Covering the sphere by equal spherical balls

Károly Böröczky, Jr.
Gergely Wintsche

Abstract

We show that for any acute φ, there exists a covering of S^d by spherical balls of radius φ such that no point is covered more than $400d\ln d$ times. It follows that the density is of order at most $d\ln d$, and even at most $d\ln\ln d$ if the number of balls is polynomial in d. If the number of equal spherical balls is $d+3$ then we determine the optimal arrangement.

At the end, we described how our and other peoples results yield estimates for the largest origin centred Euclidean ball contained in the convex hull of N points chosen from the sphere.

1 Introduction

Packings of equal balls on the sphere have been investigated rather intensively since the middle of the 20th century (see the books J.H. Conway and N.J.A. Sloane [8], and Ch. Zong [20], and the survey G. Fejes Tóth and W. Kuperberg [13]), and the optimal arrangement is known in many cases. On the other hand, covering S^d by equal spherical balls seems to be technically harder, only the case $d=2$ is well–understood (see the book L. Fejes Tóth [14]).

Theorem 1.1. *For any $d\geq 3$ and $0<\varphi<\frac{\pi}{2}$, S^d can be covered by spherical balls of radius φ in a way that no point of S^d is covered more than $400\,d\ln d$ times.*

The order of the bound in Theorem 1.1 is close to be optimal because there always exists a point that is covered by $d+1$ spheres. The proof of Theorem 1.1 is based on an idea due to P. Erdős and C.A. Rogers [12] who verified the analogous statement in the Euclidean case.

Next we consider the density of coverings. Let $\varphi<\frac{\pi}{2}$, and hence at least $d+2$ spherical balls of radius φ are needed in order to cover S^d. If $\varphi\geq\arccos\sqrt{\frac{1}{d+1}}$ then the $(d+1)$–dimensional regular crosspolytope provides a thin covering;

namely,

$$\begin{gathered} S^d \textit{ can be covered by } 2d+2 \textit{ spherical balls of radius } \varphi \geq \arccos\sqrt{\tfrac{1}{d+1}}, \\ \textit{and hence the density of the covering is less than } d+1. \end{gathered} \tag{1}$$

Therefore in order to obtain coverings with low density, we may confine our attention to the case $\varphi \leq \arccos\sqrt{\frac{1}{d+1}}$. We deduce using Theorem 1.1 that

Corollary 1.2. *For any $0 < \varphi \leq \arccos\frac{1}{\sqrt{d+1}}$, S^d can be covered by*

$$c\cos\varphi \cdot \frac{1}{\sin^d\varphi} \cdot d^{\frac{3}{2}} \ln(1 + d\cos^2\varphi)$$

spherical balls of radius φ, and the density of the arrangement is at most

$$c \cdot d\ln(1 + d\cos^2\varphi)$$

where c is an absolute constant.

Here the improvement $\ln(1 + d\cos^2\varphi)$ compared to $\ln d$ (that is essential only if φ is rather large) uses an idea of M. Kochol [16]. We note that for $\varphi < \arccos\sqrt{\frac{1}{d+1}}$, C.A. Rogers [18] verified the existence of a covering of S^d by spherical balls of radius φ whose density is at most

$$(1 + c\cdot\tfrac{\ln\ln d}{\ln d}) \cdot d \cdot \left(\ln d + \ln\tfrac{1}{\sin\varphi}\right)$$

for some absolute constant c. His proof is rather technical, and uses coverings of S^d by Euclidean $(d+1)$–dimensional caps. We will sketch an argument at the end of Section 5 that yields the upper bound

$$(1 + c\cdot\tfrac{\ln\ln d}{\ln d}) \cdot d\ln d \tag{2}$$

for the density. The optimality of Corollary 1.2 is discussed in Section 6.

Now we turn to coverings of S^d by small number of equal spherical balls. It is natural to guess that the optimal covering of $2d+2$ equal spherical balls is determined by the $(d+1)$–dimensional regular crosspolytope. More generally, we believe

Conjecture 1.3. *For $d \geq 3$ and $d+2 \leq m \leq 2d+2$, let m equal spherical balls of minimal radius cover S^d. Then the convex hull of the centres in $\mathbb{E}^{d+1}$ is the direct sum of mutually orthogonal $\lceil\frac{d+1}{m-d-1}\rceil$ and $\lfloor\frac{d+1}{m-d-1}\rfloor$–dimensional regular simplices of circumradius one, and the total number of these simplices is $m-d-1$.*

The case $d = 2$ has been actually handled earlier by L. Fejes Tóth if $m = 6$ (see [14]), and by K. Schütte if $m = 5$ (see [19]). For $m = d + 2$, Conjecture 1.3 is common knowledge (and can be verified say similarly to the proof of Theorem 1.4). The only other case known is when $d = 3$ and $m = 8$ (see L. Dalla, D.G. Larman, P. Mani–Levitska, Ch. Zong [10]). We now consider the case of $d + 3$ balls:

Theorem 1.4. *If $d+3$ equal spherical balls with minimal possible radius cover S^d then the convex hull of the centres in $\mathbb{E}^{d+1}$ is the convex hull of a $\lfloor \frac{d+1}{2} \rfloor$ and a $\lceil \frac{d+1}{2} \rceil$–dimensional regular simplex of circumradius one.*

The paper is organized as follows: Sections 2 and 3 are preparations for the proof of Theorem 1.1 in Section 4. Corollary 1.2 is verified in Section 5, and Section 6 exhibits some examples about the optimality of Corollary 1.2. After that, we prove Theorem 1.4 in Section 7. Finally, Section 8 collects estimates about the largest origin centred ball contained of polytopes of N vertices inscribed into the unit ball. This problem is equivalent to determining the Banach–Mazur distance of ellipsoids from the polytopes of at most N vertices.

2 Covering a symmetric space

P. Erdős and C.A.Rogers [12] proved essentially that $\mathbb{E}^d$ can be covered by equal spheres in a way that no point of $\mathbb{E}^d$ is contained in more than $c \cdot d \ln d$ points for some absolute constant c. Below, we prove Lemma 2.1 that formulates the core of their argument. We choose to formulate Lemma 2.1 in terms of a symmetric space X because we need it both for some torus (when the group G is just the additive group on X), and for S^{d-1} (when the group G is the orthogonal group); and in addition, the proof is the same for say tori any for any symmetric space of finite volume.

