\varrho)\}$ is d–flat.

Now we state a generalization of Theorem 3 for flat sequences. If L is some linear k–plane and $u \in S^{d-1} \cap L$, we define $D = D(K,L) \subset L$ by the formula

$$h_D(u) = \frac{1}{2} \left| \pi_{L^\perp + \mathbb{R}u}(K) \right|.$$

Theorem 6 *Let $K \in \mathcal{K}_0^d$ and Λ be a packing lattice of K. Assume that $\{C_o^n(K,\Lambda,\varrho)\}$ (or $\{C_o^n(K,\varrho)\}$) is k–flat. Then for every convergent subsequence $\{\widetilde{C}_o^{n'}(K,\Lambda,\varrho)\}$ of $\{\widetilde{C}_o^n(K,\Lambda,\varrho)\}$ ($\{\widetilde{C}_o^{n'}(K,\varrho)\}$ of $\{\widetilde{C}_o^n(K,\varrho)\}$) there exists a k–sublattice L of Λ (packing k–lattice) such that for $C_o^{n'} = C_o^{n'}(K,\Lambda,\varrho)$ ($C_o^{n'} = C_o^{n'}(K,\varrho)$) and $D = D(K,L)$, we have*

$$\lim_{n\to\infty} \widetilde{C}_o^{n'} = P_{\varrho,L,D}.$$

Clearly for given Λ there exist only finitely many choices for L. Furthermore by a result in [ABB] for $d \leq 3$ the sequence $\{C_o^n\}$ is k–flat and k–uniform for some k depending on ϱ.

Thus in the 3–dimensional case we have a complete description of the asymptotic behaviour of the shapes of optimal lattice packings.

Theorem 7 *Let $K \in \mathcal{K}_0^3$, $\varrho > 0$ and Λ be a packing lattice of K.*

[a] If $\varrho < \varrho_1(K,\Lambda)$ then for large n, $C_o^n(K,\Lambda,\varrho)$ is contained in some lattice line l (the "sausage" case).

[b] If $\varrho_1(K,\Lambda) < \varrho < \varrho_2(K,\Lambda)$ then for every convergent subsequence $\{\widetilde{C}_o^{n'}(K,\Lambda,\varrho)\}$ of $\{\widetilde{C}_o^{n}(K,\Lambda,\varrho)\}$ there exists a 2–lattice $L \subset \Lambda$ such that for the $D(K,L)$ defined above,

$$\lim_{n\to\infty} \widetilde{C}_o^{n'}(K,\Lambda,\varrho) = P_{\varrho,L,D}.$$

[c] If $\varrho > \varrho_2(K,\Lambda)$ then

$$\lim_{n\to\infty} \widetilde{C}_o^{n}(K,\Lambda,\varrho) = P_{\varrho,\Lambda,K}.$$

In particular, $\varrho_c(K,\Lambda) = \varrho_2(K,\Lambda)$.

In case of general lattice packings, one only replaces the l, L and Λ of Theorem 7 with some densest one, two and three dimensional infinite packing lattice of K. As the lattice of the densest lattice packing is not uniquely defined one has to take subsequences in the 3–dimensional case, too.

2 Mixed volumes

Let K and M be convex, compact sets. The mixed volumes $V(M,K;k)$, $k = 0,\dots,d$, of H. Minkowski are defined by the formula

$$V(\lambda M + \mu K) = \sum_{k=0}^{d} \binom{d}{k} \cdot V(M,K;k) \cdot \lambda^{d-k}\mu^k$$

for non–negative λ and μ. We review some basic properties of the mixed volumes following [S] and [BF]. The mixed volumes are continuous, homogeneous and monotonic in both variables and satisfy $V(M,K;k) = V(K,M;d-k)$ and $V(K) = V(M,K;d)$.

