Discrete point sets in the hyperbolic plane

Károly Böröczky*, Jr.

November 19, 2001

Dedicated to the 60th birthday of my father Károly Böröczky

Abstract

The main result is that if a convex compact set C in H^2 contains n point whose mutual distances are at least ϱ then

$$\frac{1}{2A(T(\varrho))} \cdot A(C) + \frac{1}{2\varrho} \cdot P(C) + 1 \geq n$$

where $T(\varrho)$ denotes the regular triangle of side length ϱ. The Euclidean analogue was initiated by A. Thue, and proved much later independently by H. Groemer and N. Oler. A generalization of the inequality above for simplicial complices in the sense of J.H. Folkman and R.L. Graham is also verified.

Let us call a convex domain D to be r*–convex* if any boundary point x of D is contained in an equidistance curve with associated distance r such that D lies in the convex side of the equidistance curve. The inequality above yields by some routine calculations that the density of non–overlapping hyperbolic circles inside an r–convex domain is less than $\frac{\pi}{2\sqrt{3}}$. This result was proved earlier by K. Bezdek.

1 Introduction

For a convex domain C in E^2, S^2 or H^2, we write $A(C)$ and $P(C)$ to denote the area and the perimeter of C, respectively. Now it is well–known in the Euclidean case that if C contains the centres of n non–overlapping unit discs then

$$\frac{1}{2\sqrt{3}} \cdot A(C) + \frac{1}{4} \cdot P(C) + 1 \geq n. \qquad (1)$$

*Research supported by OTKA

The inequality was proved by Axel Thue in [13] and [14] in a special case, while the general case is due to H. Groemer [8] and N. Oler [12]. A simple proof of (1), which also extends the inequality to simplicial complices, can be found in J.H. Folkman and R.L. Graham [6]. Actually N. Oler proved a more general inequality for translative packings of centrally symmetric domains in E^2. Since the inequality for translative packings is commonly known as Oler inequality, we call (1) Thue–Groemer inequality.

The Thue–Groemer inequality (1) is rather essential for circle packings. For example, a simple consequence is the fact proved earlier by L. Fejes Tóth [4] that the density of at least two non–overlapping unit discs inside a convex domain is always less than $\frac{\pi}{2\sqrt{3}}$. This estimate that is asymptotically optimal.

The main purpose of this paper is to generalize (1) to the hyperbolic plane. We formulate our theorem in the language of simplicial complices. The faces of a convex compact set C are the set itself, and the intersections of C with any supporting lines. We say that a finite family Σ of points, segments and triangles form a simplicial complex if the intersection of any two intersecting objects is a common face, and Σ contains the faces of any of its objects (see M.A. Armstrong [1] for the notions about simplicial complices). We write $f_i(\Sigma)$ to denote the number of i–dimensional objects in Σ. Then the Euler characteristic of Σ is defined to be

$$\chi(\Sigma) = f_2(\Sigma) - f_1(\Sigma) + f_0(\Sigma).$$

The support $\operatorname{supp}\Sigma$ of Σ is the union of the triangles, segments, and points, and we write $A(\Sigma)$ to denote the area of the support of Σ. In addition, the perimeter of Σ is defined to be

$$P(\Sigma) = \sum_{e\, edge\, of\, \Sigma} (2 - \nu(e)) \cdot |e|$$

where $\nu(e)$ is the number of triangles of Σ containing the edge e. Finally, let $T(\varrho)$ denote the regular triangle of side length ϱ. Then J.H. Folkman and R.L. Graham [6] actually proved the Euclidean version of the following result:

THEOREM 1 *Let Σ be a finite simplicial complex in H^2 such that each segment contained in the support and connecting two vertices is of length at least ϱ. Then*

$$\frac{1}{2A(T(\varrho))} \cdot A(\Sigma) + \frac{1}{2\varrho} \cdot P(\Sigma) + \chi(\Sigma) \geq f_0(\Sigma).$$

Equality holds if and only if the support of Σ can be triangulated using the same set of vertices in a way that each edge is of length ϱ.

Remark: The same statement holds in S^2 (with $\varrho \leq \frac{\pi}{4}$), and the proof is completely analogous (actually, the analogue of Proposition 2 is a direct consequence of the analogue of Proposition 3 in that case).

The convex hull of n points is a polygon, which can be triangulated using these n points as vertices. Since the area and the perimeter of convex domains are monotone, we conclude the following generalization of (1):

COROLLARY 1 *If the compact convex set C in H^2 contains n points such that the distance between any two is at least ϱ then*

$$\frac{1}{2A(T(\varrho))} \cdot A(C) + \frac{1}{2\varrho} \cdot P(C) + 1 \geq n.$$

Equality holds if and only if C is the convex hull of the n points, and C can be triangulated using the points as the vertices in a way that each edge is of length ϱ.

Remark: Naturally, the same statement holds in S^2, as well.

We note that the perimeter of a large convex domain in the hyperbolic plane can be arbitrarily close to the area, and it is far from being trivial to give good estimates on the number of points contained in a large domain.

Let us discuss the optimality of Corollary 1 (and hence of Theorem 1). Say segments of length $k \cdot \varrho$, $k \in \mathbb{N}$, show that the constant term and the coefficient of the perimeter are optimal. Next, let $T_1, \ldots, T_m$, $m \geq 2$, be non–overlapping regular triangles of side length ϱ such that T_i and T_{i+1} share a common side for $i = 1, \ldots, m-1$, and no vertex is contained in four triangles. Then the union of the triangles is an $(m+2)$–gon P, and equality holds in Theorem 1 for $n = m + 2$ and $C = P$.