Lemma 2.1. *Let G be a locally compact group such that the Haar measure μ is a probability measure, and let X be a factor of G equipped with the resulting measure $\tilde{\mu}$. We throw $N \geq 4$ copies of a measurable subset Θ of X uniformly and independently with respect to μ, and for any copies $\Theta_1, \dots, \Theta_N$, we write X_i to denote the family of points of X that are contained in exactly i out of $\Theta_1, \dots, \Theta_N$. If $\mu(\Theta) < 1$ then*

(i) $\mathbb{E}(\tilde{\mu}(X_0)) < e^{-N\tilde{\mu}(\Theta)}$;

(ii) $\mathbb{E}(\sum_{i=k}^{\infty} \tilde{\mu}(X_i)) < e^{-N\tilde{\mu}(\Theta)}$ *if* $k > 4N\tilde{\mu}(\Theta)$.

Proof: (i) simply follows by the relations $\mathbb{E}(\tilde{\mu}(X_0)) = (1 - \tilde{\mu}(\Theta))^N$ and $1 - t < e^{-t}$.

The second statement does require some argument. We may readily assume that $k \leq N$, and hence $\mu(\Theta) < \frac{1}{4}$. If $k = N$ then $\mathbb{E}(\tilde{\mu}(X_N)) = \tilde{\mu}(\Theta)^N$ and $\mu(\Theta) < \frac{1}{4}$ yield (ii). Therefore let $k < N$.

When integrating with respect to $\tilde{\mu}$ or μ, we simply write dx or dg for the variables $x \in X$ and $g \in G$, respectively, and we write χ_H to denote the indicator function of a set H. In addition, let $\Theta_i = g_i\theta$ for the i^{th} random variable $g_i \in G$. The fundamental observation is that if $1 \leq n \leq N$ then

$$\chi_{X_n}(x) = \sum_{m=n}^{N} \frac{m!(-1)^{m-n}}{n!(m-n)!} \cdot \sum_{1 \leq i_1 < \ldots i_m \leq N} \prod_{j=1}^{m} \chi_{\Theta_{i_j}}(x).$$

This formula can be verified by considering the cases when x is covered exactly by n, by less than n or more than n sets out of $\Theta_1, \ldots, \Theta_N$. Since

$$\mathbb{E}(\tilde{\mu}(X_n)) = \int_G \cdots \int_G \left(\int_X \chi_{X_n}(x)\, dx \right) dg_1 \ldots dg_N;$$
$$\int_G \cdots \int_G \int_X \prod_{j=1}^{m} \chi_{\Theta_{i_j}}(x)\, dx dg_1 \ldots dg_N = \tilde{\mu}(\Theta)^m,$$

we conclude that

$$\begin{aligned} \mathbb{E}(\tilde{\mu}(X_n) &= \sum_{m=n}^{N} \frac{m!(-1)^{m-n}}{n!(m-n)!} \cdot \frac{N!}{m!(N-n)!} \cdot \tilde{\mu}(\Theta)^m \\ &= \frac{N!}{n!(N-n)!} \cdot \tilde{\mu}(\Theta)^n (1-\tilde{\mu}(\Theta))^{N-n}. \end{aligned}$$

Adding these expressions for $n \geq k$ results in the formula

$$\begin{aligned} \mathbb{E}\left(\sum_{n=k}^{\infty} \tilde{\mu}(X_n) \right) &= \frac{N!}{k!(N-k)!} \cdot \tilde{\mu}(\Theta)^k (1-\tilde{\mu}(\Theta))^{N-k} \times \\ &\quad \times \sum_{m=0}^{N-k} \frac{k!(N-k)!}{(k+m)!(N-k-m)!} \cdot \frac{\tilde{\mu}(\Theta)^m}{(1-\tilde{\mu}(\Theta))^m}. \end{aligned}$$

The quotient of the $(m+1)^{th}$ and the m^{th} terms in the sum is

$$\frac{N-k-m}{k+m+1} \cdot \frac{\tilde{\mu}(\Theta)}{1-\tilde{\mu}(\Theta)} \leq \frac{N-k}{k+1} \cdot \frac{\tilde{\mu}(\Theta)}{1-\tilde{\mu}(\Theta)} < 1.$$

We use the corresponding geometric series to estimate the sum behind the summation symbol, and hence

$$\mathbb{E}\left(\sum_{i=k}^{\infty} \tilde{\mu}(X_i) \right) \leq \frac{N!}{k!(N-k)!} \cdot \tilde{\mu}(\Theta)^k (1-\tilde{\mu}(\Theta))^{N-k} \frac{(k+1)(1-\tilde{\mu}(\Theta))}{(k+1)-(N+1)\tilde{\mu}(\Theta)}.$$

Here $k > 4N\tilde{\mu}(\Theta)$ yields that

$$\frac{(k+1)(1-\tilde{\mu}(\Theta))}{(k+1)-(N+1)\tilde{\mu}(\Theta)} = 1 + \frac{(N-k)\tilde{\mu}(\Theta)}{(k+1)-(N+1)\tilde{\mu}(\Theta)} < \frac{4}{3}.$$

We recall that $t! = \Gamma(t+1)$ for a non–negative integer t, and

$$\left(\frac{t}{e}\right)^t \sqrt{2\pi t} < \Gamma(t+1) < \left(\frac{t}{e}\right)^t \sqrt{2\pi(t+1)} \tag{3}$$

holds according to the Stirling formula (see say E. Artin [1]). Therefore we deduce by $1-t < e^{-t}$ that

$$\mathbb{E}\left(\sum_{i=k}^{\infty}\tilde{\mu}(X_i)\right) \leq \left(\frac{eN\tilde{\mu}(\Theta)}{k(1-\tilde{\mu}(\Theta))}\right)^k \cdot e^{-N\tilde{\mu}(\Theta)} < e^{-N\tilde{\mu}(\Theta)}.$$

Q.E.D.

3 The volume of spherical balls

We need some basic estimates for the volume of spherical balls. We write $B(\varphi, x)$ to denote the spherical ball of radius φ and of centre x, and $|\cdot|$ to denote the volume (Hausdorff d–measure) on S^d. In addition, the volume of the unit Euclidean d–ball B^d is denoted by κ_d, and we note that $|S^d| = (d+1)\kappa_{d+1}$ (see say L. Fejes Tóth [14]).