∂M denotes the relative boundary of M in $\operatorname{aff} M$. First we assume that $\operatorname{int} M \neq \emptyset$. For $x \in \partial M$, denote by u_x a unit exterior normal vector at x to ∂M. Then for any convex, compact set K with support function $h_K(u)$,

$$d \cdot V(M,K;1) = \int_{\partial M} h_K(u)\,du \tag{6}$$

where the integration on ∂M is with respect to the $(d-1)$–dimensional Hausdorff–measure. In particular, we have that $d \cdot V(M,B^d;1) = S(M)$ is the surface area of M. If M is a d-polytope with facets $F_1, \dots ,F_k$, and u_i denotes the exterior unit normal to F_i then

$$\int_{\partial M} f(u)\, du = \sum_{i=1}^{k} f(u_i) \cdot |F_i|.$$

If $L = \operatorname{aff} M$ has dimension k then (see [S], p.294),

$$\binom{d}{k} \cdot V(M, K; d-k) = |M| \cdot |\pi_{L^\perp}(K)|. \tag{7}$$

If in addition K is o-symmetric then (see [ABB])

$$\binom{d}{k-1} \cdot V(M, K; d-k+1) = \frac{1}{2} \int_{\partial M} |\pi_{L^\perp + \mathbb{R}u}(K)|\, du. \tag{8}$$

3 About the Wulff-shape

Let Λ be a lattice and for $u \in S^{d-1}$, set

$$\omega_\Lambda(u) = \begin{cases} \frac{\det \Lambda}{2 \det(u^\perp \cap \Lambda)} & \text{if } \dim(u^\perp \cap \Lambda) = d-1, \\ 0 & \text{otherwise.} \end{cases}$$

According to Ehrhart's formula (see [GL]), for a lattice polytope P and $\lambda \in \mathbb{N}$ the following formula holds:

$$\det \Lambda \cdot G(\lambda\, P) = \sum_{i=0}^{d} G_i(P) \cdot \lambda^i \tag{9}$$

where

$$G_d(P) = V(P) \quad \text{and} \quad G_{d-1}(P) = S_\Lambda(P).$$

Here the lattice surface area $S_\Lambda(P)$ is defined by

$$S_\Lambda(P) = \int_{\partial P} \omega_\Lambda(u)\, du = \sum \frac{|F(v)|}{\det(v^\perp \cap \Lambda)} \tag{10}$$

where $F(v)$ denotes the facet of P with normal v and summation is with respect to all facets.

Now, let K be an o-symmetric convex body and $\varrho > 0$.

By definition, no $(d-1)$–dimensional packing can be asymptotically better than "clusters" for $\varrho > \varrho_c(K,\Lambda)$. We deduce that if $\dim(u^\perp \cap \Lambda) = d-1$ for $u \in S^{d-1}$ then

$$2\varrho_c(K,\Lambda) \cdot h_K(u) \cdot \det(u^\perp \cap \Lambda) \geq \det \Lambda.$$

For $\varrho > \varrho_c(K,\Lambda)$, we rewrite the definition of $P^*_{\varrho,\Lambda,K}$ as

$$P^*_{\varrho,\Lambda,K} = \bigcap_{u \in S^{d-1}} \{x \mid \langle u, x\rangle \leq \varrho \cdot h_K(u) - \omega_\Lambda(u)\}.$$

The condition $\varrho > \varrho_c(K,\Lambda)$ implies that $P^*_{\varrho,\Lambda,K}$ is a convex body.

Next we prove that $P^*_{\varrho,\Lambda,K}$ is a polytope, and that the old and the new definition coincide. We need some preparatory statements. Let $u_1, u_2 \in S^{d-1}$, and assume that $\langle x, u_1\rangle = h_K(u_1)$ for some $x \in K$. Then

$$h_K(u_1) - h_K(u_2) \leq \langle x, u_1\rangle - \langle x, u_2\rangle \leq \|x\| \cdot \|u_1 - u_2\|.$$

Changing the role of u_1 and u_2 yields that

$$|h_K(u_1) - h_K(u_2)| \leq R(K) \cdot \|u_1 - u_2\|. \tag{11}$$

Denote by Λ^* the dual lattice of Λ: namely, $\Lambda^* = \{x \mid \langle x, y\rangle \in \mathbb{Z} \text{ for } y \in \Lambda\}$.