Next, we assume that $k \geq 7$, and define ϱ_k in a way that the angle of $T(\varrho_k)$ is $\frac{2\pi}{k}$. Then there exists a side to side tiling of H^2 by copies of $T(\varrho_k)$ where each vertex is of degree k. We fix a vertex p of the tiling, and define P_m to be the polygon that is the convex hull of the vertices of the tiling, which can be reached by a path from p using at most m edges of the tiling. We observe that the angles of P_m are either $2\gamma(\varrho_k)$ or $3\gamma(\varrho_k)$. Readily, equality holds for P_m in Corollary 1 where n is the number of vertices of the tiling in P_m.

Next we consider two applications of Corollary 1. The first concerns the perimeter of a set that contains n points whose mutual distances are at least ϱ:

COROLLARY 2 *If the compact convex set C in H^2 contains n points such that the distance between any two is at least ϱ then*

$$P(C) \geq 0.1 \cdot n \cdot \varrho \tanh \varrho.$$

We note that $\frac{3}{4} < \tanh \varrho < 1$ if $\varrho \geq 1$, and $\frac{3}{4}\,\varrho < \tanh \varrho < \varrho$ if $\varrho \leq 1$. It will be shown in Section 4 that Corollary 2 is optimal up to a constant factor.

Finally an application of the hyperbolic Thue inequality Corollary 1 to the theory of circle packings in the hyperbolic plain is presented. Before describing the actual result, let us discuss the density of equal non–overlapping circles inside convex containers. It would be natural to aim at the so–called triangle bound: Let us consider the three circular discs of radius r centred at the vertices of $T(2r)$. We define the hyperbolic triangle bound $\sigma_{H^2}(r)$ to be the ratio of the area of the part of $T(2r)$ covered by the circular discs divided by the area of the triangle. We note that (see L. Fejes Tóth [5], p. 240)

$$\frac{\pi}{2\sqrt{3}} < \sigma_{H^2}(r) < \frac{3}{\pi}.$$

Unfortunately, if the common radius r is large then the density of a linear arragement (when the centres are aligned) with respect to its convex hull gets arbitrarily close to one, and hence no absolute bound less than one exists. In particular, the triangle bound does not hold with respect to any convex container, and hence we seek for a restricted but still natural family of containers.

Let $r > 0$. We call a convex domain D to be *r–convex* if any boundary point x of D is contained in an equidistance curve with associated distance r such that D lies in the convex side of the equidistance curve. We note that an r–convex domain is r'–convex for any $r' < r$, and a circular disc is r–convex for any $r > 0$. A typical example for r–convex domains is the *parallel domain* of radius r for a convex compact set C. This is the set of points whose distance from C is at most r. We note that for a convex domain in the hyperbolic plane, one can define the geodesic curvature almost everywhere on the boundary. Then D is r–convex if and only if the geodesic curvature of ∂D is at least $\mathrm{th}\, r$ whereever it exists. Now the main result of the paper K. Bezdek [3] can be phrased as

THEOREM 2 *If an r–convex domain D in H^2 contains at least two non–overlapping equal circles then the density of the circles in D is less than $\frac{\pi}{2\sqrt{3}}$.*

We observe that $\frac{\pi}{2\sqrt{3}}$ in Theorem 2 is the Euclidean triangle bound. Therefore large packings for small r show that Theorem 2 is optimal. On the other hand, if $r > 2\pi$ then the density of non–overlapping circles of radius r inside an r–convex domain is at most $\frac{2\pi}{r}$ (see (20) in Section 5).

In some cases, even the r–convex domain of minimal area containing a given number of non–overlapping circular discs of radius r can be determined with the help of Corollary 1. For $k \geq 7$, we define r_k in a way that the angle of $T(2r_k)$ is $\frac{2\pi}{k}$. Then the vertices and the circumcentre of the regular k–gon with side lengths $2r_k$ form $k+1$ points such that the mutual distances are at least $2r_k$. Then Corollary 1 and the isoperimetric inequality for the polygons yield the following statement:

Let D be the r_k–convex domain of minimal area that contains $k+1$ non–overlapping circular discs of radius r_k. Then the convex hull of the centres is the regular k–gon with side length $2r_k$.

Let D be a convex domain in E^2, S^2 or in H^2 containing at least two non–overlapping equal circular discs. Then the density of the circular discs in D is less than $\frac{\pi}{2\sqrt{3}}$ both in the Euclidean case (see L. Fejes Tóth [4]), and in the spherical case (see J. Molnár [10]). On the other hand, Theorem 2 can be applied if D is a circular disc in the hyperbolic case. We conclude (see also K. Bezdek [2])

COROLLARY 3 *If at least two non–overlapping equal circles are placed in a larger circle in E^2, S^2 or in H^2 then the density of the arrangement is less than $\frac{\pi}{2\sqrt{3}}$.*

We would like to recall a related result of A. Heppes: Following J. Molnár [11], we say that a (connected) domain D in E^2, S^2 or in H^2 is a $c(r)$–*domain*, if for any boundary point p there exists a circular disc of radius r that is tangent to D at p and does not overlap D in a neighbourhood of p. In addition, we write $H_3(r)$ to denote the area enclosed by three mutually tangent circular discs of radius r. Then A. Heppes [9] proved that if the $c(r)$–domain D contains n non–overlapping circular discs of radius r then area of the uncovered part is at least $2(n-1)H_3(r)$.