Lemma 3.1. *Let $0 < \varphi < \frac{\pi}{2}$, and let $x \in S^d$.*

(i) $|B(\varphi, x)| > \kappa_d \varphi \sin^{d-1}\varphi > \kappa_d \sin^d \varphi$;

(ii) $|B(\varphi, t\,x)| < t^d \cdot |B(\varphi, x)|$ *if* $1 < t < \frac{\pi}{2\varphi}$;

(iii) $\dfrac{\kappa_d}{3\cos\varphi} \cdot \sin^d\varphi < |B(\varphi, x)| < \dfrac{\kappa_d}{\cos\varphi} \cdot \sin^d\varphi$ *if* $\varphi \leq \arccos\frac{1}{\sqrt{d+1}}$.

Proof: The boundary of $B(\varphi, x)$ in S^d is the boundary of a Euclidean d–ball of radius $\sin\varphi$, as well. Therefore using polar coordinates on S^d shows that

$$|B(\varphi, x)| = d\,\kappa_d \int_0^{\varphi} \sin^{d-1} t\, dt. \tag{4}$$

We deduce that

$$\begin{aligned} \tfrac{\partial}{\partial\varphi}|B(\varphi,x)| &= d\,\kappa_d \cdot \sin^{d-1}(\varphi) > \kappa_d \cdot (\sin^{d-1}\varphi + \varphi \cdot (d-1)\sin^{d-2}\varphi\cos\varphi) \\ &= \tfrac{\partial}{\partial\varphi}(\kappa_d\varphi \cdot \sin^{d-1}\varphi), \end{aligned}$$

which in turn yields (i). Now (ii) follows by

$$\frac{\partial}{\partial t}\,|B(t\varphi,x)| = d\kappa_d\varphi\cdot\sin^{d-1}(t\varphi) < d\kappa_d\varphi\cdot t^{d-1}\sin^{d-1}\varphi < \frac{\partial}{\partial t}\,(t^d\cdot|B(\varphi,x)|).$$

In order to verify (iii), we rewrite (4) in the form

$$|B(\varphi,x)| = d\,\kappa_d\cdot\int_0^{\sin\varphi}\frac{1}{\sqrt{1-r^2}}\cdot r^{d-1}dr, \tag{5}$$

thus the upper bound is a consequence of $\frac{1}{\sqrt{1-r^2}}\le\frac{1}{\cos\varphi}$ for $r\le\sin\varphi$. On the other hand, if $(1-\frac{1}{d})\sin\varphi\le r\le\sin\varphi$ then $\tan\varphi\le\sqrt{d}$ yields

$$\frac{\sqrt{1-r^2}}{\cos\varphi}\le\frac{\sqrt{1-(1-\frac{1}{d})^2\sin^2\varphi}}{\cos\varphi}<\sqrt{1+\tfrac{2}{d}\,\tan^2\varphi}\le\sqrt{3}.$$

Since $(1-\frac{1}{d})^d<\frac{1}{e}$, we conclude that

$$|B(\varphi,x)| > d\,\kappa_d\int_{(1-\frac{1}{d})\sin\varphi}^{\sin\varphi}\frac{1}{\sqrt{1-r^2}}\cdot r^{d-1}dr > \kappa_d\left(1-\frac{1}{e}\right)\cdot\frac{1}{\sqrt{3}\cos\varphi}\cdot\sin^d\varphi.$$

Therefore $\frac{e-1}{e\sqrt{3}}>\frac{1}{3}$ yields (iii). Q.E.D.

Now the probability measure of $B(\varphi,x)$ is

$$\Omega(\varphi)=\frac{|B(\varphi,x)|}{|S^d|}.$$

We deduce by the estimates $\sqrt{\frac{d+2}{2\pi}}>\frac{\kappa_d}{\kappa_{d+1}}>\sqrt{\frac{d+1}{2\pi}}$ (see U. Betke, P. Gritzmann and J. M. Wills [4]) and by elementary calculations that

Corollary 3.2. *Let* $0<\varphi<\frac{\pi}{2}$.

(i) $\Omega(\varphi)>\dfrac{1}{\sqrt{2\pi(d+1)}}\cdot\sin^d\varphi;$

(ii) $\Omega(t\varphi)<t^d\cdot\Omega(\varphi)$ *if* $1<t<\frac{\pi}{2\varphi}$;

(iii) *if* $\varphi\le\arccos\frac{1}{\sqrt{d+1}}$ *then*
$$\frac{1}{3\sqrt{2\pi(d+1)}}\cdot\frac{1}{\cos\varphi}\cdot\sin^d\varphi<\Omega(\varphi)<\frac{1}{\sqrt{2\pi\,d}}\cdot\frac{1}{\cos\varphi}\cdot\sin^d\varphi;$$

(iv) $\dfrac{1}{3e\sqrt{2\pi}}<\Omega(\varphi)<\dfrac{1}{2}$ *if* $\varphi\ge\arccos\frac{1}{\sqrt{d+1}}$.

4 The proof of Theorem 1.1

In order to verify Theorem 1.1, first we consider the case $\varphi \geq \frac{\pi}{48d}$. The condition $\varphi \geq \frac{\pi}{48d}$ is needed in order to ensure that the probability measure of a spherical ball of radius φ is not too small when we apply Lemma 2.1. After that we project suitable Euclidean coverings stereographically onto S^d if $\varphi \leq \frac{\pi}{48d}$. In the Euclidean case, any ball B can be embedded into a torus of volume at most $2^d|B|$ according to the Minkowski–Hlawka Theorem (see say P.M. Gruber and C.G. Lekkerkerker [15]), therefore Lemma 2.1 yields the bound of right order. The argument will use the constant

$$\eta = \frac{1}{2d}.$$

Case I. $\varphi \geq \frac{\pi}{48d}$

We place spherical balls $B((1-\eta)\varphi, x_1), \ldots, B((1-\eta)\varphi, x_N)$ on S^d independently and with uniform distribution according to the probability measure μ where N will be fixed later. Let

$$k = \lceil 4N\Omega((1+\eta)\varphi) \rceil.$$

We write Y_k to denote the family of points that are covered by at least k–times by the N spherical balls $B((1+\eta)\varphi, x_1), \ldots, B((1+\eta)\varphi, x_N)$, and X_0 to denote the family of points of S^d that are not covered by the N spherical balls $B((1-\eta)\varphi, x_1), \ldots, B((1-\eta)\varphi, x_N)$. According to Lemma 3.1, we may choose $x_1, \ldots, x_N$ in a way that

$$\mu(Y_k) + \mu(X_0) < e^{-N\Omega((1+\eta)\varphi)} + e^{-N\Omega((1-\eta)\varphi)}.$$