Lemma 8 *For every $\varepsilon > 0$ and every $u \in S^{d-1}$ there exists a $v \in \Lambda^*$ such that*

$$\left\| u - \frac{v}{\|v\|} \right\| \leq \frac{\varepsilon}{\|v\|}.$$

Proof: Write u as $u = \alpha_1 w_1 + \ldots + \alpha_d w_d$ where $w_1, \ldots, w_d$ is a basis of Λ^*. We may assume that $|\alpha_i| \leq |\alpha_d|$, $i = 1, \ldots, d-1$, and by Diophantine approximation for any $\varepsilon_1 > 0$ (cf. [GL]), there exists a positive integer q such that for certain integers $p_1, \ldots, p_{d-1}$ the estimate

$$\left| \frac{\alpha_i}{\alpha_d} - \frac{p_i}{q} \right| \leq \frac{\varepsilon_1}{q}, \quad i = 1, \ldots, d-1,$$

holds. We set $v = p_1 w_1 + \ldots + p_{d-1} w_{d-1} + q\, w_d$ and find

$$\|u - \frac{\alpha_d v}{q}\| \leq \frac{c\varepsilon_1}{q} \qquad \text{and} \qquad |1 - \|\frac{\alpha_d v}{q}\|| \leq \frac{c\varepsilon_1}{q}$$

for some constant c depending only on $w_1, \ldots, w_d$. Choosing $\varepsilon_1 = \varepsilon/(2c)$ concludes the proof. $\square$

Recalling that the vector $c \in \Lambda$ is called primitive, if there exists no ν with $0 < \nu < 1$ and $\nu c \in \Lambda$, we note that if $v \in \Lambda^*$ is primitive then

$$\|v\| = \frac{\det(v^\perp \cap \Lambda)}{\det \Lambda}. \tag{12}$$

The proof of Lemma 9 was kindly provided by Uwe Schnell (see also [W1] for the case of spheres).

Lemma 9 *$P^*_{\varrho,\Lambda,K}$ is a polytope.*

Proof: We claim that

$$P^* = P^*_{\varrho,\Lambda,K} \subset \mathrm{int}\varrho K. \tag{13}$$

As P^* is compact, it is sufficient to prove that for any $u \in S^{d-1}$, $h_{P^*}(u) < \varrho\, h_K(u)$.

By Lemma 8, for any $\varepsilon > 0$ there exists a primitive $v_\varepsilon \in \Lambda^*$ such that

$$\|u - u_\varepsilon\| \leq \frac{\varepsilon}{\|v_\varepsilon\|} \tag{14}$$

for $u_\varepsilon = v_\varepsilon / \|v_\varepsilon\|$. Define $x \in P^*$ by $\langle x, u \rangle = h_{P^*}(u)$. Then the definition of P^* and (12) yield that

$$\begin{aligned} h_{P^*}(u) - \varrho\, h_K(u) &= \langle x, u \rangle - \varrho\, h_K(u) \leq |\langle x, u - u_\varepsilon \rangle| + h_{P^*}(u_\varepsilon) - \varrho\, h_K(u) \\ &\leq |\langle x, u - u_\varepsilon \rangle| + \varrho\, h_K(u_\varepsilon) - \varrho\, h_K(u) - \frac{1}{2\|v_\varepsilon\|}. \end{aligned}$$

We conclude by (11) and (14) for suitable ε the validity of

$$h_{P^*}(u) - \varrho\, h_K(u) \leq \frac{cR(K)\varepsilon}{\|v_\varepsilon\|} + \frac{c\varrho R(K)\varepsilon}{\|v_\varepsilon\|} - \frac{1}{2\|v_\varepsilon\|} < -\frac{1}{4\|v_\varepsilon\|}, \tag{15}$$

and hence (13) holds.

Thus we find that there exists some positive β such that

$$P^* = \bigcap_{\substack{v \in \Lambda^* \ primitive \\ \|v\| < \beta}} \left\{ x \mid \langle x, v \rangle \leq \varrho\, h_K(v) - \tfrac{1}{2} \right\},$$

which in turn yields that P^* is a polytope. $\square$

For any convex body M, define

$$\Omega(M) = \frac{\int_{\partial M} (\varrho\, h_K(u) - \omega(u))\, du}{V(M)^{(d-1)/d}}$$

where $\omega(u)$ is readily Borel measurable.