The paper is organized as follows: Sections 2 and 3 verify Theorem 1, Section 4 discusses Corollary 2, and finally Theorem 2 is proved in Section 5.

2 Some basic properties of triangles and quadrilaterals

We write $\gamma(\varrho)$ to denote the angle of the regular triangle $T(\varrho)$ of side length ϱ. We need some technical estimate on certain quadrilaterals (see Proposition 3) and on regular triangles, which estimates are rather trivial in the Euclidean plane. For sake of completeness, we show (but see also L. Fejes Tóth [5], p. 244)

PROPOSITION 1 *Assume that each side of the triangle T is at least ϱ, while the circle passing through the vertices of T has radius at most ϱ. Then*

$$A(T) \geq A(T(\varrho)),$$

and equality holds if and only if T is congruent to $T(\varrho)$.

Proof: We may readily assume that two sides of T have length ϱ, and hence the largest angle of T is less than $2\gamma(\varrho)$, and two equal angles greater than $\frac{1}{2}\gamma(\varrho)$. Cutting T into half, we obtain a triangle T', which has a side with length ϱ, the opposite angle is a right angle, and the other two angles α and β are between $\frac{1}{2}\gamma(\varrho)$ and $\gamma(\varrho)$.

Now $A(T') = \frac{\pi}{2} - \alpha - \beta$ is minimal exactly when $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ maximal. Here the law of cosine yields that $\tan\alpha\tan\beta = \frac{1}{\operatorname{ch}\varrho}$. Therefore $\tan\alpha + \tan\beta$ is maximal when $|\tan\alpha - \tan\beta|$ is maximal; namely, if T' is the half of $T(\varrho)$.

Q.E.D.

The next step is a special case of Proposition 3:

PROPOSITION 2 *Let the angle α of the triangle T be enclosed by the sides with length ϱ and b. If $\alpha \leq \gamma(\varrho)$, and the angle opposite to the side with length ϱ is a right angle then*

$$\frac{1}{2A(T(\varrho))} \cdot A(T) + \frac{1}{2\varrho} \cdot b \geq \frac{1}{2}.$$

Proof: Denoting the angle opposite to b by β, the laws of sine and cosine yield that

$$\tan\alpha \cdot \tan\beta \cdot \operatorname{ch}\varrho = 1 \quad \text{and} \quad \operatorname{th} b = \operatorname{th}\varrho \cdot \cos\alpha. \tag{2}$$

We consider β and b as the function of α, and hence

$$\beta' = \frac{-\operatorname{ch}\varrho}{1 + \operatorname{sh}^2 \cdot \sin^2\alpha}, \tag{3}$$

$$b' = \frac{-\operatorname{sh}\varrho \cdot \operatorname{ch}\varrho \cdot \sin\alpha}{1 + \operatorname{sh}^2 \cdot \sin^2\alpha}. \tag{4}$$

Now Proposition 2 states that $f(\alpha) \geq \frac{1}{2}$ holds for

$$f(\alpha) = \frac{1}{2A(T(\varrho))} \cdot A(T) + \frac{1}{2\varrho} \cdot b.$$

The derivative of $f(\alpha)$ can be evaluated using the relation $A(T) = \frac{\pi}{2} - \alpha - \beta$; namely,

$$\begin{aligned} f'(\alpha) &= \frac{1}{1+\sin^2\alpha \cdot \operatorname{sh}^2\varrho} \times \\ &\times \left(\frac{\operatorname{ch}\varrho - 1}{2A(T(\varrho))} - \frac{\operatorname{sh}\varrho \cdot \operatorname{ch}\varrho}{2\varrho} \cdot \sin\alpha - \frac{\operatorname{sh}^2\varrho}{2A(T(\varrho))} \cdot \sin^2\alpha \right). \end{aligned}$$

The expresssion in the parentheses is monotone decreasing in α. Therefore there exists some $0 < \alpha_0 \leq \gamma(\varrho)$ such that $f'(\alpha)$ is positive if $0 < \alpha < \alpha_0$, and negative if $\alpha_0 < \alpha < \gamma(\varrho)$. In particular, the minimum of $f(\alpha)$ on $[0, \gamma(\varrho)]$ is attained either at $\alpha = 0$ or at $\alpha = \gamma(\varrho)$. Since $f(\alpha) = \frac{1}{2}$ for both values of α, the proof of Proposition 2 is now complete. Q.E.D.

The main auxiliary result is the following innocent looking statement:

PROPOSITION 3 *Let the points p and q be lying on a semicircle bounded by the line l, and assume that the radius of the semicircle is at most ϱ, while $d(p,q) \geq \varrho$. We write $\tilde{p}$ and $\tilde{q}$ to denote the orthogonal projection of p and q onto l, respectively, and Q to denote the quadrilateral determined by $p, q, \tilde{p}, \tilde{q}$. Then*

$$\frac{1}{2A(T(\varrho))} \cdot A(Q) + \frac{1}{2\varrho} \cdot d(\tilde{p}, \tilde{q}) \geq \frac{1}{2}.$$

Proof: We may assume that $d(p,q) = \varrho$. We denote the radius and the centre of the semicirle by r and o, respectively. Let 2ω be the largest angle of the triangle whose two sides are of length r, and the third side is of length ϱ.