We claim that if $N\Omega((1-\eta)\varphi) \geq 8d \ln d$ then

$$\mu(Y_k) + \mu(X_0) < \Omega(\eta\varphi), \tag{6}$$

which inequality is readily the consequence of the esimate

$$e^{-N\Omega((1-\eta)\varphi)} < \tfrac{1}{2}\,\Omega(\eta\varphi).$$

It follows by Corollary 3.2 that

$$\ln \frac{2}{\Omega(\eta\varphi)} < \ln \frac{2}{\eta^d \Omega(\varphi)} < \ln 2 + d \ln(2d) + \ln \sqrt{2\pi(d+1)} + d \ln \tfrac{1}{\sin\varphi} < 8d \ln d,$$

which in turn yields the claim (6). Therefore we choose $N = \left\lceil \frac{8d \ln d}{\Omega((1-\eta)\varphi)} \right\rceil$, and hence say $k < 400\, d \ln d$ holds according to $\frac{(1+\eta)^d}{(1-\eta)^d} < e^2$ and Corollary 3.2.

Let us verify that $B(\varphi, x_1), \ldots, B(\varphi, x_N)$ form a suitable cover of S^d. If $x \in S^d$ is not covered by $B(\varphi, x_1), \ldots, B(\varphi, x_N)$ then $B(\eta\varphi, x)$ is disjoint from

$B((1-\eta)\varphi, x_1), \ldots, B((1-\eta)\varphi, x_N)$, contradicting (6). On the other hand, if some y is contained in at least k spherical balls out of $B(\varphi, x_1), \ldots, B(\varphi, x_N)$ then Y_k contains $B(\eta\varphi, y)$, which is absurd, as well. This completes the proof if $\varphi \geq \frac{\pi}{48d}$.

Case II. $\varphi \leq \frac{\pi}{48d}$

First we provide a covering in Euclidean space. We also write $B(x,r)$ to denote the ball of centre x and radius r in $\mathbb{E}^d$. It will be always clear from the context when we mean spherical, and when Euclidean balls.

Proposition 4.1. *For any $r > 0$, there exists a (periodic) covering of $\mathbb{E}^d$ by balls $\{B(x_i, (1-\eta)\,r)\}$ such that no point of $\mathbb{E}^d$ is contained in more than $200\,d\ln d$ out of the balls $\{B(x_i, (1+\eta)\,r)\}$.*

Proof: According to the Minkowski–Hlawka Theorem (see say P.M. Gruber and C.G. Lekkerkerker [15]), there exists a lattice Λ such that the balls of radius $(1+2\eta)\,r$ centred at the points of Λ form a packing, and $\det\Lambda \leq 2^d(1+2\eta)^d\kappa_d\cdot r^d$. Therefore the torus $T = \mathbb{E}^d/\Lambda$ contains embedded Euclidean balls of radius $(1+2\eta)\,r$, and having a suitable cover of T yields the required cover of $\mathbb{E}^d$.

We write μ to denote the invariant probability measure on T, and hence

$$\mu(\eta r\, B^d) \geq \frac{\eta^d}{2^d(1+2\eta)^d} > \frac{1}{e\, 4^d d^d}.$$

We place balls $B((1-2\eta)\varphi, x_1), \ldots, B((1-2\eta)\varphi, x_N)$ on T independently and with uniform distribution according to the probability measure μ where N will be fixed later. Let

$$k = \lceil 4N\mu((1+2\eta)r\, B^d)\rceil.$$

We write Y_k to denote the family of points that are covered by at least k–times by the N balls $B((1+2\eta)\varphi, x_1), \ldots, B((1+2\eta)\varphi, x_N)$, and X_0 to denote the family of points of S^d that are not covered by the N balls $B((1-2\eta)\varphi, x_1), \ldots, B((1-2\eta)\varphi, x_N)$. According to Lemma 3.1, we may choose $x_1, \ldots, x_N$ in a way that

$$\mu(Y_k) + \mu(X_0) < e^{-N\mu((1+2\eta)r\, B^d)} + e^{-N\mu((1-2\eta)r\, B^d)}.$$

We claim that if $N\mu((1-2\eta)r\, B^d) \geq 3d\ln d$ then

$$\mu(Y_k) + \mu(X_0) < \mu(\eta r\, B^d). \tag{7}$$

Now $d \geq 3$ yields that

$$\ln\frac{2}{\mu(\eta r\, B^d)} < \ln(2e) + d\ln d + d\ln 4 < 3d\ln d,$$

and hence $e^{-N\mu((1-2\eta)r\, B^d)} < \frac{1}{2}\cdot\mu(\eta r\, B^d)$. In turn, we conclude the claim (7).

Since $\frac{(1+2\eta)^d}{(1-2\eta)^d} < e^{\frac{5}{2}}$ follows by $2\eta < \frac{1}{2}$, we may choose N in a way that (7) holds and $k < 200$. Therefore the proof of Proposition 4.1 can be completed by the arguments given in Case I.

Q.E.D.

Using stereographic projection and coverings in Euclidean d-space, we now construct suitable coverings of S^d by spherical balls of radius $\varphi \leq \frac{\pi}{48d}$.

Let p and q be a pair of opposite points of S^d, and we think about p as the north pole, and about q as the south pole. First we construct the covering for the southern hemisphere $\{x \in S^d : \langle x, q \rangle \geq 0\}$.

For $i = 1, \ldots, 4d$, we define

$$\Theta_i = \left\{ x \in S^d \ : \cos \tfrac{(i-1)\pi}{8d} \leq \langle x, q \rangle \leq \cos \tfrac{i \cdot \pi}{8d} \right\}.$$

We identify the tangent hyperplane to S^d at q with $\mathbb{E}^d$. For any object X contained in $S^d \backslash p$, we write $\tilde{X}$ to denote the stereographic projection of X into $\mathbb{E}^d$ from p. In addition, if $x \in S^d \backslash p$ then $\alpha(x)$ denotes the angle of x and q. We observe that if $\gamma : [0,1] \to S^d \backslash p$ is a continuously differentiable curve then the length of $\tilde{\gamma}$ is

$$\int_0^1 |\tilde{\gamma}'(t)| \, dt = \int_0^1 \left(1 + \tan^2 \tfrac{\alpha(\gamma(t))}{2} \right) \cdot |\gamma'(t)| \, dt. \tag{8}$$