Lemma 10 *If $M \in \mathcal{K}^d$ then $\Omega(M) \geq \Omega(P_{\varrho,\Lambda,K})$ with equality only if M is homothetic to $P_{\varrho,\Lambda,K}$.*

Proof: As $\Omega(\lambda M) = \Omega(M)$ for all $\lambda > 0$, it is equivalent to prove the statement for $P^* = P^*_{\varrho,\Lambda,K}$ instead of $P_{\varrho,\Lambda,K}$. By the definition of P^*,

$$h_{P^*}(u) \leq \varrho\, h_K(u) - \omega_\Lambda(u)$$

for any $u \in S^{d-1}$ where equality holds if u is an outer facet normal of P^*. We deduce by (6) that

$$V(M)^{(d-1)/d}\Omega(M) \geq d \cdot V(M, P^*; 1),$$

and equality holds if $M = P^*$. Assume that M is not homothetic to P^*. Then Minkowski's inequality (see [S]) and $V(P^*, P^*; 1) = V(P^*)$ yield that

$$V(M)^{d-1}\Omega(M)^d \geq d^d V(M, P^*; 1)^d > d^d V(M)^{d-1} V(P^*) = V(M)^{d-1}\Omega(P^*)^d,$$

and hence $\Omega(M) > \Omega(P^*)$. □

Remark 11 *Lemma 10 is actually the celebrated theorem of Wulff.*

The next lemma is used to show that there are reasonably dense packings.

Lemma 12 *Let Q be a polytope such that all normals to facets are multiples of vectors in Λ^*. Then there exist polytopes Q^n such that $Q^n \to Q$, and for suitable λ_n, we have $G_\Lambda(\lambda_n Q^n) = n$, $n = \lambda_n^d V(Q^n)/\det\Lambda + \lambda_n^{d-1} S_\Lambda(Q) + o(\lambda_n^{d-1})$.*

Proof: We have to show that for every $\varepsilon > 0$ there exists an n_0, such that for all $n \geq n_0$ there are polytopes Q^n and reals λ_n, such that $G_\Lambda(\lambda_n Q^n) = n$, $\Delta^H(Q, Q^n) < \varepsilon$ and

$$\lambda_n^d \cdot \frac{V(Q^n)}{\det\Lambda} + \lambda_n^{d-1} S_\Lambda(Q) - \varepsilon\lambda_n^{d-1} < n < \lambda_n^d \cdot \frac{V(Q^n)}{\det\Lambda} + \lambda_n^{d-1} S_\Lambda(Q) + \varepsilon\lambda_n^{d-1}.$$

We do this by an explicit construction of Q^n. First we observe that there is a polytope $\hat{Q}$, such that $\Delta^H(\hat{Q}, Q) \leq \varepsilon/2$, $S_\Lambda(Q) - S_\Lambda(\hat{Q}) \leq \varepsilon/2$ and a suitable multiple $\mu\hat{Q}$ is a lattice polytope. We find by Ehrhart's formula (9) that for positive integers m, we have

$$G_\Lambda(m\mu\hat{Q}) = V(\hat{Q})(m\mu)^d / \det\Lambda + S_\Lambda(\hat{Q})(m\mu)^{d-1} + O(m^{d-2}).$$

Thus we find $G_\Lambda((m+1)\mu\hat{Q}) - G_\Lambda(m\mu\hat{Q}) = O(m^{d-1})$. For given n we now choose m such that $G_\Lambda(m\mu\hat{Q}) \leq n < G_\Lambda((m+1)\mu\hat{Q})$. Let e be an arbitrary vertex of $(m+1)\mu\hat{Q}$. Then there is a lattice hyperplane H with normal $v \in \Lambda^*$, v primitive, supporting $(m+1)\mu\hat{Q}$ at e.

Denoting by H_i the hyperplane parallel to H translated by iv, we have that H_i cuts a pyramid T_i off $(m+1)\mu\hat{Q}$, if we choose $i = o(m)$ and m sufficiently large. T_i satisfies $G_\Lambda(T_i) = ci^d + O(i^{d-1})$ and $G_\Lambda(T_i) - V(T_i)/\det\Lambda = O(i^{d-1})$. Thus there is a suitable i such that

$$G(((m+1)\mu\hat{Q}) \setminus T_i) \leq n < G(((m+1)\mu\hat{Q}) \setminus T_{i-1}).$$

By taking $\lambda_n = (m+1)\mu$ and $\lambda_n Q^n$ to be the convex hull of the closure of $((m+1)\mu\hat{Q}) \setminus T_i$ and a suitable polytope in H_{i-1} the construction of Q^n is completed.