As a parameter, we use the angle α of l and the perpendicular bisector of the segment pq, where $\omega \leq \alpha \leq \frac{\pi}{2}$. We assume that p is not farther from l than q, and hence $\angle(po\tilde{p}) = \alpha - \omega$ and $\angle(qo\tilde{p}) = \alpha + \omega$.

Let us investigate

$$f(\alpha) = \frac{1}{2A(T(\varrho))} \cdot A(Q) + \frac{1}{2\varrho} \cdot d(\tilde{p}, \tilde{q}),$$

and the proposition is equivalent with $f(\alpha) \geq \frac{1}{2}$. We define $\beta_1 = \angle(op\tilde{p})$ and $b_1 = d(o, \tilde{p})$, while we write β_2 and b_2 to denote the signed angle $\angle(oq\tilde{q})$ and the signed distance $d(o, \tilde{q})$, where β_2 and b_2 are positive (negative) if $\alpha + \omega < \frac{\pi}{2}$ ($\alpha + \omega > \frac{\pi}{2}$). Then both $A(Q)$ and $d(\tilde{p}, \tilde{q})$ can be expressed applying (2) to the triangles $op\tilde{p}$ and $oq\tilde{q}$, and we deduce by (3) and (4) that

$$f'(\alpha) = \frac{1}{2A(T(\varrho))} \cdot (\beta_2 - \beta_1)' + \frac{1}{2\varrho} \cdot (b_1 - b_2)'$$

$$
\begin{aligned}
&= \frac{\operatorname{sh} r \cdot \operatorname{ch} r \cdot \sin\omega \cdot \cos\alpha}{(1+\sin(\alpha+\omega)\cdot \operatorname{sh}^2 r)(1+\sin(\alpha-\omega)\cdot \operatorname{sh}^2 r)} \times \\
&\quad \times \left(-\frac{\operatorname{sh}^2 r}{\varrho} \cdot \sin^2\alpha + \frac{2\operatorname{sh} r \cdot \cos\omega}{A(T(\varrho))} \cdot \sin\alpha + \frac{1+\operatorname{sh}^2 r \cdot \sin^2\omega}{\varrho} \right).
\end{aligned}
$$

Now the expression in the parentheses is quadratic with negative main coefficient in $\sin\alpha$, and positive for $\sin\alpha = 0$. Therefore there exists an $\omega \le \alpha_0 \le \frac{\pi}{2}$ such that $f'(\alpha) > 0$ for $\omega < \alpha < \alpha_0$, while $f'(\alpha) < 0$ for $\alpha_0 < \alpha < \frac{\pi}{2}$. We conclude that it is sufficient to check whether $f(\alpha) \ge \frac{1}{2}$ if $\alpha = \omega$ or $\alpha = \frac{\pi}{2}$. In case $\alpha = \omega$, Proposition 2 yields the inequality.

Thus let $\alpha = \frac{\pi}{2}$, and hence the perpendicular bisector of pq is perpendicular to l, and divides Q into two equal halves Q' and Q''. We write x to denote the distance of the midpoint of pq and o, and by φ the acute angle of Q'. Applying the laws of sine and cosine to Q' yield the formulae

$$
\operatorname{th}\tfrac{1}{2}\varrho = \operatorname{ch} x \cdot \operatorname{th} b_1 \quad \text{and} \quad \operatorname{sh}\tfrac{1}{2}\varrho \cdot \operatorname{th} x \cdot \tan\varphi = 1.
$$

Next we consider the function

$$
g(x) = \frac{1}{2A(T(\varrho))} \cdot A(Q) + \frac{1}{2\varrho} \cdot d(\tilde{p}, \tilde{q}),
$$

and we verify that $g(x) \ge \frac{1}{2}$. Here $A(Q) = 2(\frac{\pi}{2} - \varphi)$ and $d(\tilde{p}, \tilde{q}) = 2b_1$, and hence

$$
\begin{aligned}
g'(x) &= \frac{1}{1+\operatorname{ch}^2\frac{1}{2}\varrho \cdot \operatorname{sh}^2 x} \times \\
&\quad \times \left(\frac{\operatorname{sh}\frac{1}{2}\varrho}{A(T(\varrho))} - \frac{\operatorname{sh}\frac{1}{2}\varrho \cdot \operatorname{sh}\frac{1}{2}\varrho \cdot \operatorname{sh} x}{\varrho} \right).
\end{aligned}
$$

The derivative $g'(x)$ is positive if $x < x_0$ and negative if $x > x_0$ for some $x_0 > 0$. Therefore it is sufficient to check $g(x)$ at $x = 0$, and if x is the height h of $T(\varrho)$. If $x = h$ then Q containes a copy of $T(\varrho)$, and hence $g(h) > \frac{1}{2}$. Since $g(0) = \frac{1}{2}$, Proposition 3 finally follows.

Q.E.D.

3 The Thue type inequality

We should prove that if Σ is a finite simplicial complex in H^2 satisfying that each segment contained in the support and connecting two vertices has length at least ϱ then

$$
\frac{1}{2A(T(\varrho))} \cdot A(\Sigma) + \frac{1}{2\varrho} \cdot P(\Sigma) + \chi(\Sigma) \ge f_0(\Sigma). \tag{5}
$$

The proof is based on a cell decomposition related to the Delone tiling. This decomposition is rather complicated, but it allows us to minimize the amount of computation, what is essential in the hyperbolic plane. Let us recall that a *cell complex* Θ is the collection of (possibly non–convex) polygons, segments and points such that the sides and the vertices of the polygons, and the endpoints of the segments are contained in Θ, and if the intersection of two elements is non–empty then it is a common face (see M.A. Armstrong [1]). The simplest examples are the simplicial complices. The support $\operatorname{supp}\Theta$ is the union of the elements of Θ, and Θ can be considered as a *cell decomposition* of its support. We write $f_i(\Theta)$ to denote the number of i–dimensional elements of Θ, and define the Euler charactheristic of Θ to be

$$\chi(\Theta) = f_2(\Theta) - f_1(\Theta) + f_0(\Theta).$$

Then the Euler charactheristic is independent of the support (see M.A. Armstrong [1]).