We start with the spherical ball Θ_1. According to Proposition 4.1, there exists a covering of $\widetilde{\Theta}_1$ by Euclidean balls $\{B(\tilde{x}_j, (1-\eta)\,\varphi)\}$ such that no point of $\mathbb{E}^d$ is contained in more than $200\, d \ln d$ out of the balls $\{B(\tilde{x}_j, (1+\eta)\,\varphi)\}$. We may assume that each of the balls $B(\tilde{x}_j, (1-\eta)\,\varphi)$ intersect $\widetilde{\Theta}_1$. It follows by (8) that corresponding family $\{B(x_j, \varphi)\}$ of spherical balls covers Θ_1. Since $\tan^2 \frac{\alpha(x)}{2} < \eta$ if $\alpha(x) \leq \frac{\pi}{8d} + 2\varphi$, we conclude that no point of S^d is contained in more than $200\, d \ln d$ out of the balls $\{B(x_j, \varphi)\}$.

Next let $2 \leq i \leq 4d$. We define $\psi_i = \frac{(2i-1)\pi}{16}$, what is $\alpha(x)$ if x is contained in the "median" level of Ω_i. Using Proposition 4.1 as above, we cover $\widetilde{\Theta}_i$ by Euclidean balls $\{B(\tilde{y}_j, (1-\eta)\, r_i)\}$ such that no point of $\mathbb{E}^d$ is contained in more than $200\, d \ln d$ out of the balls $\{B(\tilde{x}_j, (1+\eta)\, r_i)\}$ where

$$r_i = \left(1 + \tan^2 \frac{\psi_i}{2} \right) \cdot \varphi.$$

We may assume that each of the balls $B(\tilde{y}_j, (1-\eta)\, r_i)$ intersect $\widetilde{\Theta}_i$. Now if x is contained in a spherical ball of radius φ that intersects Ω_i then $|\alpha(x) - \psi_i| \leq \frac{5\pi}{24d}$.

On the other hand,

$$\frac{\left|(1+\tan^2\frac{\psi_i}{2})-(1+\tan^2\frac{\alpha(x)}{2})\right|}{1+\tan^2\frac{\psi_i}{2}} \leq (\tan\frac{\psi_i}{2}+\tan\frac{\alpha(x)}{2})\times$$
$$\times\frac{1+\tan\frac{\psi_i}{2}\tan\frac{\alpha(x)}{2}}{1+\tan^2\frac{\psi_i}{2}}\cdot\tan\frac{5\pi}{48d} < \eta,$$

where we used the fact that $\psi_i < \frac{\pi}{2}$. It follows that the family of spherical balls $\{B(y_j,\varphi)\}$ cover Ω_i, but no point of S^d is covered by more than $200\,d\ln d$ times.

Finally, the regions Ω_i were defined wide enough that even after repeating the same procedure for the northern hemisphere, no point of S^d is covered by more than $400\,d\ln d$ times by the spherical balls of radius φ. Q.E.D.

5 The proof of Corollary 1.2

In this section, $c_1, c_2, \dots$ always denote some positive absolute constants. Theorem 1.1 yields right away that there exists a covering of S^d by spherical balls of radius φ of density at most $c_1 \cdot d\ln d$ for any φ. If $\varphi \leq \arccos\frac{1}{d+1}$ then Lemma 3.1 yields that the number of spherical balls is at most

$$c_2\,\cos\varphi\cdot\frac{1}{\sin^d\varphi}\cdot d^{\frac{3}{2}}\ln d. \tag{9}$$

Therefore we may assume that

$$d>15 \quad\text{and}\quad \tfrac{1}{\sqrt{d+1}} \leq \cos\varphi \leq \tfrac{1}{4^{\frac{1}{4}}(d+1)^{\frac{3}{8}}}.$$

Let $m=\lceil 3(d+1)^2\cos^4\varphi\rceil$ and $k=\left\lceil\frac{d+1}{m}\right\rceil$. The idea is first to take pairwise orthogonal linear subspaces $L_1,\dots,L_k$ that span $\mathbb{E}^{d+1}$, and satisfy $\dim L_i = m$ for $i=1,\dots,\left\lfloor\frac{d+1}{m}\right\rfloor$, and then to cover each $L_i\cap S^d$ using Theorem 1.1.

We define the acute angle ψ by the relation $\cos\psi=\sqrt{k}\cdot\cos\varphi$. According to Theorem 1.1, $L_i\cap S^d$ can be covered by N_i spherical $(m-1)$–balls of radius ψ where

$$N_i \leq c_3\,\cos\psi\cdot\frac{1}{\sin^{\frac{m-1}{2}}\psi}\cdot m^{\frac{3}{2}}\ln m.$$

Since for any $u\in S^d$, the length of the orthogonal projection of u into some L_i is at least $\frac{1}{\sqrt{k}}$, we obtain a covering of S^d by $n=\sum_1^k N_i$ spherical balls of radius φ where

$$n \leq c_3\left\lceil\frac{d+1}{m}\right\rceil^{\frac{3}{2}}\cdot\cos\varphi\cdot\frac{1}{\left(1-\left\lceil\frac{d+1}{m}\right\rceil\cos^2\varphi\right)^{\frac{m-1}{2}}}\cdot m^{\frac{3}{2}}\ln m.$$

We observe that

$$m \le 4(d+1)^2 \cos^4 \varphi \le \sqrt{d+1}, \text{ and hence } \left\lceil \tfrac{d+1}{m} \right\rceil \le \tfrac{5}{4} \cdot \tfrac{d+1}{m}, \tag{10}$$

which in turn yields that $\left\lceil \frac{d+1}{m} \right\rceil \cos^2 \varphi < \frac{1}{2}$. On the other hand, $1 - t > e^{-t-t^2}$ holds for $0 < t < \frac{1}{2}$, therefore

$$n < c_4\, d^{\frac{3}{2}} \cdot \cos\varphi \cdot \frac{e^{\frac{m-1}{2} \cdot \left\lceil \frac{d+1}{m} \right\rceil^2 \cos^4 \varphi}}{e^{-\frac{m-1}{2} \cdot \left\lceil \frac{d+1}{m} \right\rceil \cos^2 \varphi}} \cdot \ln 4(d+1)^2 \cos^4 \varphi.$$

Here $\frac{m-1}{2} \left\lceil \frac{d+1}{m} \right\rceil^2 \cos^4 \varphi \le \frac{25}{96}$ by (10). In addition, we claim that

$$e^{-\frac{m-1}{2} \cdot \left\lceil \frac{d+1}{m} \right\rceil \cos^2 \varphi} > \sin^d \varphi. \tag{11}$$

Since $\left\lceil \frac{d+1}{m} \right\rceil < \frac{d+1}{m} + 1$ and $m^2 \le d+1$ yield that $\frac{m-1}{2} \cdot \left\lceil \frac{d+1}{m} \right\rceil < d$, we deduce (11) by $e^{-dt} > (1-t)^d$. In turn, we conclude Corollary 1.2. Q.E.D.