$\square$

4 Packings with respect to a given lattice

The proof of Theorem 3 relies on one hand on Theorem B in [BB]. This provides an estimate of the number of lattice points in convex bodies, which we quote as

Lemma 13 *Let Λ be a lattice, $\{\widetilde{C}^n\}$ be a sequence of convex bodies such that $C = \lim_{n\to\infty} \widetilde{C}^n$ and $\{\lambda_n\}$ be a sequence of reals such that $\lim_{n\to\infty} \lambda_n = \infty$. Then*

$$\#(\lambda_n \widetilde{C}^n \cap \Lambda) \leq \lambda_n^d \frac{V(\widetilde{C}^n)}{\det\Lambda} + \lambda_n^{d-1} S_\Lambda(C) + o(\lambda_n^{d-1}).$$

The other basic fact we need is the main result of [ABB] about k–flat sequences, which we quote as

Lemma 14 *If $\varrho > 0$ and $\{C^n\}$, $C^n \in \mathcal{P}_n(\Lambda)$, is a k–flat sequence satisfying $\lim_{n\to\infty} \delta(K, C_n, \varrho) = \delta_k(K, \Lambda, \varrho)$ then $\{C^n\}$ is uniform. The analogous statement holds for general lattice packings.*

We call a k–flat sequence $\{C^n\}$, $C^n \in \mathcal{P}_n(\Lambda)$, asymptotically optimal if

$$\delta(K, C^n, \varrho) \geq \delta_k(K, \Lambda, n, \varrho) + o\left(\frac{1}{n^{1/k}}\right).$$

Theorem 15 *Let $\varrho > \varrho_c(K,\Lambda)$ for a packing lattice Λ of K and assume that $\{C^n\}$, $C^n \in \mathcal{P}_n(\Lambda)$, is an asymptotically optimal d–flat sequence. Then*

$$\lim_{n\to\infty} \widetilde{C}^n = P_{\varrho,\Lambda,K}.$$

Proof: Let $\{C^n\}$ be an asymptotically optimal d–flat sequence. As $\{\widetilde{C}^n\}$ is uniformly bounded, we may assume by possibly taking a suitable subsequence $\{C^{n'}\}$, that $\widetilde{C}^{n'} \to C$, and have to show that $C = P_{\varrho,\Lambda,K}$. Set $\lambda_{n'} = R(C^{n'})$, and find as an immediate consequence of the d–flatness that

$$n' = \lambda_{n'}^d \cdot \frac{V(\widetilde{C}^{n'})}{\det\Lambda} + o(\lambda_{n'}^d). \tag{16}$$

or

$$\lambda_{n'} = \left(\frac{n' \det\Lambda}{V(\widetilde{C}^{n'})}\right)^{1/d} + o({n'}^{1/d}). \tag{17}$$

We obtain by Lemma 13

$$n' \le \lambda_{n'}^d \cdot \frac{V(\widetilde{C}^{n'})}{\det\Lambda} + \lambda_{n'}^{d-1} S_\Lambda(C) + o(\lambda_{n'}^{d-1}). \tag{18}$$

Using (6), (16) and (18) we find

$$\begin{aligned}
\delta(K, C^{n'}, \varrho) &= \frac{n'V(K)}{V(\operatorname{conv} C^{n'} + \varrho K)} \\
&\le \frac{V(K)\left(\lambda_{n'}^d V(\widetilde{C}^{n'})/\det\Lambda + \lambda_{n'}^{d-1} S_\Lambda(C) + o(\lambda_{n'}^{d-1})\right)}{\lambda_{n'}^d V(\widetilde{C}^{n'}) + \lambda_{n'}^{d-1} dV(\widetilde{C}^{n'}, \varrho K; 1) + o(\lambda_{n'}^{d-1})} \\
&= \frac{V(K)}{\det\Lambda} + \lambda_{n'}^{-1} \cdot \frac{V(K)\left(S_\Lambda(C) - dV(C,\varrho K;1)\right)}{V(C)} + o(\lambda_{n'}^{-1}) \\
&= \frac{V(K)}{\det\Lambda} - {n'}^{-1/d} \cdot \frac{V(K)\Omega(C)}{(\det\Lambda)^{1/d} V(C)^{(d-1)/d}} + o({n'}^{-1/d}).
\end{aligned} \tag{19}$$

Furthermore we have by Lemma 12, that there is a sequence $\{C^n\}$ such that $\widetilde{C}^n \to P_{\varrho,\Lambda,K}$ and equality holds in (19). The theorem is an immediate consequence, as

$$\frac{V(K)\Omega(C)}{(\det\Lambda)^{1/d} V(C)^{(d-1)/d}} > \frac{V(K)\Omega(P_{\varrho,\Lambda,K})}{(\det\Lambda)^{1/d} V(P_{\varrho,\Lambda,K})^{(d-1)/d}}$$

for $C \neq P_{\varrho,\Lambda,K}$ by Lemma 10.