Let us start the actual proof of Theorem 1. Removing a free edge e ($\nu(e) = 0$) of Σ increases $\chi(\Sigma)$ by 1, while decreases $P(\Sigma)$ by at least 2ϱ. Therefore we may assume that any edge or vertex of Σ is contained in some triangle. We call an edge of Σ a *boundary edge* if it is contained in exactly one two–dimensional cell.

We say that a point x sees a point y if the connecting segment lies in the support of Σ. Since $\chi(\Sigma)$ depends only on the support of Σ (see M.A. Armstrong [1]), not on the particular triangulation chosen, we may assume that any point x in the support of Σ sees a vertex v of Σ with $d(x, v) < \varrho$.

A circle with centre y and radius r is denoted by $B(y, r)$. We call a circle $B(y, r)$ to be an *empty circle* if

- $y \in \operatorname{supp}\Sigma$, and
- no vertex of Σ that lies in the interior of $B(y, r)$ sees y.

We observe that if $B(y, r)$ is an empty circle then $r < \varrho$. The following simple observation is the main tool in our argument:

PROPOSITION 4 *If $B(y, r)$ is an empty circle, and the triangle $T \subset B(y, r)$ satisfies that y is a vertex of T, two sides of T lie in* $\operatorname{supp}\Sigma$*, and T intersects the interior of* $\operatorname{supp}\Sigma$ *then* $T \subset \operatorname{supp}\Sigma$.

We call an empty circle $B(y, r)$ to be

- a *supporting circle of Delone type* if at least three vertices of Σ visible from y lie on $\partial B(y, r)$; and

- a *supporting circle of boundary type* if y lies on a boundary edge, at least two vertices of Σ visible from y lie on $\partial B(y,r)$, and y is not contained in the convex hull of these vertices.

We observe that a supporting circle may be of Delone type and of boundary type at the same time.

Now in order to prove Theorem 1, we construct a cell decomposition Σ_2 of $\operatorname{supp}\Sigma$ using the same set of vertices in the following way: First we create a related complex Σ_0 whose maximal cells are segments or convex polygons assigned to supporting circles of either type where polygon is assigned if the supporting circle is of Delone type. We will fix a triangulation of Σ_1 of Σ_0 that is a subcomplex of Σ_2. The rest of the two–cells of Σ_2 are possibly non–convex polygons that are in bijective correspondence with the boundary edges of Σ, which are not already edges of Σ_0. If such a polygon P is assigned to the boundary edge e then a side of P is either e, or corresponds to some supporting circle $B(y,r)$ of boundary type with $y \in e$.

To any supporting circle $B(y,r)$, we assign the convex hull C of the vertices of Σ that lie on $\partial B(y,r)$ and see y. Then C is a segment or a polygon, and Proposition 4 yields that

$$\operatorname{conv}\{y, C\} \subset \operatorname{supp}\Sigma. \tag{6}$$

Now let $B(y,r)$ run through the family of all supporting circles, and let Σ_0' be the collection of all the assigned sets C. We define Σ_0 to be the union of Σ_0' and the family of all faces of the elements of Σ_0'.

We claim that Σ_0 is a cell complex. We suppose that two maximal elements C_1 and C_2 of Σ_0 intersect in their relative interiors, and seek a contradiction. We denote the corresponding supporting circles by $B(y_1,r_1)$ and $B(y_2,r_2)$, respectively. Since the common secant of the two circles does not separate C_1 and C_2, say C_1 has a vertex v that is contained in the interior of $B(y_2,r_2)$. Let $p \in C_1 \cap C_2$. We may apply Proposition 4 to the triangle pvy_2 and to the empty circle $B(y_2,r_2)$ according to (6), and hence y_2 sees v. This is absurd, therefore the sets in Σ_0 form a cell complex.

Let Σ_1 be the simplicial complex resulting by triangulating the two–cells of Σ_0 using the same set of vertices. Now we consider $\operatorname{supp}\Sigma \backslash \operatorname{supp}\Sigma_1$. Let $B(y,r)$ be a supporting circle such that y is contained in a boundary edge e of Σ, and y does not lie in the convex hull of the vertices of Σ lying on $\partial B(y,r)$ that are visible from y. In this case, $B(y,r)$ intersects e in a diameter, and there exist an edge zw of Σ_0 visible from y, which separates y from the rest of the visible vertices of Σ on the boundary of $B(y,r)$. Then we associate zw to e, and observe that by Proposition 4, the quadrilateral

$Q = \mathrm{conv}\{z, w, e \cap B(y, r)\}$ satisfies

$$Q \subset \mathrm{supp}\,\Sigma \quad \text{and} \quad Q \cap \mathrm{supp}\,\Sigma_0 = \mathrm{conv}\{z, w\}. \tag{7}$$

Moving y along e and considering empty circles centred at y show that the boundary edges of Σ_0 associated to e form a broken line, and adding e to them results in the boundary of a possibly non–convex polygon, which we call a *cell of boundary type*. Now (7) yields that the cell is contained in $\mathrm{supp}\,\Sigma$, and we define Σ_2 to be the cell complex determined by Σ_0 and the cells of boundary type. Σ_2 is actually a cell complex because of (7).