Remark 5.1. *If $\varphi < \frac{\pi}{2}$ then there exists a covering by spherical balls of radius φ whose density is at most*

$$\left(1 + c \cdot \tfrac{\ln \ln d}{\ln d}\right) \cdot d \ln d. \tag{12}$$

This estimate is a slight improvement of Corollary 1.2 if say $\varphi < \arccos \frac{1}{\sqrt[4]{d}}$ and d is large.

We sketch the proof that is based on an idea of C.A. Rogers [17]. Let $\eta = \frac{1}{d \ln d}$. We first distribute N centres $x_1, \ldots, x_N$ on S^d independently and with uniform distribution according to the probability measure on S^d for suitable N. According to Lemma 2.1, we may choose the centres in a way that the probability measure of the part of S^d not covered by balls $B((1-\eta)\,\varphi, x_i)$, $i = 1, \ldots, N$, is at most $e^{-N\Omega((1-\eta)\,\varphi)}$. Next we place as many non–overlapping copies of spherical balls of radius $\eta\,\varphi$ into the uncovered part as possible, say the spherical balls $B(\eta\,\varphi, y_i)$, $i = 1, \ldots, M$. Since $1 - t < e^{-t}$ holds for positive $t < 1$, we deduce that

$$M \le \frac{(1 - \Omega((1-\eta)\,\varphi))^N}{\Omega(\eta\,\varphi)} < \frac{e^{-N \cdot \Omega((1-\eta)\,\varphi)}}{\Omega(\eta\,\varphi)}.$$

Now for any $z \in S^d$, the spherical ball $B(\eta\,\varphi, z)$ intersects either some $B((1-\eta)\,\varphi, x_i)$ or some $B(\eta\,\varphi, y_j)$, and hence we obtain a covering of S^d by $n = N + M$ spherical balls of radius φ where

$$n \le N + \frac{e^{-N \cdot \Omega((1-\eta)\,\varphi)}}{\Omega(\eta\,\varphi)}.$$

Derivation shows that the optimal N is

$$N = \left\lfloor \frac{1}{\Omega((1-\eta)\,\varphi)} \cdot \ln \frac{\Omega((1-\eta)\,\varphi)}{\Omega(\eta\,\varphi)} \right\rfloor.$$

Since readily $\Omega((1-\eta)\,\varphi) < \frac{1}{2}$, we deduce that the density of the covering by n spherical balls of radius φ is

$$n \cdot \Omega(\varphi) < \frac{\Omega(\varphi)}{\Omega((1-\eta)\,\varphi)} \cdot \left(\ln \frac{\Omega((1-\eta)\,\varphi)}{\Omega(\eta\,\varphi)} + \sqrt{e} \right).$$

Finally we apply the estimate of Corollary 3.2 (ii). Since $\eta = \frac{1}{d \ln d}$, it follows that

$$n \cdot \Omega(\varphi) < \frac{1}{(1-\eta)^d} \cdot \left(\ln \frac{1}{\eta^d} + \sqrt{e} \right) < (1 + \tfrac{2}{\ln d}) \cdot (d \ln d + d \ln \ln d + \sqrt{e}).$$

In turn, we conclude (12).

6 Examples

Let us discuss the optimality of the estimates in Corollary 1.2 where we always assume that d is large:

Example 6.1. *If $\gamma \geq 1$ and $\arccos \frac{\gamma}{\sqrt{d+1}} \leq \varphi < \frac{\pi}{2}$ then some $e^{c\,\gamma^2} \cdot d$ spherical balls of radius φ cover S^d where c is a absolute constant; namely, the number of balls needed is linear in d.*

We note that at least $d+2$ spherical balls are needed to cover S^d.

Example 6.2. *If $\frac{1}{\sqrt{d+1}} \leq \cos\varphi \leq \frac{\ln d}{\sqrt{d+1}}$ then there exists a covering by spherical balls of radius φ with density at most $c\,d \ln \ln d$ where c is a absolute constant.*

For example, this case includes when the number of balls in the optimal covering is polynomial in d; namely, at most d^α for an absolute constant $\alpha \geq 2$. More precisely, $d^{\ln d + 2}$ balls are sufficient.

Example 6.3. *If $\varphi \leq \arcsin \sqrt{\frac{1}{d+1}}$ then the density of any finite covering of S^d by spherical balls of radius φ is at least $c\,d$ where c is a positive absolute constant.* Therefore the upper bound in Corollary 1.2 is close to be optimal up to a factor of constant times $\ln d$.

We verify the lower bound for the density on S^d by comparing it to the Euclidean simplex bound. For a spherical d–simplex T of circumradius at most φ, the $d+1$ spherical balls of radius φ centred at the vertices cover T. We define the *associated density* to be the ratio of the sum of the volumes of the sectors of the $d+1$ balls, lying in the simplex, to the volume of the simplex. For some a finite covering of S^d by spherical balls of radius φ, we write P to denote the convex hull (in $\mathbb{E}^{d+1}$) of the centres, and we consider a triangulation of ∂P using the same set of vertices. The radial projections of these Euclidean simplices on S^d results in a spherical triangulation of S^d, which triangulation is known as *Delone triangulation*.

Let T be a spherical d–simplex of the Delone triangulation. Since the original system of spherical balls cover S^d, we deduce that the spherical circumradius of T is at most φ. Now the Euclidean d–simplex T' whose vertices coincide with the vertices of T, is of circumradius at most $\sin\varphi$. In addition, the Euclidean d–ball in aff T' centred at a vertex v of T' and of radius $\sin\varphi$ satisfies that its radial projection onto S^d is contained in the spherical ball centred at v and of radius φ. According to the simplex bound by H.S.M. Coxeter, L. Few and C.A. Rogers [9], the associated density for T' (which can be defined analogously) is minimal for the regular simplex, and hence the associated density for T' is at least $c'\,d$ for a positive absolute constant c'. Since $\cos\varphi \geq \sqrt{1-\frac{1}{d+1}}$, we conclude that the associated density for the spherical simplex T is at least $c\,d$. The same estimate for each Delone d–simplex yields the density bound $c\,d$ for the covering of S^d.