$\square$

The optimal sequences are proven to be flat in some important cases. We quote here the results of [ABB] (see also [BHW1] or [BS] for [a]).

Lemma 16

[a] If $\varrho > \varrho_c(K,\Lambda)$ then $\{C_o^n(K,\Lambda,\varrho)\}$ is d–flat.

[b] If $d = 3$ and $\varrho_{k-1}(K,\Lambda) < \varrho < \varrho_k(K,\Lambda)$, $k = 1,2,3$, then $\{C_o^n(K,\Lambda,\varrho)\}$ is k–flat.

The analoguous statements hold for general packings.

Theorem 3[a] is an immediate consequence, since $\{C_o^n(K,\Lambda,\varrho)\}$ is asymptotically optimal.

Remark 17 We sketch the changes needed for the proof of Theorem 6 in the case of a given lattice Λ. Under the conditions, $\{C^n\}$ is k–flat and uniform in some linear lattice k–plane L. Now the same arguments work as for Theorem 3, only using $D(K,L)$ instead of K, and applying (8) instead of (6).

5 General lattice packings

Proof of Theorem 3[b]. Let $C^n = C_o^n(K,\varrho)$. We have $C^n \subset \mathcal{P}_n(\Lambda_n)$ for some packing lattice Λ_n. $\varrho > \varrho_c(K)$ gives again that $r(\operatorname{conv} C^n) \to \infty$. Hence $\operatorname{conv} C^n$ is a convex body which can be dissected into at least $n-d$ lattice d–simplices of Λ_n. It follows that $V(\operatorname{conv} C^n) \geq n \cdot \det \Lambda_n/(2d!)$ for large n, and we deduce that $\det \Lambda_n$ is bounded by the minimality of $V(\operatorname{conv} C^n + \varrho K)$.

Let now $\widetilde{C}^{n'}$ be a convergent subsequence of $\{\widetilde{C}^{n'}\}$. Since each $\Lambda_{n'}$ is a packing lattice for K, we may assume by Mahler's theorem (see [GL]) and possibly taking a suitable subsequence that $\Lambda_{n'}$ tends to a packing lattice Λ. By the definition of the convergence of lattices there exists a sequence $T_{n'}$ of linear operators with $\Lambda_{n'} = T_{n'}^{-1}\Lambda$ and $T_{n'}^{-1}$ tends to identity. Then $T_{n'}$ tends to identity, too. We set $D^{n'} = T_{n'}C^{n'}$.

We observe that $r(\operatorname{conv} C^{n'}) \to \infty$ implies that $V(\operatorname{conv} C^{n'} + \varrho K) \sim n' \det \Lambda$ (see [GL]), and hence Λ is a densest packing lattice by the minimality of $V(\operatorname{conv} C^n + \varrho K)$. In particular, $\det T_{n'} \leq 1$, and hence $V(\operatorname{conv} D^{n'}) \leq V(\operatorname{conv} C^{n'})$. Since $\{C^n\}$ is uniform, we deduce that

$$\begin{aligned} V(\operatorname{conv} D^{n'}) + d \cdot V(\operatorname{conv} D^{n'}, \varrho K; 1) \\ \leq V(\operatorname{conv} C^{n'}) + d \cdot V(\operatorname{conv} C^{n'}, \varrho K; 1) + o\left(n^{(d-1)/d}\right), \end{aligned}$$

which in turn yields that $\{D^{n'}\}$ is asymptotically optimal with respect to Λ. Therefore $\{\widetilde{D}^{n'}\}$ tends to $P_{\varrho,\Lambda,K}$ by Theorem 15, and hence $\{\widetilde{C}^{n'}\}$ tends to $P_{\varrho,\Lambda,K}$, too. $\square$

In order to prove Theorems 6 and 7, one only needs the changes sketched at the end of the previous section.

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