We deduce using Proposition 4 that any edge of Σ_2 intersecting the interior of $\mathrm{supp}\,\Sigma$ lies in exactly two two–cells of Σ_2. Thus the two–cells of Σ_2 at a vertex of Σ cover a neighbourhood in $\mathrm{supp}\,\Sigma$ of that vertex, and hence the supports of Σ and Σ_2 coincide. In particular, the Euler characteristic of Σ, Σ_1 and Σ_2 are the same.

We note that the set of vertices of Σ, Σ_1 and Σ_2 are the same. We write e_0, e_1 and e_2 to denote the number of edges of Σ_1 that are contained in exactly zero, one and two triangles, respectively. Then $3f_2(\Sigma_1) = 2e_2 + e_1$, and hence the definition of the Euler characteristic yields that

$$\frac{1}{2}\, f_2(\Sigma_1) + e_0 + \frac{1}{2}\, e_1 + \chi(\Sigma_1) = n.$$

If T is a triangle of Σ_1 then the sides of T are at least $2r$, and the circle passing through the vertices of T has radius at most $2r$. Therefore $A(T) \geq A(T(2r))$ by Proposition 1, and hence

$$\frac{1}{2A(T(2r))} \cdot A(\Sigma_1) + e_0 + \frac{1}{2} \cdot e_1 + \chi(\Sigma_1) \geq n. \tag{8}$$

Let us return to Σ, or what is equivalent from our point of view, to Σ_2. Consider a boundary cell Π associated to the boundary edge e of Σ. Project orthogonally each side of Π but e onto e. Then Proposition 3 can be applied to the resulting quadrilaterals. On the other hand, a boundary edge of Σ_1 is an edge of C, or the side of a boundary cell, or both. Therefore Theorem 1 follows by $\chi(\Sigma_2) = \chi(\Sigma_1)$ and (8).

Q.E.D.

4 The inequality for the perimeter

We verify the following somewhat stronger form of Corollary 2:
If the compact convex set C in H^2 contains n points such that the distance

between any two is at least ϱ then

$$P(C) > \begin{cases} \frac{1}{2}\cdot\varrho\cdot n & \text{if } \varrho\geq 3.5;\\ 0.1\cdot\varrho^2\cdot n & \text{if } \varrho\leq 3.5 \ \text{ and } \ n\geq 4. \end{cases} \tag{9}$$

Let $\varrho \geq 3.5$. We may assume that C is the convex hull of the n points. Since C can be divided into $n-2$ triangles, its area is at most $(n-2)\,\pi$. On the other hand, $A(T(\varrho)) > \frac{2}{3}\,\pi$ if $\varrho \geq 3.5$, thus the hyperbolic Thue inquality Corollary 1 yields (9). We note that the segment of length $(n-1)\varrho$ (hence of perimeter $2(n-1)\varrho$) contains n points such that the distance between any two is at least ϱ.

Next let $\varrho \leq 3.5$. The paralleldomain of C of radius $r = \frac{1}{2}\varrho$ contains n non–overlapping circles of radius r, and hence (see (10))

$$\mathrm{ch}\, r\cdot A(C) + \mathrm{sh}\, r\cdot P(C) + 2\pi(\mathrm{ch}\, r - 1) \geq n\cdot 2\pi(\mathrm{ch}\, r-1).$$

Since $A(C) < P(C)$ according to the isoperimetric inequality, we deduce that

$$P(C) \geq \tfrac{3}{4}\,n\cdot\frac{2\pi(\mathrm{ch}\, r-1)}{\mathrm{ch}\, r+\mathrm{sh}\, r} > 0.8\cdot(\mathrm{ch}\, r-1) > 0.1\cdot\varrho^2.$$

In order to show the optimality of this estimate (up to a constant factor), it is suffient to consider the case when ϱ is very small. Then say the regular triangle T with angle $\frac{2\pi}{7}$ contains $\frac{\text{const.}}{\varrho^2}$ points with mutual distances are at least ϱ. Using T as a building block, arbitrary large convex domain C can be built for fixed small ϱ such that C contains n points whose mutual distances are at least ϱ, and $P(C) < \text{const.}\cdot\varrho^2\cdot n$.

5 The density bound

Let an r–convex domain D_n contain $n\geq 2$ non–overlapping circles of radius r. We write C_n to denote the convex hull of the centres. We may assume that D_n is the paralleldomain of C_n of radius r, and hence the area of D_n is

$$\mathrm{ch}\, r\cdot A(C_n) + \mathrm{sh}\, r\cdot P(C_n) + 2\pi(\mathrm{ch}\, r-1). \tag{10}$$

We deduce by Corollary 1 that Theorem 2 is a consequence of the inequality

$$\begin{aligned} &\left(\frac{1}{2A(T(2r))}\cdot A(C_n) + \frac{1}{4r}\cdot P(C_n) + 1\right)\cdot 2\pi(\mathrm{ch}\, r-1)\\ < \quad &\frac{\pi}{2\sqrt{3}}\cdot\left(\mathrm{ch}\, r\cdot A(C_n) + \mathrm{sh}\, r\cdot P(C_n) + 2\pi(\mathrm{ch}\, r-1)\right). \end{aligned} \tag{11}$$