7 Covering S^d by d+3 balls

The proof of Theorem 1.4 use Euclidean tools, and depends on the following observation:

Lemma 7.1. *Some family of spherical balls of radius $\varphi < \frac{\pi}{2}$ cover S^d if and only if the convex hull of the centres in $\mathbb{E}^{d+1}$ contains a ball centred at the origin of radius $\cos\varphi$.*

Proof: We write $B(r,x)$ to denote the spherical ball of radius r and of centre x. First we assume that spherical balls of radius $\varphi < \frac{\pi}{2}$ cover S^d, and write P to denote the convex hull of the centres. The condition $\varphi < \frac{\pi}{2}$ yields that P contains the origin in its interior. If F is a facet of P then aff F cuts off a spherical ball $B(\psi, y)$ with $\psi < \frac{\pi}{2}$. Now y is a contained in a spherical ball of radius φ around a vertex of P, and hence $\psi \leq \varphi$. Therefore the Euclidean distance of aff F from the origin is at least $\cos\varphi$.

On the other hand, let P be a polytope whose vertices lie on S^d, and we assume that P contains the Euclidean ball centred at the origin of radius $\cos\varphi$. For any $y \in S^d$, P intersects the convex hull (in $\mathbb{E}^{d+1}$) of $B(\varphi, y)$, and hence $B(\varphi, y)$ contains a vertex v of P. Therefore $y \in B(\varphi, v)$. Q.E.D.

In order to verify Theorem 1.4, we seek the polytope inside the unit ball with the largest inscribed ball regardless of the place of the centre of the inscribed ball, and show that the centre of the inscribed ball is actually the origin for the optimal polytope. Therefore let $P \subset B^{d+1}$ be a polytope with at most $d+3$ vertices such that $r(P)$ is maximal, and let B be an inscribed ball of P.

If P has only $d+2$ vertices then it is a $(d+1)$–simplex. We choose a vertex v, and a point $u \in S^d$ such that the segment vu intersect the interior of P. Then the convex hull P' of P and u has $d+3$ vertices, and each facet of P' that touches B contains v. Therefore B can be enlarged from v in a way that the resulting ball is still contained in P', what is a contradiction.

Therefore P has actually $d+3$ vertices. According to the Radon Theorem, P is the convex hull of a k–dimensional simplex S and an m–dimensional simplex T such that $k+m=d+1$ holds for $k,m\geq 1$, and $S\cap T\neq\emptyset$. It follows that S and T intersect in one point that we denote by q, and q is not a vertex of P.

Next we suppose that a vertex v of P is contained in the interior of B^{d+1}, say v is a vertex of T. We choose a v' outside of P such that the ray $v'v$ intersects the interior of P, and for any facet F of P that does not contain v, the point v' lies in the open halfspace determined by $\operatorname{aff} F$ that intersects P. Now we define P' to be the convex hull of P and v', and observe that P' has $d+3$ vertices, as well. We choose a vertex w of T different from v. Since any facet of P contains either v or w, we deduce that any facet of P' contains either v' or w. On the other hand, no facet of P' containing v' touches B, thus B can be enlarged from w in a way that the resulting ball is still contained in P'. This contradiction yields that each vertex of P lies on S^d.

Finally, we suppose that there exist an edge vw of say S and a vertex u of P such that $d(u,v)\neq d(u,w)$. Let H be the hyperplane that bisects the edge vw perpendicularly, and we write X' to denote the image of any object X by the Steiner symmetrization with respect to H (see say T. Bonnesen and W. Fenchel [5]). We observe that $P'\subset B^{d+1}$, and P' is a polytope with $d+3$ vertices. Since $B'\subset P'$, we deduce that P' is an optimal polytope contained in B^{d+1}, as well. On the other hand, the vertex u' of P' lies in the interior of B^{d+1}, what is absurd. Therefore S and T are regular k and m–simplices, respectively, whose affine hulls are orthogonal, and meet at the common circumcentre of S and T.

We deduce that P has a unique inscribed ball that is centred at the origin and touches each facet. Any facet C of P is the convex hull of a $(k-1)$–face F of S and an $(m-1)$–face G of T. We observe that the distance of F and G from the origin is $\frac{1}{k}$ and $\frac{1}{m}$, respectively, and $F\subset(\operatorname{lin} G)^{\perp}$ and $G\subset(\operatorname{lin} F)^{\perp}$. Denoting the centroids of F and G by p and q, respectively, the distance of the origin from C is the distance of the origin from the segment pq. Since the segments op and oq are orthogonal, and of length $\frac{1}{k}$ and $\frac{1}{m}$, respectively, we deduce that $r(P)=\frac{1}{k^2+m^2}$. The condition $k+m=d+1$ yields that the sum k^2+m^2 is minimal if k and m are as close to $\frac{d+1}{2}$ as possible. Therefore the dimensions of S and T are $\lfloor\frac{d+1}{2}\rfloor$ and $\lceil\frac{d+1}{2}\rceil$ in this or in the reverse order. In turn, we conclude Theorem 1.4 by Lemma 7.1. Q.E.D.

We note that similar applications of the Radon Theorem occur in various context, say I. Bárány [2] applies it to the minimal ratio of the maximum distance to the minimal distance among $d+2$ points in $\mathbb{E}^d$.

8 Inradius of the inscribed polytopes

We have proved various estimates on the density of coverings by equal spherical balls. We would like to summarize these and some related results from another perspective; namely, we consider the largest origin centred Euclidean d–ball contained in the convex hull of a given number of points on S^{d-1}. More precisely, the covering problem on S^{d-1} can be formulated as follows: What is the minimal radius $\varphi(N)$ of N equal spherical balls that cover S^{d-1}? According to Lemma 7.1, $\cos\varphi(N)$ is the maximal radius of the Euclidean balls centred at the origin that are contained in the convex hull of N points chosen from S^{d-1}. Equivalently, $\frac{1}{\cos\varphi(N)}$ is the Banach–Mazur distance of ellipsoids from polytopes of at most N vertices.