The reason why (11) holds is that the contribution of the perimeter lowers the density. We define $\omega(r)$ by

$$\omega(r) = 2(2\sqrt{3} - \pi) \cdot \left(\frac{\operatorname{sh} r \cdot r}{\operatorname{ch} r - 1} - \sqrt{3} \right)^{-1}.$$

It is chosen in a way that if $p \geq \omega(r) \cdot r$ then

$$\left(\frac{1}{4r} \cdot p + 1 \right) \cdot 2\pi(\operatorname{ch} r - 1) \leq \frac{\pi}{2\sqrt{3}} \cdot (\operatorname{sh} r \cdot p + 2\pi(\operatorname{ch} r - 1)). \tag{12}$$

We note that

$$c(r) = \frac{\operatorname{ch} r - 1}{r \operatorname{sh} r} \text{ is monotone decreasing and } \lim_{r \to 0} c(r) = \frac{1}{2}. \tag{13}$$

Thus $\omega(r)$ is bounded above by its limit $\frac{2 \cdot (2\sqrt{3} - \pi)}{2 - \sqrt{3}}$ at zero, and hence

$$\omega(r) < 6 - 2\sqrt{3} < 4. \tag{14}$$

If $n = 2$ then C_n is a segment with $P(C_n) \geq 4r$, and combining (12) and (14) yields (11).

Therefore let $n \geq 3$. Now (12) tells us that it is sufficent to prove

$$\begin{aligned} &\frac{\operatorname{ch} r - 1}{A(T(2r))} \cdot A(C_n) + \frac{\operatorname{ch} r - 1}{2r} \cdot (P(C_n) - \omega(r) \cdot r) \\ < \quad &\frac{\operatorname{ch} r}{2\sqrt{3}} \cdot A(C_n) + \frac{\operatorname{sh} r}{2\sqrt{3}} \cdot (P(C_n) - \omega(r) \cdot r). \end{aligned} \tag{15}$$

Here $\frac{\operatorname{ch} r - 1}{2r} < \frac{\operatorname{sh} r}{2\sqrt{3}}$ by (13). On the other hand, if $R > 0$ is chosen in a way that $P(C_n) = 2\pi \cdot \operatorname{sh} R$ then $A(C_n) \leq 2\pi \cdot (\operatorname{ch} R - 1)$ holds by isoperimetric inequality. Therefore $2\pi \operatorname{sh} R \geq 6r$, and (15) is a consequence of the inequality

$$\begin{aligned} &\left(\frac{\operatorname{ch} r - 1}{A(T(2r))} - \frac{\operatorname{ch} r}{2\sqrt{3}} \right) \cdot 2\pi(\operatorname{ch} R - 1) \\ < \quad &\left(\frac{\operatorname{sh} r}{2\sqrt{3}} - \frac{\operatorname{ch} r - 1}{2r} \right) \cdot (2\pi \operatorname{sh} R - \omega(r) \cdot r). \end{aligned} \tag{16}$$

For fixed $\theta > 0$, checking the derivative shows that $\frac{\operatorname{sh} R - \theta}{\operatorname{ch} R - 1}$ is monotone increasing if $\frac{\operatorname{ch} R - 1}{\operatorname{sh} R} < \theta$, and monotone decreasing if $\frac{\operatorname{ch} R - 1}{\operatorname{sh} R} > \theta$. On the other hand, $\frac{\operatorname{ch} R - 1}{\operatorname{sh} R}$ is monotone increasing. Therefore in (16), it is sufficient to consider the cases $2\pi \operatorname{sh} R = 6r$ and $R = \infty$.

First let $2\pi \operatorname{sh} R = 6r$. Then $\operatorname{ch} R = \sqrt{1 + \frac{9r^2}{\pi^2}}$, and we deduce by (14) that

$$2\pi \cdot \operatorname{sh} R - \omega(r) \cdot r \geq 6r - \omega(r) \cdot r \geq 6r - \omega(r) \cdot r > 2\sqrt{3} \cdot r.$$

Note that $t < \frac{\pi}{3} \cdot \sin t$ holds for $0 < t < \frac{\pi}{6}$. Thus applying the substitution $t = \arcsin \frac{1}{2\operatorname{ch} r}$ in $A(T(2r)) = \pi - 6 \cdot \arcsin \frac{1}{2\operatorname{ch} r}$ yields that

$$\frac{\operatorname{ch} r - 1}{A(T(2r))} < \frac{\operatorname{ch} r}{\pi}. \tag{17}$$

Therefore instead of (16), it is sufficient to prove that

$$\left(2 - \frac{\pi}{\sqrt{3}}\right) \operatorname{ch} r \left(\sqrt{1 + \frac{9r^2}{\pi^2}} - 1\right) < \left(\operatorname{sh} r - \frac{\sqrt{3}(\operatorname{ch} r - 1)}{r}\right) \cdot r. \tag{18}$$

We split the argument into two cases. If $r \leq 1.2$ then the bounds $\frac{\operatorname{ch} r - 1}{r} < \frac{1}{2} \cdot \operatorname{sh} r$ (see (13)) and $\sqrt{1 + \frac{9r^2}{\pi^2}} < 1 + \frac{9r^2}{2\pi^2}$ show that it is sufficient to consider the inequality

$$\left(2 - \frac{\pi}{\sqrt{3}}\right) \cdot \frac{9}{2\pi^2} < \left(1 - \frac{\sqrt{3}}{2}\right) \cdot \frac{\operatorname{th} r}{r}.$$

Now this inequality holds for $r = 1.2$, and $\frac{\operatorname{th} r}{r}$ is monotone decreasing, therefore (18) holds in this case.

On the other hand, if $r \geq 1.2$ then $\frac{\operatorname{ch} r - 1}{r} \leq c(1.2) \cdot \operatorname{sh} r$ holds by (13). Using the bound $\sqrt{1 + \frac{9r^2}{\pi^2}} < 1 + \frac{3r}{\pi}$, (18) follows from

$$\left(2 - \frac{\pi}{\sqrt{3}}\right) \cdot \frac{3}{\pi} < \left(1 - c(1.2) \cdot \sqrt{3}\right) \cdot \operatorname{th} r.$$

This inequality holds for $r = 1.2$, and $\operatorname{th} r$ is monotone increasing, which in turn verifies (18) for all r.

Finally, let R tend to infinity in (16). Since $\frac{\operatorname{ch} r - 1}{2r} < \frac{\operatorname{sh} r}{4}$ by (13), we are done if

$$\frac{\operatorname{ch} r - 1}{A(T(2r))} < \frac{1}{2\sqrt{3}} \cdot \operatorname{ch} r + \left(\frac{1}{2\sqrt{3}} - \frac{1}{4}\right) \cdot \operatorname{sh} r. \tag{19}$$

Estimating the left hand side by (17), it is sufficient to verify.

$$\frac{1}{\pi} < \frac{1}{2\sqrt{3}} + \left(\frac{1}{2\sqrt{3}} - \frac{1}{4}\right) \cdot \operatorname{th} r.$$

This inequality holds for $r = 1.1$, and hence for $r \geq 1.1$. Therefore let $r \leq 1.1$. Then we rewrite (19) in the form

$$\frac{\operatorname{ch} r - 1}{\frac{1}{2\sqrt{3}} \cdot \operatorname{ch} r + \left(\frac{1}{2\sqrt{3}} - \frac{1}{4}\right) \cdot \operatorname{sh} r} < A(T(2r)).$$

We have equality for $r = 0$, therefore it is sufficient to verify the same inequality for the derivatives; namely, that

$$\frac{(2\sqrt{3} - 3) \cdot (\operatorname{ch} r - 1) + 2\sqrt{3} \cdot \operatorname{sh} r}{\operatorname{ch}^2 r + (2 - \sqrt{3}) \cdot \operatorname{ch} r \cdot \operatorname{sh} r + (\frac{7}{4} - \sqrt{3}) \cdot \operatorname{sh}^2 r} < \frac{6 \cdot \operatorname{sh} r}{\operatorname{ch} r \sqrt{4 \operatorname{ch}^2 r - 1}}.$$

We may actually forget about $(\frac{7}{4} - \sqrt{3}) \cdot \operatorname{sh}^2 r$ in the dominator, and verify

$$\begin{aligned} (2\sqrt{3} - 3)\,(\operatorname{ch} r - 1)\sqrt{4 \operatorname{ch}^2 r - 1} + 2\sqrt{3} \operatorname{sh} r \sqrt{4 \operatorname{ch}^2 r - 1} \\ < \quad 6 \operatorname{ch} r \cdot \operatorname{sh} r + (12 - 6\sqrt{3}) \operatorname{sh}^2 r. \end{aligned}$$

Here $r \leq 1.1$ yields that

$$(2\sqrt{3} - 3)\,(\operatorname{ch} r - 1)\sqrt{4 \operatorname{ch}^2 r - 1} < (11 - 6\sqrt{3}) \operatorname{sh}^2 r,$$

because the inequality holds for $r = 1.1$, and the quotient of the left hand side and the right hand side is monotone increasing. So the last inequality left to prove is

$$2\sqrt{3}\sqrt{4 \operatorname{ch}^2 r - 1} < 6 \operatorname{ch} r + \operatorname{sh} r.$$

Squaring and rearranging show that this formula actually identically holds for any r, and in turn we deduce (19). Therefore the proof of Theorem 2 is finally complete.

Q.E.D.

Finally, we show that Theorem 2 gives a very crude estimate if r large; namely, *if $r \geq 1.75$ then for any packing of at least two non–overlapping circles of radius r inside an r–convex domain D, the density δ^* satisfies*

$$\delta^* \leq \frac{2\pi}{r}. \tag{20}$$

We write C to denote the convex hull of the centres. Now D contains the paralleldomain of C of radius r, and hence Corollary 1 yields that

$$\begin{aligned} \delta^* \quad &\leq \quad \frac{(\frac{1}{2A(T(2r))} \cdot A(C) + \frac{1}{4r} \cdot P(C) + 1) \cdot 2\pi(\operatorname{ch} r - 1)}{\operatorname{ch} r \cdot A(C) + \operatorname{sh} r \cdot P(C) + 2\pi(\operatorname{ch} r - 1)} \\ &< \quad \frac{(\frac{1}{2A(T(2r))} \cdot A(C) + \frac{1}{4r} \cdot P(C) + 1) \cdot 2\pi}{P(C)}. \end{aligned}$$

Here $\frac{2}{3}\pi \leq A(T(2r)) < \pi$, while $P(C) \geq n \cdot r$ holds according to (9). In additon, C can be triangulated into at most $n-2$ triangles, and hence its area is at most $(n-2)\,\pi$. In turn, we conclude (20).

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