If $N \geq \sqrt{2}^d$ then Corollary 1.2 and the trivial estimate $N \cdot \Omega(\varphi(N)) \geq 1$ yield

$$1 - \cos\varphi(N) = (1 + e(N,d)) \cdot \tfrac{1}{2}\, N^{\frac{-2}{d-1}}, \qquad \text{where } \; |e(N,d)| < N^{\frac{-2}{d-1}}. \tag{13}$$

Therefore let $d + 1 \leq N \leq \sqrt{2}^d$. Concerning lower bounds,

$$\cos\varphi(N) > \begin{cases} \left(1 - \frac{\ln N}{d} - o(1)\right) \cdot \sqrt{\frac{2\ln\frac{N}{d}}{d}} & \text{if } d^2 \leq N \leq \sqrt{2}^d; \\ c \cdot \sqrt{\frac{\ln\frac{N}{d}}{d}} & \text{if } d+1 \leq N \leq d^2, \end{cases} \tag{14}$$

where $o(1)$ denotes a function of d that tends to zero as d tends to infinity, and c is a positive absolute constant. Here if $d+1 \leq N \leq 2d$ then the arrangements described in Conjecture 1.3 provide the lower bound, while the case $2d < N \leq \sqrt{2}^d$ is a consequence of Corollary 1.2 and the estimates

$$e^{\frac{d}{2}\cdot\cos^2\varphi(N)} < \sin^{-d}\varphi(N) < e^{\frac{d}{2}\cdot\cos^2\varphi(N)\cdot(1+\cos^2\varphi(N))}.$$

Let us turn to upper bounds when $d + 1 \leq N \leq \sqrt{2}^d$. The trivial estimate $N \cdot \Omega(\varphi(N)) \geq 1$ yields that

$$\cos\varphi(N) < \sqrt{\frac{2\ln N}{d}}. \tag{15}$$

According to the lower bounds above, this estimate is a asymptotically optimal if $N \geq d^{\ln d}$, say, and good up to a multiplicative constant $\sqrt{2}$ if $N \geq d^2$. On the other hand, a better upper bound is needed if N is close to be linear in d. It is known that if $2d \leq N \leq d^2$ then

$$\cos\varphi(N) < c \cdot \sqrt{\frac{\ln\frac{N}{d}}{d}} \tag{16}$$

for some absolute constant c. It is still open whether $\cos\varphi(N) < c\cdot\sqrt{\frac{\ln\frac{N}{d}}{d}}$ if $d+2 < N < 2d$. A very satisfying way to prove this estimate would be by verifying Conjecture 1.3.

Next we discuss the two known arguments leading to (16). Both of them give an upper bound for the maximal relative volume $V(d,N)$ of a polytope with N vertices that is contained in the unit ball B^d. We note that readily

$$\cos\varphi(N) < \sqrt[d]{V(d,N)}.$$

Now B. Carl [6] (see also B. Carl and A. Pajor [7]) verified (16) using deep methods from the local theory of Banach spaces. Independently, I. Bárány and Z. Füredi [3] obtained the same estimate. Here we show how their elementary argument yields (16) because (16) is not stated explicitly in the paper, and even the experts seem to be unaware the fact that the proof of I. Bárány and Z. Füredi [3] leads to (16).

Theorem 3 of I. Bárány and Z. Füredi [3] states that

$$V(d,N) \le \binom{N}{k+1}\cdot\frac{\sqrt{k+1}}{k!}\left(1+\frac{1}{k}\right)^{\frac{k}{2}}\cdot\frac{\kappa_{d-k}}{\kappa_d}\left(\frac{d-k}{d\cdot k}\right)^{\frac{d-k}{2}} \tag{17}$$

holds for any $k=1,\dots,d-1$. This theorem is a consequence of the (elementary) fact that if S is a d simplex of circumradius one and $x\in S$ then S has a k–face F such that the closest point y of aff F to x lies in F and $d(x,y)\le\sqrt{\frac{d-k}{d\cdot k}}$. Since $\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ (see say L. Fejes Tóth [14]), the Stirling formula (3) and (17) yield that

$$V(d,N) < c_0^k\cdot N\cdot\frac{N^k d^{\frac{k}{2}}}{k^{\frac{d+3k}{2}}}$$

for some absolute constant c_0. In particular, if $\alpha=\frac{d}{k}$ then

$$\sqrt[d]{V(d,N)} < c_0^{\frac{1}{\alpha}}\cdot N^{\frac{1}{d}}\cdot\frac{N^{\frac{1}{\alpha}}\cdot d^{\frac{1}{2\alpha}}}{\sqrt{k}\cdot k^{\frac{3}{2\alpha}}} = c_0^{\frac{1}{\alpha}}\cdot N^{\frac{1}{d}}\cdot e^{\frac{\ln\frac{N}{d}}{\alpha}}\cdot\alpha^{\frac{3}{2\alpha}}\cdot\frac{1}{\sqrt{k}}.$$

Therefore choosing $k=\left\lfloor\frac{d}{2\ln\frac{N}{d}}\right\rfloor$ (as it is implicitly suggested by I. Bárány and Z. Füredi [3]) yields (16).

Finally, we note that estimating $V(d,N)$ does not give sufficient information about $\cos\varphi(N)$ if N is close to $d+1$ because the volume of the regular simplex inscribed into the unit balls larger than $\frac{1}{2^d}$ times the volume of the inscribed regular crosspolytope.

Remark 8.1. We would like to point out that if $d^2 \leq N < 1.05^d$ then the argument above gives

$$\sqrt[d]{V(d,N)} < \left(1 + \frac{4\ln N}{d} + c_1 \cdot \frac{\ln\ln N}{\ln N}\right) \cdot \sqrt{\frac{2e\ln\frac{N}{d}}{d}}$$

for some absolute constant c_1. Here we stop at 1.05^d because say for $1.05^d < N < \sqrt{2}^d$, the bound $V(d,N) < \frac{N}{2^d}$ (see Gy. Elekes [11]) slowly takes over. Comparing to (15), we see that the quotient of the upper and the lower bound for $\cos\varphi(N)$ is at most asymptotically $\sqrt{e}$.

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About Authors

Károly Böröczky, Jr. is at the Rényi Institute, Budapest, PO. Box. 127, H–1364 Hungary. *e-mail:* carlos@renyi.hu

Gergely Wintsche is at the Lóránd Eötvös University, Teacher Training Faculty, Budapest, Pázmány P. sétány 1/c, 1117 Hungary. *e-mail:* wgerg@ludens.elte.hu

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